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On partial asymptotic stability and instability. III (Energy-like Ljapunov functions)

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Dedicated to Professor László Leindler on his 50th birthday

1. Introduction

The main tools in the proofs of the theorems in [1]—[2] were certain invariance principles. Their applications were made possible by the special structure of the basic differential equation: in [1] and [2] the equations were autonomous and asymptotically autonomous, respectively.

In this paper we study the case, when there are no direct restrictions on the right-hand side of the equation. Theorems of such type have been established first by A. M. LJAPUNOV [3]. Besides some other conditions, he required the Ljapunov function V to be decrescent, i.e. $V(x,t)\to 0$ uniformly in $t\in R_+$ as $x\to 0$. V. V. RUMJANCEV [4] generalized these theorems to partial asymptotic stability. Since the uncontrolled part z of the coordinates may be unbounded along motions, the condition that V be decrescent has come into these generalizations even in a stronger form: the condition "$V(y,z,t)\to 0$ uniformly in $(z,t)\in R^n\times R_+$ as $y\to 0$" are required in them. Sometimes in practice it is very difficult to construct such a Ljapunov function. For example, the total mechanical energy of a mechanical system is decrescent with respect to the velocities only in that case when no potential forces act on the system. Namely, let us consider again the motion of a material point in a constant field of gravity along a surface under the action of frictional forces [1], [2]. It is a very reasonable conjecture that if the surface is a cup looking upward then the equilibrium is asymptotically stable with respect to the velocities. However, using the generalizations established in [4] one can prove this property only in that case when the surface is a horizontal plane.

In mechanics the total mechanical energy, i.e. the sum of the kinetic and potential energy is often used as a Ljapunov function for stability investigations. These

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applications inspired us to give a sufficient condition for partial asymptotic stability using a non-decrescent Ljapunov function which is the sum of two auxiliary functions. The result will be applied to the study of the conditions of the asymptotic stability with respect to the velocities of equilibrium states in mechanical systems under the action of dissipative and potential forces depending also on the time.

2. The main theorems

Consider the differential equation

$$\dot{x} = X(x, t) \quad (t \in R_+, x \in R^k). \tag{2.1}$$

Let $x=(y, z)$ be a partition of the vector $x \in R^k$ ($y \in R^m$, $z \in R^n$, $1 \leqq m \leqq k$, $n=k-m$) and suppose that the right-hand side of (2.1) satisfies the same conditions as in [1] (see Section 2), i.e. the function X is defined on the set $\Gamma_y(H)$:

$$\Gamma_y(H) := G_y(H) \times R_+, \quad G_y(H) := \{(y, z) \in R^m \times R^n : |y| < H\} \ (0 < H \leqq \infty),$$

it is continuous in x and measurable in t, and satisfies the Carathéodory condition locally. The solutions of (2.1) are z-continuable, and $x=0$ is a solution of the equation, i.e. $X(0, t) \equiv 0$ for all $t \in R_+$.

Let $\mathcal{K}$ be the class of continuous strictly increasing functions $a: R_+ \to R_+$ such that $a(0)=0$.

For formulating our main result a new concept is needed. A continuous function $\varphi: R_+ \to R_+$ is said to be *integrally positive* (see [5], [6]) if $\int_I \varphi(t)\, dt = \infty$ whenever

$$I = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i], \quad \text{and} \quad \alpha_i < \beta_i < \alpha_{i+1}, \quad \beta_i - \alpha_i \geqq \delta > 0$$

hold for all $i=1, 2, \ldots$ with some positive constant δ.

Denote by $[\alpha]_+$ and $[\alpha]_-$ the positive and the negative parts of the real number α, respectively, i.e. $[\alpha]_+ := \max\{0, \alpha\}$, $[\alpha]_- := \max\{0, -\alpha\}$.

Theorem 2.1. *Suppose that there exist two functions V_1, V_2: $\Gamma_y(H) \to R$ which are continuous, locally Lipschitzian and satisfy the following conditions on the set $\Gamma_y(H)$:*

(i) $V(x, t) := V_1(x, t) + V_2(x, t) \geqq 0$;

(ii) $V_1(x, t) \geqq 0$;

(iii) *the derivative of V with respect to (2.1) admits an estimate*

$$\dot{V}(x, t) \leqq -\varphi(t) c(V_1(x, t))$$

with some $c \in \mathcal{K}$ and some integrally positive function φ: $R_+ \to R_+$;

(iv) *for every* $\alpha, \alpha_1>0$ *and for every continuous function* $\xi: R_+\to R^k$ *the inequalities* $V(\xi(t),t)\leqq\alpha$, $V_1(\xi(t),t)\geqq\alpha_1$ $(t\in R_+)$ *imply that the function*

$$\int_0^t [\dot V_2(\xi(s),s)]_{+(-)}\,ds$$

is uniformly continuous on R_+, *where the symbol* $[\cdot]_{+(-)}$ *means that either the positive part* $[\cdot]_+$ *or the negative part* $[\cdot]_-$ *is considered for all* $s\in R_+$.

