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Ingham—Jessen's inequality for deviation means

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1. Introduction

Let $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{N}$ denote the set of real numbers, positive real numbers and natural numbers, respectively. Let $I\subset\mathbf{R}$ be an interval and let M and N be discrete symmetric means on I. (See PÁLES [11].) We say that M and N satisfy the Ingham—Jessen's inequality if

$$M(N(x_{11}, \ldots, x_{1n}), \ldots, N(x_{m1}, \ldots, x_{mn})) \leqq$$

$$\leqq N(M(x_{11}, \ldots, x_{m1}), \ldots, M(x_{1n}, \ldots, x_{mn}))$$

i.e. if

$$M^i(N^j(x_{ij})) \leqq N^j(M^i(x_{ij})) \tag{1}$$

for $x_{ij}\in I$, $i\in\{1, \ldots, m\}$, $j\in\{1, \ldots, n\}$, $n, m\in\mathbf{N}$.

This inequality was considered first by JESSEN [7] and INGHAM in the case when M and N are power means.

Define, for $a\in\mathbf{R}$, $x=(x_1, \ldots, x_n)\in\mathbf{R}_+^n$ $(n\in\mathbf{N})$, the a-th power mean $M_a(x)=$ $=M_a^i(x_i)=M_a(x_1, \ldots, x_n)$ by

$$\left(\sum_{i=1}^n x_i^a/n\right)^{1/a}, \quad \text{if} \quad a \neq 0,$$

$$\left(\prod_{i=1}^n x_i\right)^{1/n}, \quad \text{if} \quad a = 0.$$

Now the result obtained by JESSEN [7] can be formulated as follows (see HARDY—LITTLEWOOD—PÓLYA [6, Th. 26, p. 31]):

Theorem A. *Let* $a, b\in\mathbf{R}$. *In order that the inequality*

$$M_b^i(M_a^j(x_{ij})) \leqq M_a^j(M_b^i(x_{ij})) \tag{2}$$

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be valid for any $x_{ij}\in\mathbf{R}_+$, $i\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$, $n,m\in\mathbf{N}$, *it is necessary and sufficient that*

$$a \leqq b. \tag{3}$$

There are a lot of investigations concerning this result. JESSEN [8] considered a more complicated inequality than (1) for power means. (See [6, Th. 137, p. 101].) KALMAN [9] proved a more general inequality than (2). TOYAMA [14] investigated the ratio of the right and left hand sides of (2). Fixing n and m, he gave the greatest lower and least upper bounds of this ratio.

A natural way of generalizing the inequality (2) is to investigate (1) for more general classes of means than power means. In [13], the author considered inequality (1) for homogeneous quasiarithmetic means with continuous weight function. These are the means defined as follows (see ACZÉL—DARÓCZY [1]): for $a,p\in\mathbf{R}$, $x=(x_1,\dots,x_n)\in\mathbf{R}_+^n$, $n\in\mathbf{N}$, let

$$M_a(x)_p = M_a^i(x_i)_p = M_a(x_1,\dots,x_n) = \Big(\sum_{i=1}^n x_i^{a+p}\Big/\sum_{i=1}^n x_i^p\Big)^{1/a}, \quad \text{if} \quad a\neq 0,$$

$$= \exp\Big(\sum_{i+1}^n x_i^p \ln x_i\Big/\sum_{i=1}^n x_i^p\Big), \quad \text{if} \quad a=0.$$

For $t\in\mathbf{R}$, denote $t^+=\max\{t,0\}$, $t^-=\max\{-t,0\}$.

Concerning these mean values the author obtained the following result (see [13]):

Theorem B. *Let* $a,b,p,q\in\mathbf{R}$. *In order that the inequality*

$$M_b^i(M_a^j(x_{ij})_p)_q \leqq M_a^j(M_b^i(x_{ij})_q)_p \tag{4}$$

be valid for any $x_{ij}\in\mathbf{R}_+$, $i\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$, $n,m\in\mathbf{N}$, *it is necessary and sufficient that*

$$p-a^- \leqq q-b^- \leqq p+a^+ \leqq q+b^+, \quad (p-a^-)(q-b^-)(p+a^+)(q+b^+)=0. \tag{5}$$

It is easy to see that if $p=q=0$ then we obtain the power means, furthermore, (4) and (5) turn into (2) and (3), respectively.

