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On multiplicative functions that are q-additive

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1. Introduction. We shall say that a complex-valued function $f(n)$ defined on the set of natural numbers is multiplicative if $f(ab)=f(a)f(b)$ holds for every coprime pairs a and b. Let $\mathcal{M}$ be the class of multiplicative functions.

Let $q\geqq 2$ be a fixed integer. Every positive integer can be represented in the form

$$n=a_0+a_1q+\ldots+a_sq^s,\quad a_s\neq 0,\quad a_i\in\{0,1,\ldots,q-1\} \tag{1.1}$$

uniquely. We shall say that a complex-valued function $g(n)$ defined on the set of nonnegative integers is q-additive if $g(0)=0$ and

$$g(n)=g(a_0)+g(a_1q)+\ldots+g(a_sq^s) \tag{1.2}$$

for n having the representation (1.1). Let $\mathcal{A}_q$ be the class of q-additive functions. This notion has been introduced by A. O. GELFOND [1].

It is obvious that $g(n)$ is a q-additive function if and only if

$$g(Aq^k+r)=g(Aq^k)+g(r) \tag{1.3}$$

whenever $0\leqq r<q^k$, $A\geqq 0$. It is obvious that $f(n)=n$ is a multiplicative and q-additive function. The zero function has the same properties.

Our main purpose is to determine all functions in $\mathcal{A}_q\cap\mathcal{M}$.

Theorem 1. *Let $f\in\mathcal{A}_q\cap\mathcal{M}$, $f(q)\neq 0$. Then $f(n)=n$.*

We shall give all the multiplicative functions $f(n)$ with period q, i.e. those for which

$$f(n+q)=f(n) \tag{1.4}$$

holds for every $n>0$.

Let $q=Q_1^{\omega_1}\ldots Q_r^{\omega_r}$, where $Q_1,\ldots,Q_r$ are distinct primes.

Main lemma. *Let $f\in\mathcal{M}$ satisfy* (1.4), $f(1)=1$.

(1) *Then $f(n)=\chi(n)$ for every n coprime to q, where χ is a multiplicative character* mod q.

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Let $\chi=\chi_1\cdot\ldots\cdot\chi_r$, *where* χ_i *is a multiplicative character* mod $Q_i^{\omega_i}$ $(i=1, 2, \ldots, r)$. *The components* χ_i *are determined by* χ. *Let*

$$\psi_i = \prod_{\substack{j=1\\ j\neq i}}^{r} \chi_j \quad (i = 1, \ldots, r), \tag{1.5}$$

Let $Q_i^{\varepsilon_i}$ *be the smallest period of* χ_i. *Then* $0\leqq\varepsilon_i\leqq\omega_i$.

(2) *We have*

$$f(Q_i^{\omega_i+h}) = \begin{cases} f(Q_i^{\omega_i})\psi_i(Q_i^h) & \text{if } \chi_i \text{ is principal character,} \\ 0 & \text{if } \chi_i \text{ is not the principal character} \end{cases} \tag{1.6}$$

$(h=1, 2, \ldots)$.

(3) *If* $f(Q_i^{\lambda_i})\neq 0$ *for some* $\lambda_i\in[1, \omega_i-1]$, *then* $\lambda_i\leqq\omega_i-\varepsilon_i$.

Conversely, if $f\in\mathcal{M}$ *satisfies* (1), (2), (3) *then* (1.4) *holds.*

Remark. The assertion stated here may be known. It has an auxiliary character for us. A. Sárközy [2] considered multiplicative arithmetic functions satisfying a linear recursion.

Theorem 2. *Let* $f\in\mathcal{M}\cap\mathcal{A}_q$, $f(1)=1$, $f(q)=0$. *Then* f *is a periodic function with period* q, $f(Q_i^{\omega_i})=0$ *for at least one* i. *The assertions* (1), (2), (3) *in the Main lemma are satisfied.*

Conversely, if these conditions hold and $f\in\mathcal{M}$, *then* $f\in\mathcal{A}_q$.