Then for every solution $x(t)$ *of* (2.1) *defined for all t large enough* $\lim_{t\to\infty} V_1(x(t),t)=0$, *and* $V_2(x(t),t)$ *has a finite limit as* $t\to\infty$.

Proof. Define the functions $v_1(t):=V_1(x(t),t)$, $v_2(t):=V_2(x(t),t)$, $v(t):=v_1(t)+v_2(t)$. Obviously, $v(t)$ is nonincreasing and bounded from below, so $\lim_{t\to\infty} v(t)=:v_0$ exists and is finite. It is sufficient to show that $\lim_{t\to\infty} v_1(t)=0$.

Suppose the contrary. Then, in consequence of (i)—(iii) we have

$$\liminf_{t\to\infty} v_1(t) < \limsup_{t\to\infty} v_1(t) =: v_1^* \leqq \infty,$$

$$v_{2*} := \liminf_{t\to\infty} v_2(t) = v_0 - v_1^* < v_0 = \limsup_{t\to\infty} v_2(t).$$

Now we show the existence of a sequence of disjoint intervals on which the variations of the function $v_2(t)$ are bounded from below by a positive constant. Indeed, let $\varepsilon:=v_1^*/4>0$ if $v_1^*<\infty$, and let $\varepsilon>0$ be arbitrary if $v_1^*=\infty$. There exists a $T\in R_+$ such that $v_0\leqq v(t)<v_0+\varepsilon$ for all $t\geqq T$. For the sake of definiteness let us suppose that "plus" sign stands in condition (iv) of the theorem. Obviously, an appropriate sequence $T<t_1'<t_1''<\ldots<t_i'<t_i''<\ldots$ has the properties

$$v_1(t_i')=3\varepsilon,\quad v_1(t_i'')=\varepsilon,\quad \varepsilon\leqq v_1(t)\leqq 3\varepsilon \quad \text{for}\quad t\in[t_i',t_i'']\quad (i=1,2,\ldots).$$

Since $v_2(t)=v(t)-v_1(t)$, we obtain

$$v_2(t_i')\leqq v_0-2\varepsilon,\quad v_2(t_i'')\geqq v_0-\varepsilon\quad (i=1,2,\ldots).$$

Consequently,

$$0<\varepsilon\leqq v_2(t_i'')-v_2(t_i')\leqq \int_{t_i'}^{t_i''} [\dot V_2(x(t),t)]_+\,dt\quad (i=1,2,\ldots).$$

Hence, because of condition (iv), it follows that $t_i''-t_i'\geqq\delta>0$ $(i=1,2,\ldots)$ with some constant δ. By condition (iii) this implies $v(t)\to-\infty$, which is a contradiction.

The theorem is proved.

If the function V_1 in the theorem is even positive definite with respect to y, then for every solution $x(t)=(y(t),z(t))$ defined for all t large enough $y(t)\to 0$ as $t\to\infty$. If, in addition, $V_1(0,t)\equiv V_2(0,t)\equiv 0$ and $V_2(x,t)\geqq 0$, then $V=V_1+V_2$ is a positive y-definite Ljapunov-function to (2.1), so the zero solution is even y-stable [4], which leads to the following

Corollary 2.1. *Suppose that there exist two Ljapunov functions*

$$V_1, V_2: \Gamma_y(H') \to R \quad (0 < H' < H)$$

satisfying the following conditions on the set $\Gamma_y(H')$:

(i) $V_2(x, t) \geqq 0$;

(ii) *there is a function* $a_1 \in \mathscr{K}$ *such that*

$$a_1(|y|) \leqq V_1(y, z, t).$$

Suppose, in addition, that conditions (iii)—(iv) *in Theorem* 2.1 *are also satisfied.*

Then the zero solution of (2.1) *is asymptotically y-stable and for every solution* $x(t)$ *with sufficiently small initial values the function* $V_2(x(t), t)$ *has a finite limit as* $t \to \infty$.