JESSEN [8] investigated (1) for quasiarithmetic means, too. However he obtained only necessary conditions. (See [6, Th. 136, p. 100].) The aim of the present note is to discuss (1) under very general circumstances. We consider inequality (1) for deviation means. This class of means has many interesting properties (see e.g. DARÓCZY [3], [4], DARÓCZY—PÁLES [5], PÁLES [10], [11], [12]) and contains the well-known classes of means (e.g. power means, quasiarithmetic means, quasiarithmetic means with weight function (ACZÉL—DARÓCZY [1], BAJRAKTAREVIC [2])). If M and N are deviation means then under certain regularity assumptions we obtain necessary and sufficient conditions in order that (1) be valid. We also consider (1) for homogeneous deviation means. In this case the necessary and sufficient conditions are simpler. In the last section we mention some open problems.

2. Notations, definitions and auxiliary results

Let $I \subset \mathbf{R}$ be an open interval. Now, we introduce the basic subclass of deviation functions.

Definition 1. A function $E: I^2 \to \mathbf{R}$ is called a $*$-deviation if it satisfies the following properties:

(i) E is twice differentiable on I^2.

(ii) $\partial E(x, t)/\partial t = \partial_2 E(x, t) < 0$ for $x, t \in I$.

(iii) $E(t, t) = 0$ for $t \in I$.

The class of $*$-deviations will be denoted by $\mathscr{E}(I)$. For $*$-deviations on I we shall also need the following usefull notation: If $E \in \mathscr{E}(I)$ then define E^* by

$$E^*(x, t) = -E(x, t)/\partial_2 E(t, t), \quad x, t \in I.$$

The following theorem and definition is due to Daróczy [3], [4].

Theorem C. *Let* $E \in \mathscr{E}(I)$, $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$. *Then there exists a unique number* t_0 *in* I *such that*

$$\operatorname{sgn} \sum_{i=1}^{n} E(x_i, t) = \operatorname{sgn}(t_0 - t) \tag{6}$$

for $t \in I$ *and*

$$\min_{1 \le i \le n} x_i \le t_0 \le \max_{1 \le i \le n} x_i. \tag{7}$$

Definition 2. Let $E \in \mathscr{E}(I)$, $n \in \mathbf{N}$, $x = (x_1, \ldots, x_n) \in I^n$, and consider the equation

$$\sum_{i=1}^{n} E(x_i, t) = 0. \tag{8}$$

Then, by Theorem C, there exists a unique solution $t = t_0$ of (8) and this solution is called the E-deviation mean of x and is denoted by $\mathfrak{M}_E(x)$ or $\mathfrak{M}_E^i(x_i)$ or $\mathfrak{M}_E(x_1, \ldots, x_n)$. (7) shows that $\mathfrak{M}_E(x)$ is indeed a mean value of x.

Remark. The proof of Theorem C can be found in [3], [4]. However, it can easily be proved using the facts that the function

$$t \to \sum_{i=1}^{n} E(x_i, t), \quad t \in I,$$

is continuous, strictly monoton decreasing and changes sign on the interval I.

The class of $*$-deviations is contained in the class of deviations introduced by Daróczy [3], [4]. Theorem C and Definition 2 can very easily be extended to deviations (see [3]).