2. Proof of the Main lemma. Let us assume that $f\in\mathcal{M}, f(1)=1$ and (1.4) holds. It is well known that $f(n)=\chi(n)$ for $(n, q)=1$, χ is a character mod q.

Let i be fixed, $h>0$, $R=q\cdot Q_i^{-\omega_i}$, $n=Q_i^{\omega_i+h}\cdot r$, $(r, q)=1$. Then $n+q=Q_i^{\omega_i}(Q_i^h\cdot r+R)$. Since $(Q_i^h\cdot r+R, q)=1$, we have

$$f(n+q) = f(Q_i^{\omega_i})f(Q_i^h r+R) = f(Q_i^{\omega_i})\chi(Q_i^h r+R) =$$
$$= f(Q_i^{\omega_i})\chi_i(Q_i^h r+R)\psi_i(Q_i^h r+R) = f(Q_i^{\omega_i})\chi_i(Q_i^h r+R)\psi_i(Q_i^h r).$$

Here we observed that ψ_i is a character mod R. Similarly,

$$f(n) = f(Q_i^{\omega_i+h})\chi_i(r)\psi_i(r).$$

Since $\psi_i(r)\neq 0$, therefore from (1.4) we get

$$f(Q_i^{\omega_i+h})\chi_i(r) = f(Q_i^{\omega_i})\psi_i(Q_i^h)\chi_i(Q_i^h r+R). \tag{2.1}$$

This gives immediately that $f(Q_i^{\omega_i})=0$ if and only if $f(Q_i^{\omega_i+h})=0$. Let us assume that $f(Q_i^{\omega_i})\neq 0$. Then $f(Q_i^{\omega_i+h})\neq 0$ for $h=1, 2, \ldots$. By choosing $h=\omega_i$ and observing that $\chi_i(Q_i^{\omega_i}\cdot r+R)=\chi_i(R)$, from (2.1) we get that $\chi_i(r)$ is constant on the set $(r, q)=1$. Since χ_i is a character mod $Q_i^{\omega_i}$, its values depend on r (mod $Q_i^{\omega_i}$), consequently $\chi_i(r)$ is constant for $(r, Q_i)=1$, and so χ_i is the principal character. This proves (2).

Let now $n=Q_i^{\lambda_i}\cdot x, (x,q)=1, 1\leqq\lambda_i<\omega_i$. Then $n+q=Q_i^{\lambda_i}[x+Q_i^{\omega_i-\lambda_i}\cdot R]$. From (1.4) we get

$$f(n)=f(Q_i^{\lambda_i})\chi_i(x)\psi_i(x)=f(n+q)=f(Q_i^{\lambda_i})\chi_i(x+Q_i^{\omega_i-\lambda_i}R)\psi_i(x).$$

Let us assume that $f(Q_i^{\lambda_i})\neq 0$. Then

$$\chi_i(x)=\chi_i(x+Q_i^{\omega_i-\lambda_i}R)(x,q)=1. \tag{2.2}$$

Let $x_0=Ry$, $(y,Q_i)=1$, $x=x_0+t\cdot Q_i^{\varepsilon_i}$, $(t,R)=1$. Hence it follows that $(x,q)=1$, consequently from (2.2) we get

$$\chi_i(x_0)=\chi_i(x)=\chi_i(x+Q_i^{\omega_i-\lambda_i}R)=\chi_i(x_0+Q_i^{\omega_i-\lambda_i}R),$$

and so

$$\chi_i(y)\chi_i(R)=\chi_i(y+Q_i^{\omega_i-\lambda_i})\chi_i(R)\quad\text{for}\quad (y,Q_i)=1.$$

Since $\chi_i(R)\neq 0$, this gives that $Q_i^{\omega_i-\lambda_i}$ is a period of χ_i and so $\omega_i-\lambda_i\geqq\varepsilon_i$. By this (3) is proved.

Now we prove the second assertion. Assume that $f\in\mathcal{M}$ and (1), (2), (3) hold.