Let $x = (y', z')$ be another partition of the vector $x \in R^k$: $y' \in R^{m'}$, $z' \in R^{n'}$, $m \leqq m' \leqq \leqq k$, $n' = k - m'$. If V_1 is decrescent with respect to y', then condition (iii) becomes simpler:

Corollary 2.2. *Suppose that there exist two Ljapunov functions*

$$V_1, V_2: \Gamma_y(H') \to R \quad (0 < H' < H)$$

satisfying the following conditions on the set $\Gamma_y(H')$:

(i) $V_2(x, t) \geqq 0$;

(ii) *there are functions* $a_1, b_1 \in \mathscr{K}$ *such that*

$$a_1(|y|) \leqq V_1(y, z, t) \leqq b_1(|y'|);$$

(iii) *an inequality*

$$\dot{V}(x, t) \leqq -\varphi(t) c(|y'|)$$

holds with some $c \in \mathscr{K}$ *and some integrally positive* φ: $R_+ \to R_+$;

(iv) *for every* $\alpha, \beta > 0$ *the function*

$$\int_0^t \sup \{[\dot{V}_2(y', z', s)]_{+(-)}: (y', z') \in M_{\alpha,\beta}(s)\} \, ds$$

is unformly continuous on R_+, *where*

$$M_{\alpha,\beta}(s) := \{(y', z') \in R^{m'} \times R^{n'}: V(y', z', s) \leqq \alpha, \quad |y'| \geqq \beta\}.$$

Then the statement of Corollary 2.1 *holds.*

Remark 2.1. In condition (iii) of Theorem 2.1 (and Corollaries 2.1—2.2) φ is integrally positive which, roughly speaking, means that it cannot be small in average in any period as $t\to\infty$. It can be formulated also in the following way: for every $\delta>0$

$$\liminf_{t\to\infty}\int_t^{t+\delta}\varphi(s)\,ds>0.$$

Therefore, for example, every nonnegative continuous periodic function not vanishing on any interval is integrally positive. But, obviously, a function tending to zero as $t\to\infty$ cannot be integrally positive even if its integral equals infinity. However, by experiences asymptotic stability may appear also in this case.

Let us relax the condition of integral positivity. We say that a continuous function $\varphi\colon R_+\to R_+$ is *weakly integrally positive* [6] if $\int_I\varphi=\infty$ whenever

$$I=\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i),\quad \alpha_i<\beta_i<\alpha_{i+1},\quad \beta_i-\alpha_i\geqq\delta>0,\quad \alpha_{i+1}-\beta_i\leqq\gamma\quad (i=1,2,\ldots)$$

hold with some positive constants δ, γ.

It is easy to see that any nonincreasing function whose integral on R_+ equals infinity is weakly integrally positive.

We say that equation (2.1) has *property* P with respect to V_1, V_2 if for every $\varepsilon,\alpha>0$ there exist $\eta>0$, $T\in R_+$ such that for every solution $x(t)$ of (2.1) the point $(x(t),t)$ cannot be contained in the set

$$M(\varepsilon,\alpha,\eta):=\{(x,t)\colon\ \alpha\leqq V_1(x,t)+V_2(x,t)<\alpha+\varepsilon,\ V_1(x,t)\geqq\eta\}$$

during any period longer than T.

Analysing the proof of Theorem 2.1 one can show that property P makes it possible to choose the sequence $\{(t_i',t_i'')\}$ in the proof so that the inequality $t_{i+1}'-t_i''\leqq\gamma$ holds for all $i=1,2,\ldots$. Consequently, possessing property P we can assume the function φ in condition (iii) of Theorem 2.1 to be weakly integrally positive instead of integrally positive.

Property P is often guaranteed by means of another auxiliary function [5], [6], [14]. For example, if for some ε, α, η there exists a function $W\colon M(\varepsilon,\alpha,\eta)\to R$ such that for every continuous function $\zeta\colon R_+\to R_+$ with $(\zeta(t),t)\in M(\varepsilon,\alpha,\eta)$ $(t\in R_+)$ the function $W(\zeta(t),t)$ is bounded from above and the condition

$$\lim_{\sigma\to\infty}\int_{t_0}^{t_0+\sigma}\dot W(\zeta(t),t)\,dt=\infty$$

holds uniformly with respect to $t_0\in R_+$, then (2.1) has property P.