Examples. 1. Let $\varphi\colon I\to\mathbf{R}$ be a twice differentiable function with positive first derivative and let $f\colon I\to\mathbf{R}_+$ be a positive, twice differentiable function. Set

$$E_{\varphi,f}(x,t)=f(x)\big(\varphi(x)-\varphi(t)\big),\quad x,t\in I. \tag{9}$$

It is obvious that $E_{\varphi,f}\in\mathcal{E}(I)$. If $n\in\mathrm{N}$, $x=(x_1,\dots,x_n)\in I^n$ then $\mathfrak{M}_{E_{\varphi,f}}(x)$ has the following form:

$$\mathfrak{M}_{E_{\varphi,f}}(x)\doteq M_\varphi(x)_f=\varphi^{-1}\Big(\sum_{i=1}^n f(x_i)\varphi(x_i)\Big/\sum_{i=1}^n f(x_i)\Big)$$

i.e. $\mathfrak{M}_{E_{\varphi,f}}$ is a quasiarithmetic mean with weight function (see Bajraktarević [2]). If $f(x)=1$ then $\mathfrak{M}_{E_{\varphi,f}}$ becomes the quasiarithmetic mean M_φ (see Hardy—Littlewood—Pólya [6]).

2. Let $a,p\in\mathbf{R}$ and set

$$E_{a,p}(x,t)=\begin{cases} x^p(x^a-t^a)/a, & \text{if } a\neq 0,\\ x^p(\ln x-\ln t), & \text{if } a=0.\end{cases} \tag{10}$$

Now, for $x\in I^n$, we obtain that $\mathfrak{M}_{E_{a,p}}(x)=M_a(x)_p$. If $p=0$ then we get the power means.

Now we prove a sequence of lemmas which will be needed later on.

Lemma 1. *Let $E\in\mathcal{E}(I)$. Then, for fixed $n\in\mathrm{N}$,*

$$(x_1,\dots,x_n)\to\mathfrak{M}_E(x_1,\dots,x_n) \tag{11}$$

is a continuously differentiable function on I^n and

$$\partial_i\mathfrak{M}_E(x_1,\dots,x_n)=-\partial_1E(x_i,\mathfrak{M}_E(x))\Big/\Big(\sum_{j=1}^n\partial_2E(x_j,\mathfrak{M}_E(x))\Big) \tag{12}$$

for $x=(x_1,\dots,x_n)\in I^n$. (Here ∂_i denotes the partial differentiation with respect to the i-th variable.)

Proof. Let $x_0=(x_{01},\dots,x_{0n})\in I^n$ be fixed and denote by t_0 the mean value $\mathfrak{M}_E(x_0)$. Let

$$F(x,t)=\sum_{i=1}^n E(x_i,t)$$

for $x=(x_1,\dots,x_n)\in I^n$ and $t\in I$. By our assumption (i) on deviations belonging to $\mathcal{E}(I)$, F is continuously differentiable in a neighborhood of (x_0,t_0). (ii) in Definition 1 implies that

$$\partial_t F(x_0,t_0)=\sum_{i=1}^n\partial_2E(x_o,t_0)<0,$$

and we know that $F(x_0, t_0)=0$. Thus the conditions of the implicit function theorem are satisfied. Consequently, the function (11) determined by the equation

$$F(x, \mathfrak{M}_E(x)) = 0$$

is differentiable at the point x_0 and its derivative has the form

$$D\mathfrak{M}_E(x_0) = -(\partial_t F(x_0, t_0))^{-1}\partial_x F(x_0, t_0)$$

i.e. (12) is satisfied at x_0.

The continuity of the function (19) follows from (i). This completes the proof of the lemma.

Lemma 2. *Let $E\in\mathcal{E}(I)$. Then, for $x, t\in I$,*

$$\lim_{n\to\infty} (n-1)\big(\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1})-t\big) = E^*(x, t). \tag{13}$$

We omit the proof of this lemma since it is proved in Daróczy [3], [4].