We shall assume that $f(Q_i^{\omega_i})=0$ for $i=1,\dots,s$ and $f(Q_i^{\omega_i})\neq 0$ for $i=s+1,\dots,r$, allowing that one of these classes is empty. Then the characters $\chi_{s+1},\dots,\chi_r$ are principal characters with the moduli $Q_j^{\omega_j}$ $(j=s+1,\dots,r)$, respectively.

Let

$$\alpha(n)=\chi_1(n)\cdot\ldots\cdot\chi_s(n),\quad \beta(n)=\chi_{s+1}(n)\cdot\ldots\cdot\chi_r(n),$$

$$q_1=\prod_{i=1}^{s}Q_i^{\omega_i},\quad q_2=\prod_{i=s+1}^{r}Q_i^{\omega_i},\quad q_1^*=\prod_{i=1}^{s}Q_i^{\varepsilon_i}.$$

Then $\beta(n)$ is the principal character with the modulus q_2, $\alpha(n)$ is a character with the modulus q_1, that is periodic with the period q_1^*. Furthermore we may observe that $\psi_i(Q_i^h)=\alpha(Q_i^h)$ for $i=s+1,\dots,r$, $h\geqq 0$.

To prove (1.4) we take $n=m\eta$, $n+q=a\zeta$, where $(\eta,q)=1$, $(\zeta,q)=1$ and m and a are composed from the prime factors of q. Let

$$m=\Big(\prod_{i=1}^{s}Q_i^{\beta_i}\Big)\Big(\prod_{i=s+1}^{k}Q_i^{\beta_i}\Big)\Big(\prod_{i=k+1}^{v}Q_i^{\beta_i}\Big)\Big(\prod_{i=v+1}^{r}Q_i^{\beta_i}\Big)=\Pi_1\Pi_2\Pi_3\Pi_4$$

and

$$a=\Big(\prod_{i=1}^{s}Q_i^{\gamma_i}\Big)\Big(\prod_{i=s+1}^{k}Q_i^{\gamma_i}\Big)\Big(\prod_{i=k+1}^{v}Q_i^{\gamma_i}\Big)\Big(\prod_{i=v+1}^{r}Q_i^{\gamma_i}\Big)=R_1R_2R_3R_4,$$

where in Π_2 $\beta_i>\omega_i$, in Π_3 $\beta_i=\omega_i$, and in Π_4 $\beta_i<\omega_i$. Hence it follows that in R_2 $\gamma_i=\omega_i$, in R_3 $\gamma_i\geqq\omega_i$, in R_4 $\gamma_i=\beta_i$. Consequently $R_4=\Pi_4$. Let U and V be defined by the relations

$$\Pi_2=R_2\Pi Q_i^{\beta_i-\omega_i}=R_2U,\quad R_3=\Pi_3\Pi Q_i^{\gamma_i-\omega_i}=\Pi_3V.$$

If $\beta_i \geqq \omega_i$ for at least one $i \in [1, s]$, then $f(n)=0$, $\gamma_i \geqq \omega_i$ and so $f(n+q)=0$, i.e. (1.4) is true. If $\omega_i - \varepsilon_i < \beta_i < \omega_i$, then $\beta_i = \gamma_i$, $f(Q_i^{\omega_i})=0$, consequently $f(n) = f(n+q)=0$. So we may assume that $\beta_i \leqq \omega_i - \varepsilon_i$ for $i=1, \dots, s$. Hence it follows that $\beta_i = \gamma_i$. $\Pi_1 = R_1$. Let us consider now the relation

$$q = a\zeta - m\eta = R_1 R_2 R_3 R_4 \zeta - \Pi_1 \Pi_2 \Pi_3 \Pi_4 \eta = \Pi_1 \Pi_4 R_2 \Pi_3 \{V\zeta - U\eta\}.$$

Since $(\Pi_4 R_2 \Pi_3, q_1)=1$, $\Pi_1 | q_1/q_1^*$, we get that $V\xi \equiv U\eta \pmod{q_1^*}$. Furthermore

$$f(m\eta) = f(\Pi_1)f(\Pi_4)f(R_2 U)f(\Pi_3)f(\eta) = f(\Pi_1)f(\Pi_4)f(R_2)f(\Pi_3)\alpha(U)\alpha(\eta),$$

$$f(a\zeta) = f(\Pi_1)f(\Pi_4)f(R_2)f(\Pi_3 V)\alpha(\zeta) = f(\Pi_1)f(\Pi_4)f(R_2)f(\Pi_3)\alpha(V)\alpha(\zeta).$$

By observing that $\alpha(U)\alpha(\eta)=\alpha(V)\alpha(\zeta)$, we get (1.4).