3. Applications

I. Consider the generalized Liénard equation

$$\ddot{x}+a(t)g(x,\dot{x})\dot{x}+b(t)f(x)=0, \tag{3.1}$$

where the functions $a\colon R_+\to R_+$, $g\colon R^2\to R_+$, $f\colon R\to R$ are continuous, $b\colon R_+\to R_+$ is continuously differentiable, and $b(t)>0$, $F(x):=\int_0^x f(u)\,du\geqq 0$ for all $t\in R_+$, $x\in R$. This equation, which describes the oscillation of a material point round the origin $x=0$, has been investigated by many authors [7]—[12]. In [13] we obtained sufficient conditions for the asymptotic stability if $0\leqq a(t)\leqq A_0$, $0<b_0\leqq b(t)\leqq B_0$ $(t\in R_+)$, and for the asymptotic x-stability provided that $\lim_{t\to\infty} b(t)=\infty$. Using our results proved in Section 2 of the present paper we can sharpen these theorems and get sufficient conditions for the asymptotic $\dot{x}$-stability, too.

First we define the auxiliary functions

$$V_1(\dot{x}):=\dot{x}^2/2,\quad V_2(x,t):=b(t)F(x).$$

The derivatives of $V:=V_1+V_2$ and V_2 read as follows:

$$\dot{V}(x,\dot{x},t)=-a(t)g(x,\dot{x})\dot{x}^2+\dot{b}(t)F(x)=-2a(t)g(x,\dot{x})V_1(\dot{x})+\big(\dot{b}(t)/b(t)\big)V_2(x,\dot{x},t),$$

$$\dot{V}_2(x,\dot{x},t)=\big(\dot{b}(t)/b(t)\big)V_2(x,\dot{x},t)+b(t)\dot{x}f(x).$$

Applying Corollary 2.1 we obtain the following

Corollary 3.1. *Suppose that*

(i) $\dot{b}(t)\leqq 0$ *and* $\dot{b}(t)/b(t)$ *is bounded from below on* R_+;

(ii) $a(t)$ *is integrally positive on* R_+;

(iii) *for every* $0<c_2<C_2$ *there is a* $g_0>0$ *such that*

$$g(u,v)\geqq g_0\quad (u\in R,\ c_2\leqq |v|\leqq C_2);$$

(iv) $f(x)$ *is bounded on* R.

Then the zero solution of (3.1) *is asymptotically $\dot{x}$-stable and for every solution $x(t)$ the function $b(t)F\big(x(t)\big)$ has a finite limit.*

Some conditions in this corollary become simpler if the solutions are guaranteed to be bounded.

Corollary 3.2. *Suppose that*

(i) $-\beta_0\leqq\dot{b}(t)\leqq 0$, $b(t)\geqq b_0>0$ $(t\in R_+)$;

(ii) $a(t)$ *is integrally positive on* R_+;

(iii) $g(u,v)>0$ *if* $v\neq 0$;

(iv) $\lim_{|x|\to\infty} F(x)=\infty$.

Then the zero solution of (3.1) *is asymptotically $\dot{x}$-stable and for every solution $x(t)$ the function $F(x(t))$ has a finite limit.*

Proof. By conditions (i), (iv) we have $\dot{V}(u, v, t) \leqq 0$, $\lim_{|u|+|v| \to \infty} V(u, v, t) = \infty$. Consequently, x and $\dot{x}$ are bounded on R_+ along every solution, so the conditions of Corollary 3.1 are satisfied.

Surprisingly, the boundedness, even the stability with respect to x, can be guaranteed by a modification of condition (ii) provided $g(u, v) > 0$ on R^2

Corollary 3.3. *Suppose that*

(i) $\dot{b}(t) \leqq 0$ *and* $\dot{b}(t)/b(t)$ *is bounded from below on* R_+;

(ii) $a(t) > 0$ *for* $t \in R_+$, *and* $\int_0^\infty dt/a(t) < \infty$;

(iii) $g(u, v) > 0$ *for* $(u, v) \in R^2$.

Then the zero solution of (3.1) *is stable, asymptotically stable with respect to* $\dot{x}$, *and every solution* $x(t)$ *with sufficiently small initial values* $|x(t_0)|$, $|\dot{x}(t_0)|$ *has a finite limit as* $t \to \infty$.