Lemma 3. *Let $E\in\mathcal{E}(I)$. Then, for $x, t\in I$,*

$$\lim_{n\to\infty} (n-1)\partial_1\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = \partial_1 E^*(x, t). \tag{14}$$

Proof. Let $x, t\in I$ be arbitrary. For $n\in\mathbf{N}$, let

$$t_n = \mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}). \tag{15}$$

Applying Lemma 1, we have

$$\partial_1\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = -\frac{\partial_1 E(x, t_n)}{\partial_2 E(x, t_n)+(n-1)\partial_2 E(t, t_n)}.$$

Hence

$$(n-1)\partial_1\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = -\frac{\partial_1 E(x, t_n)}{\partial_2 E(x, t_n)/(n-1)+\partial_2 E(t, t_n)}. \tag{16}$$

By Lemma 2,

$$\lim_{n\to\infty} t_n = t. \tag{17}$$

Therefore, taking the limit $n\to\infty$ in (16) we obtain (14).

Lemma 4. *Let $E\in\mathcal{E}(I)$. Then, for $x, t\in I$,*

$$\lim_{n\to\infty} (n-1)\Big(\sum_{i=2}^{n} \partial_i\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1})-1\Big) = \partial_2 E^*(x, t). \tag{18}$$

Proof. Let $x, t\in I$ be arbitrary and let t_n be defined by (15). Applying Lemma 1 we have

$$\partial_i\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = -\frac{\partial_1 E(t, t_n)}{\partial_2 E(x, t_n)+(n-1)\partial_2 E(t, t_n)}$$

for $2 \leq i \leq n$. Hence, after a simple calculation, we obtain

$$(19)\qquad \begin{aligned} &(n-1)\Big(\sum_{i=2}^{n} \partial_i \mathfrak{M}_E(x, \underbrace{t, \ldots, t}_{n-1}) - 1\Big) = \\ &= -\frac{\partial_2 E(x, t_n) + (n-1)(\partial_1 E(t, t_n) + \partial_2 E(t, t_n))}{\partial_2 E(x, t_n)/(n-1) + \partial_2 E(t, t_n)}. \end{aligned}$$

Since $E(t, t) = 0$, we have

$$(20)\qquad \partial_1 E(t, t) + \partial_2 E(t, t) = 0 \quad \text{for} \quad t \in I.$$

If $t \neq x$ then t_n is strictly between x and t. Applying (20) and Lemma 2, we obtain

$$(21)\qquad \begin{aligned} &\lim_{n\to\infty} (n-1)(\partial_1 E(t, t_n) + \partial_2 E(t, t_n)) = \\ &= \lim_{n\to\infty} (n-1)(t_n - t)\left(\frac{\partial_1 E(t, t_n) - \partial_1 E(t, t)}{t_n - t} - \frac{\partial_2 E(t, t_n) - \partial_2 E(t, t)}{t_n - t}\right) = \\ &= E^*(x, t)(\partial_2\partial_1 E(t, t) + \partial_2\partial_2 E(t, t)). \end{aligned}$$

It is easy to see that (21) remains valid if $x = t = t_n$. Now, applying (17) and (21), we can calculate the limit of the right hand side of (19). We get

$$\begin{aligned} &\lim_{n\to\infty} (n-1)\Big(\sum_{i=2}^{n} \partial_i \mathfrak{M}_E(x, \underbrace{t, \ldots, t}_{n-1}) - 1\Big) = \\ &= -\frac{\partial_2 E(x, t) + E^*(x, t)(\partial_2\partial_1 E(t, t) + \partial_2\partial_2 E(t, t))}{\partial_2 E(t, t)} = \partial_2 E^*(x, t). \end{aligned}$$

The proof is complete.

Remark. There is a simple connection between Lemma 2 and Lemmas 3 and 4. Differentiating (13) with respect to x and t we obtain (14) and (18), respectively.

3. The main result

In this section we give necessary and sufficient conditions in order that (1) be satisfied with $M = \mathfrak{M}_F$, $N = \mathfrak{M}_E$ where $F, E \in \mathscr{E}(I)$.