Thus the proof of the Main lemma is complete.

3. Proof of Theorem 1.

Lemma 1. *If $f \in \mathcal{A}_q \cap \mathcal{M}$, $f(1)=1$, then*

$$f(nq^\alpha) = f(n)f(q^\alpha) \tag{3.1}$$

holds for every nonnegative n and α.

Proof. (3.1) is obviously true if $n=0$ or $\alpha=0$. Let $\alpha>0$ and $n>0$. Let us assume that $n=q^\beta$, or $n<q^\alpha$ and $n|q^s$ for a suitable large s. Then $(n, q^\alpha+1)=1$, and hence

$$f(nq^\alpha)+f(n) = f(nq^\alpha+n) = f(n)f(q^\alpha+1) = f(n)f(q^\alpha)+f(n),$$

i.e. (3.1) holds. Let $n<q^\alpha$, $n=n_1n_2$, where $(n_1, q)=1$ and all prime divisors of n_2 divide q. Then

$$f(nq^\alpha) = f(n_1)f(n_2q^\alpha) = f(n_1)f(n_2)f(q^\alpha) = f(n)f(q^\alpha).$$

Let now $n=a_0+a_1q+\dots+a_sq^s$ be an arbitrary positive integer. By using the q-additive property and the results proved earlier we get

$$f(nq^\alpha) = f(a_0q^\alpha+\dots+a_sq^{\alpha+s}) = f(a_0q^\alpha)+\dots+f(a_sq^{\alpha+s}) =$$
$$= f(q^\alpha)[f(a_0)+\dots+f(a_sq^s)] = f(q^\alpha)f(n).$$

The proof of Lemma 1 is finished.

Now we prove Theorem 1. From (3.1) it is obvious that $f(q^\beta)=(f(q))^\beta$ $(\beta=1, 2, \dots)$. Assume that $f(q)\neq 0$. We shall prove that

$$f(2n+1) = f(n+1)+f(n), \quad f(2n) = 2f(n), \tag{3.2}$$

which immediately yields the desired result $f(n)=n$.

Let n be fixed, α be large. Since $(q^\alpha+n+1, q^\alpha+n)=1$, we have

$$f((q^\alpha+n+1)(q^\alpha+n)) = f(q^\alpha+n+1)f(q^\alpha+n) =$$
$$= (f(q^\alpha)+f(n+1))(f(q^\alpha)+f(n)) = f(q^\alpha)^2+f(q^\alpha)(f(n)+f(n+1))+f(n)f(n+1).$$

Furthermore

$$f((q^\alpha+n+1)(q^{2\alpha}+n)) = f(q^\alpha)+f((2n+1)q^\alpha)+f(n(n+1)).$$

Hence we get immediately that $f(2n+1) = f(n+1)+f(n)$. To prove the second relation in (3.2) we consider

$$f(2nq+1) = f(nq+1)+f(nq),$$

whence it follows that

$$f(2n)f(q)+f(1) = 2f(n)f(q)+f(1),$$

and from $f(q)\neq 0$ we get that $f(2n)=2f(n)$.

Theorem 1 is proved.

4. Proof of Theorem 2. Let $f\in\mathcal{A}_q\cap\mathcal{M}$, $f(1)=1$, $f(q)=0$. Since $f(q)=0$, from Lemma 1 we get that $f(nq)=0$ for every n, consequently f is a periodic function with period q. The necessity of the conditions is obvious from the Main lemma. But they are also sufficient, since a periodic multiplicative function f with $f(q)=0$ is q-additive, and so the sufficiency is an immediate consequence of the Main lemma as well.

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