Proof. All the conditions of Corollary 3.4 in [12] are obviously met by $q := x$, $\dot{q} := \dot{x}$, $A(q) := 1$, $\Pi(t, q) := b(t) F(q)$, $Q(t, q, \dot{q}) := -a(t) g(q, \dot{q}) \dot{q}$, $\alpha = 1$. Consequently, the zero solution of (3.1) is stable and every solution has a finite limit as $t \to \infty$. By Schwarz's inequality, condition (ii) implies the function $a(t)$ to be integrally positive, so all the condition of Corollary 3.1 are satisfied.

The case of nondecreasing function $b(t)$ will be treated for a mechanical system of arbitrary degree of freedom.

II. Consider a holomorphic mechanical system of r degrees of freedom with time-independent constraints under the action of potential and dissipative forces depending on the time, too. Let the motions be described by the Lagrangian equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -g^2 \frac{\partial P^*}{\partial q} + Q \quad (q, \dot{q} \in R^r). \tag{3.2}$$

Here we use the same notations as in [1]—[2]: $T = T(q, \dot{q}) := (1/2)\dot{q}^T A(q) \dot{q}$ is the kinetic energy; $P(q, t) := g^2(t) P^*(q)$ denotes the potential energy in which g: $R_+ \to (0, \infty)$ and $P^*: R^r \to R_+$ are continuously differentiable functions. By $Q = Q(q, \dot{q}, t)$ we denote the resultant of frictional and gyroscopic forces. This means that $Q^T(q, \dot{q}, t)\dot{q} \leqq 0$ for all values of the variables. Assume that $q = \dot{q} = 0$ is an equilibrium state to system (3.2) and $P^*(0) = 0$.

In [1], [2] we investigated the stationary case (i.e. P and Q were independent of t). In the instationary case it gives rise to difficulties that the total mechanical energy is not constant along the motions and the invariance principle cannot be applied any more [5], [6]. L. SALVADORI [14] has given a sufficient condition for the asymptotic stability of the equilibrium $q=\dot{q}=0$ of (3.2) with respect to the coordinates q. Now we are interested in the asymptotic behaviour of the velocities provided that the dissipation is complete in a certain sense. Particularly, we seek for conditions of the asymptotic stability of the equilibrium $q=\dot{q}=0$ with respect to the velocities.

If g is nonincreasing the results proved in paragraph I of the present section for the case of one degree of freedom can be generalized to (3.2). In the sequel g is not supposed to be monotone.

By the transformation $\dot{q}=g(t)y$, introduced by L. SALVADORI [14], system (3.2) can be rewritten into the form

$$\dot{q} = g(t)y \tag{3.3}$$
$$\frac{d}{dt}\frac{\partial T^*}{\partial y} - g\frac{\partial T^*}{\partial q} = -\frac{\dot{g}}{g}\frac{\partial T^*}{\partial y} - g\frac{\partial P^*}{\partial q} + \frac{Q^*}{g},$$

where

$$T^*(q, y) := (1/2)y^{\mathrm{T}}A(q)y, \quad Q^*(q, y, t) = Q(q, g(t)y, t).$$

Denote by $\lambda(q)$ and $\Lambda(q)$ the smallest and the largest eigenvalue of the positive symmetric matrix $A(q)$, respectively. For $M>0$ let the set $E_M \subset R^r$ be defined by

$$E_M := \{q \in R^r\colon\ P^*(q) \leqq M\}.$$

The derivative of the function $H(q, y) := T^* + P^*$ with respect to (3.3) is

$$\dot{H}(q, y, t) = -2(\dot{g}/g)T^* + (1/g)yQ^*.$$

Making the choice $V_1 := T^*, V_2 := P^*$, from Theorem 2.1 we obtain a lemma, which may be of some independent interest.

Lemma 3.1. *Suppose that for every $M>0$ there exist a function $\varphi\colon R_+ \to R_+$ and a constant L such that $\varphi + 2\dot{g}/g$ is integrally positive and the following conditions are satisfied:*

(i) $Q^{\mathrm{T}}(q, \dot{q}, t)\dot{q} \leqq -\varphi(t)\Lambda(q)|\dot{q}|^2$ *for all* $q \in E_M$, $\dot{q} \in R^r$, $t \in R_+$;

(ii) $|\mathrm{grad}\, P^*(q)| \leqq L[\lambda(q)]^{1/2}$ $(q \in E_M)$;

(iii) *the function* $\int_0^t g(s)\, ds$ *is uniformly continuous on* R_+.