Theorem 1. *Let* $E, F \in \mathscr{E}(I)$. *The inequality*

$$(22)\qquad \mathfrak{M}_F^i(\mathfrak{M}_E^j(x_{ij})) \leq \mathfrak{M}_E^j(\mathfrak{M}_F^i(x_{ij}))$$

holds for any $x_{ij} \in I$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, $n, m \in \mathbf{N}$, *if and only if*

$$(23)\qquad \begin{aligned} &E^*(x, y)\partial_1 F^*(y, u) + E^*(z, u)\partial_2 F^*(y, u) \leq \\ &\leq F^*(x, z)\partial_1 E^*(z, u) + F^*(y, u)\partial_2 E^*(z, u) \end{aligned}$$

for each $x, y, z, u \in I$.

Proof. Necessity. Let $x, y, z, u \in I$ be arbitrary and let $n, m \in \mathbf{N}$. Define x_{ij} $(1 \leqq i \leqq m, 1 \leqq j \leqq n)$ as follows:

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} = \begin{pmatrix} x & y & \dots & y \\ z & u & \dots & u \\ \vdots & \vdots & & \vdots \\ z & u & \dots & u \end{pmatrix}. \tag{24}$$

Now introduce the following notations:

$$a(n) = \mathfrak{M}_E(x, \underbrace{y, \dots, y}_{n-1}), \qquad A(n) = \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1}),$$

$$b(m) = \mathfrak{M}_F(x, \underbrace{z, \dots, z}_{m-1}), \qquad B(m) = \mathfrak{M}_F(y, \underbrace{u, \dots, u}_{m-1}).$$

Applying inequality (22) for the x_{ij}'s defined by (24) we obtain

$$\mathfrak{M}_F(a(n), \underbrace{A(n), \dots, A(n)}_{m-1}) \leqq \mathfrak{M}_E(b(m), \underbrace{B(m), \dots, B(m)}_{n-1}) \tag{25}$$

for $n, m \in \mathbf{N}$. If $m \to \infty$ then, using Lemma 2, one can easily see that both sides of (25) tend to $A(n)$. Therefore we calculate the following limits:

$$\lim_{m \to \infty} (m-1)\big(\mathfrak{M}_F(a(n), \underbrace{A(n), \dots, A(n)}_{m-1}) - A(n)\big) \doteq R(n), \tag{26}$$

$$\lim_{m \to \infty} (m-1)\big(\mathfrak{M}_E(b(m), \underbrace{B(m), \dots, B(m)}_{n-1}) - A(n)\big) \doteq L(n). \tag{27}$$

To calculate (26) apply Lemma 2. Then we obtain

$$R(n) = F^*(a(n), A(n)). \tag{28}$$

It is a bit more complicated to determine $L(n)$. By Lemma 2 we have

$$\lim_{m \to \infty} (m-1)(b(m) - z) = F^*(x, z), \quad \lim_{m \to \infty} (m-1)(B(m) - u) = F^*(y, u).$$

Hence, using Lemma 1 and the differentiability of $\mathfrak{M}_E$, we get

$$\begin{aligned} L(n) &= \lim_{m \to \infty} (m-1)\big(\mathfrak{M}_E(b(m), \underbrace{B(m), \dots, B(m)}_{n-1}) - \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1})\big) = \\ &= \partial_1 \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1}) F^*(x, z) + \sum_{i=2}^{n} \partial_i \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1}) F^*(y, u). \end{aligned} \tag{29}$$

Inequality (25) implies that, for $n \in \mathbf{N}$,

$$R(n) \leqq L(n). \tag{30}$$

Using Lemmas 3 and 4 we easily obtain that both sides of (30) tend to $F^*(y, u)$. Therefore we calculate the following limits:

$$\lim_{n\to\infty}(n-1)\big(R(n)-F^*(y,u)\big) \doteq R^*, \quad \lim_{n\to\infty}(n-1)\big(L(n)-F^*(y,u)\big) \doteq L^*.$$

Applying Lemma 2 we have

$$\lim_{n\to\infty}(n-1)\big(a(n)-y\big) = E^*(x,y), \quad \lim_{n\to\infty}(n-1)\big(A(n)-u\big) = E^*(z,u).$$

Hence, using the differentiability of F^*, we get

$$\begin{aligned} R^* &= \lim_{n\to\infty}(n-1)\big(F^*(a(n),A(n))-F^*(y,u)\big) = \\ &= \partial_1 F^*(y,u)E^*(x,y)+\partial_2 F^*(y,u)E^*(z,u). \end{aligned} \tag{31}$$

Applying Lemmas 3 and 4 we have

$$\lim_{n\to\infty}(n-1)\partial_1\mathfrak{M}_E(z,\underbrace{u,\dots,u}_{n-1}) = \partial_1 E^*(z,u),$$

$$\lim_{n\to\infty}(n-1)\Big(\sum_{i=1}^{n}\partial_i\mathfrak{M}_E(z,\underbrace{u,\dots,u}_{n-2})-1\Big) = \partial_2 E^*(z,u).$$

Consequently

$$L^* = \partial_1 E^*(z,u)F^*(x,z) = \partial_2 E^*(z,u)F^*(y,u). \tag{32}$$

Inequality (30) implies $R^* \leqq L^*$. This completes the proof of (23).

Sufficiency. Let $n, m \in \mathrm{N}$ and $x_{ij} \in I$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. Further, let

$$y_i = \mathfrak{M}_E^j(x_{ij}), \quad z_j = \mathfrak{M}_F^i(x_{ij}), \quad u = \mathfrak{M}_E^j(\mathfrak{M}_F^i(x_{ij})) = \mathfrak{M}_E^j(z_j). \tag{33}$$

Apply (23) for $x=x_{ij}$, $y=y_i$, $z=z_j$ and add the inequalities obtained. Then we get

$$\begin{aligned} &\sum_{i=1}^{m}\Big\{\partial_1 F^*(y_i,u)\sum_{j=1}^{n}E^*(x_{ij},y_i)\Big\}+\sum_{i=1}^{m}\partial_2 F^*(y_i,u)\sum_{j=1}^{n}E^*(z_j,u) \leqq \\ &\leqq \sum_{j=1}^{n}\Big\{\partial_1 E^*(z_j,u)\sum_{i=1}^{m}F^*(x_{ij},z_j)\Big\}+\sum_{j=1}^{n}\partial_2 E^*(z_j,u)\sum_{i=1}^{n}F^*(y_i,u). \end{aligned} \tag{34}$$

Using (33) and Definition 2 we have

$$\sum_{j=1}^{n}E^*(x_{ij},y_i)=0, \quad \sum_{i=1}^{m}F^*(x_{ij},z_j)=0, \quad \sum_{j=1}^{n}E^*(z_j,u)=0.$$

Therefore (34) simplifies to the following inequality

$$0 \leqq \sum_{j=1}^{n}\partial_2 E^*(z_j,u)\sum_{i=1}^{m}F^*(y_i,u). \tag{35}$$

As we have seen in the proof of Lemma 4,

$$\partial_2 E^*(x, t) = -\frac{\partial_2 E(x, t)+E^*(x, t)\big(\partial_2\partial_1 E(t, t)+\partial_2\partial_2 E(t, t)\big)}{\partial_2 E(t, t)}.$$

Hence, by property (ii) of $*$-deviations,

$$\sum_{j=1}^{n} \partial_2 E^*(z_j, u) = -\sum_{j=1}^{n} \partial_2 E(z_j, u)/\partial_2 E(u, u) < 0. \tag{36}$$

(35) and (36) imply

$$\sum_{i=1}^{m} F(y_i, u) \leqq 0. \tag{37}$$

It follows from Theorem C and from (37) that $\mathfrak{M}_F^i(y_i) \leqq u$ i.e. (22) holds.

Remark. Applying Theorem 1 for the deviations defined by (9) we can easily obtain necessary and sufficient conditions for (1) if M and N are quasiarithmetic means with weight function.

4. Homogeneous means

Let $E \in \mathscr{E}(\mathbf{R}_+)$. The E-deviation mean $\mathfrak{M}_E$ is said to be homogeneous if

$$\mathfrak{M}_E(tx_1, \dots, tx_n) = t\mathfrak{M}_E(x_1, \dots, x_n)$$

for $t, x_1, \dots, x_n \in \mathbf{R}_+$, $n \in \mathbf{N}$.

Concerning homogeneous deviation means Daróczy [3] obtained the following result:

Theorem D. *Let $E \in \mathscr{E}(\mathbf{R}_+)$. Then $\mathfrak{M}_E$ is homogeneous if and only if*

$$E^*(x, t) = tE^*(x/t, 1)$$

for $x, t \in \mathbf{R}_+$.

For homogeneous deviation means Theorem 1 simplifies to the following form.

Theorem 2. *Let $E, F \in \mathscr{E}(\mathbf{R}_+)$ and assume that $\mathfrak{M}_E$ and $\mathfrak{M}_F$ are homogeneous means. Then the inequality (22) is valid for any $x_{ij} \in \mathbf{R}_+$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, $n, m \in \mathbf{N}$, if and only if*

$$t\partial_1^2 E^*(t, 1)\partial_1 F^*(s, 1) \leqq \partial_1 E^*(t, 1) s\partial_1^2 F^*(s, 1) \tag{38}$$

for $s, t \in \mathbf{R}_+$.

Proof. Since $\mathfrak{M}_E$ and $\mathfrak{M}_F$ are homogeneous means, Theorem D implies

$$E^*(x, t) = tE^*(x/t, 1) \doteq te(x/t), \quad F^*(x, t) = tF^*(x/t, 1) \doteq tf(x/t)$$

for $x, t \in \mathbf{R}_+$. By our assumptions on E and F we have that e and f are twice differentiable functions. Then, applying Theorem 1, we obtain that (22) holds if and only if

$$y/uf'(y/u)\{e(x/y)-e(z/u)\} \leqq z/ue'(z/u)\{f(x/z)-f(y/u)\} \tag{39}$$

for $x, y, z, u \in \mathbf{R}_+$.

Replacing x/u, y/u and z/u by r, s and t, respectively, we get

$$0 \leqq te'(t)\{f(r/t)-f(s)\}-sf'(s)\{e(r/s)-e(t)\} \tag{40}$$

for $r, s, t \in \mathbf{R}_+$. It is easy to check that (40) is equivalent to (39). Therefore the proof of the theorem will be complete if we show that (40) holds if and only if (38) is satisfied. Fixing s and t, we denote by $g(r)$ the right hand side of (40).

If (40) is satisfied then $r=st$ is the place of minimum of g. Hence $g''(st) \geqq 0$. This yields (38).

In the other direction we prove that

$$g'(r)(r-st) \geqq 0 \quad \text{for} \quad r>0. \tag{41}$$

Then $g(st)=0$ and (41) implies $g(r)>0$ for $r>0$.

Applying (38) for $s=r/t$ it can be easily seen that the function

$$t \to e'(t)f'(r/t), \quad t>0,$$

is monotone decreasing. Therefore, in the case $r<st$,

$$e'(t)f'(r/t) \leqq e'(r/s)f'(s),$$

i.e. (41) is satisfied in this case. In the case $r>st$ the proof of (41) is similar. The theorem is proved.

Remark. Applying Theorem 2 for the homogeneous means $M=M_{b,q}$, $N=M_{a,p}$ one can easily prove Theorem B. (For details see [13].)

5. Open problems and final remarks

Consider the following more general inequality than (1):

$$M_1^i(N_1^j(x_{ij})) \leqq N_2^j(M_2^i(x_{ij})) \tag{42}$$

for $x_{ij} \in I$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, $n, m \in \mathbf{N}$. (Here M_1, M_2, N_1, N_2 are discrete symmetric means on I.) The following conditions are necessary (but not sufficient) in order that (42) be satisfied:

$$M_1 \leqq M_2, \quad N_1 \leqq N_2, \tag{43}$$

$$N_1 \leqq M_2. \tag{44}$$

If we take $n=1$ and $m=1$ in (42) then we obtain (43). To prove that (44) is also a necessary condition we substitute into (42) the following matrix:

$$(x_{ij}) = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_2 & x_3 & x_4 & \dots & x_1 \\ \vdots & \vdots & \vdots & & \vdots \\ x_n & x_1 & x_2 & \dots & x_{n-1} \end{pmatrix}$$

(whenever $x_1, \dots, x_n \in I$); then we obtain (44).

The system of inequalities (43), (44) is also a sufficient condition if M_1, M_2, N_1, N_2 are power means. (See JESSEN [7], HARDY—LITTLEWOOD—PÓLYA [6, Th. 137, p. 101].) However, it is not sufficient in other classes of means. Finally, we formulate a condition which is sufficient in the class of deviation means.

Let $E_1, E_2, F_1, F_2 \in \mathscr{E}(I)$. If there exist functions $A_1, A_2, B_1, B_2\colon I^2 \to \mathbf{R}$ such that, for $x_1, \dots, x_n \in I$, $n \in \mathrm{N}$,

$$\sum_{j=1}^{n} B_2(x_j, \mathfrak{M}_{E_2}^j(x_j)) \leqq 0$$

and

$$E_1(x, y)A_1(y, u) + E_2(z, u)A_2(y, u) \leqq F_2(x, z)B_1(z, u) + F_1(y, u)B_2(z, u)$$

for $x, y, z, u \in I$ then (42) is satisfied for $M_1 = \mathfrak{M}_{F_1}$, $M_2 = \mathfrak{M}_{F_2}$, $N_1 = \mathfrak{M}_{E_1}$, $N_2 = \mathfrak{M}_{E_2}$. The proof of this proposition is similar to the proof of the sufficiency part of Theorem 1. We remark that this sufficient condition is also necessary if $E_1 = E_2 = E$, $F_1 = F_2 = F$. (That was the case investigated in Theorem 1.)

References

[1] J. ACZÉL—Z. DARÓCZY, Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind, *Publ. Math. Debrecen*, **10** (1963), 171—190.

[2] M. BAJRAKTAREVIC, Sur une équation fonctionnelle aux valeurs moyennes, *Glasnik Mat. Fiz. Astr.*, **13** (1958), 243—248.

[3] Z. DARÓCZY, Über eine Klasse von Mittelwerten, *Publ. Math. Debrecen*, **19** (1972), 211—217.

[4] Z. DARÓCZY, A general inequality for means, *Aequationes Math.*, **7** (1972), 16—21.

[5] Z. DARÓCZY—ZS. PÁLES, On comparison of mean values, *Publ. Math. Debrecen*, **29** (1982), 107—116.

[6] G. H. HARDY—J. E. LITTLEWOOD—G. PÓLYA, *Inequalities*, Cambridge Univ. Press (New York—London, 1952).

[7] B. JESSEN, Om Uligheder imellem Potensmiddelwaerdier, *Mat. Tidsskrift*, **B** (1931), No. 1.

[8] B. JESSEN, Bemearkinger om konvekse Funktioner og Uligheder imellem Middelwaerdier. I—II, *Mat. Tidsskrift*, **B** (1931), No. 2., Nos. 3—4.

[9] J. A. KALMAN, On the inequality of Ingham and Jessen, *J. London Math. Soc.*, **33** (1958), 306—311.

[10] Zs. PÁLES, On complementary inequalities, *Publ. Math. Debrecen*, **30** (1983), 75—88.
[11] Zs. PÁLES, Characterization of quasideviation means, *Acta Math. Acad. Sci. Hungar.*, **40** (1982), 243—260.
[12] Zs. PÁLES, General inequalities for quasideviation means, *Studia Sci. Math.*, submitted.
[13] Zs. PÁLES, Inequalities for homogeneous means depending on two parameters, in: *General Inequalities* 3 (E. F. Beckenbach and W. Walter, eds.) Birkhäuser Verlag (Basel—Boston—Stuttgart, 1983); pp. 107—122.
[14] H. TOYAMA, On the inequality of Ingham and Jessen, *Proc. Japan Acad.*, **24** (1948), 10—12.

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