Then for every motion $q=q(t)$ *of* (3.2) *we have*

$$\dot{q}(t) = o\big(g(t)/[\lambda(q(t))]^{1/2}\big), \quad P^*(q(t)) \to \text{const.} \quad (t \to \infty). \tag{3.4}$$

If it is "a priori" known that $q(t)$ is bounded along every motion (e.g. $P^*(q) \to \infty$ as $|q| \to \infty$), then $\Lambda(q)$ can be replaced by 1 in (i), condition (ii) is not needed and one can state $\dot{q}(t) = o(g(t))$ $(t \to \infty)$.

Let us now consider the case of the viscous friction, i.e. if $Q(q, \dot{q}, t) = -B(q, t)\dot{q}$ where B is a symmetric positive semi-definite matrix; the smallest eigenvalue of it we denote by $\beta(q, t)$.

Theorem 3.1. *Suppose that for every $M>0$ the following conditions are satisfied:*

(i) *there acts viscous friction on the system such that "the dissipation of the energy is integrally complete", i.e. the function*

$$\inf\{\beta(q, t)/A(q, t)\colon\ q\in E_M\}+2\dot{g}(t)/g(t)$$

is integrally positive;

(ii) $\inf\{\lambda(q)\colon\ q\in E_M\}>0$;

(iii) *the functions g and* grad $P^*(q)$ *are bounded on R_+ and E_M, respectively.*

Then the equilibrium $q=\dot{q}=0$ of (3.2) *is asymptotically stable with respect to the velocities.*

Finally, we examine the case of "weakly integrally complete dissipation" starting from Remark 2.1. In order to guarantee property P, let us consider the auxiliary function $W(q, y)=y^{\mathrm{T}}A(q)\operatorname{grad} P^*(q)$. If g is nondecreasing, A and grad P^* are continuously differentiable, then the derivative $\dot{W}_{(3.3)}$ can be estimated as follows:

$$\dot{W}_{(3.3)}(q, y, t) \leqq -g(t)[\operatorname{grad} P^*(q)]^2+ \tag{3.5}$$
$$+g(t)\left\{d(|y|)\left[\frac{\dot{g}(t)}{g^2(t)}F_1(q)+F_2(q)\right]+\frac{Q(q, g(t)y, t)}{g^2(t)}F_3(q)\right\},$$

where $d\in\mathscr{K}$ and $F_i\colon R^r\to R_+$ are appropriate continuous functions.

Theorem 3.2. *Suppose that in some neighbourhood $N\subset R^r$ of the origin the following conditions are satisfied:*

(i) *$q=0$ is the only equilibrium position of* (3.2) *in N;*

(ii) *there acts viscous friction on the system with "weakly integrally complete dissipation", i.e. the function*

$$\varphi(t) := \inf\{\beta(q, t)\colon\ q\in N\}$$

is weakly integrally positive on R_+;

(iii) *the function*

$$\frac{1}{t}\int_0^t \sup\{\|B(q, s)\|\colon\ q\in N\}\,ds$$

is bounded on R_+;

(iv) *the function g is nondecreasing and bounded on R_+.*

Then the equilibrium state $q=\dot{q}=0$ *of* (3.2) *is stable, asymptotically stable with respect to the velocities, and for every motion* $q(t)$ *with sufficiently small initial values* $P^*(q(t))\to\text{const.}$ *as* $t\to\infty$.

Proof. By (i) and condition $P^*(q)\geqq 0$ function P is positive definite. Consequently, the solution $q=y=0$ of (3.3) is stable and $q(t)\in N$ for all $t\geqq t_0$ provided that $|q(t_0)|$, $|\dot{q}(t_0)|$ are sufficiently small. In accordance with Remark 2.1 we have only to prove the existence of property P with respect to T^* and P^*.

Let $\varepsilon>0$, α, η $(0<\eta<\alpha)$ be given, and define

$$S(\alpha,\eta) := \{(q,y)\colon q\in N,\ T^*(q,y)\leqq\eta,\ P^*(q)\geqq\alpha-\eta\}.$$

Condition (i) implies that

$$m := \inf\{[\operatorname{grad} P^*(q)]^2\colon P^*(q)\geqq\alpha-\eta>0,\ q\in N\}>0.$$

Since

$$\int_0^t (\dot{g}(s)/g^2(s))\,ds = 1/g(0)-1/g(t)\leqq\text{const.}\quad (t\geqq 0),$$

by condition (iii) and inequality (3.5) we have the estimate

$$\dot{W}_{(3.3)}(q,y,t)\leqq -g(t)\{m-[c_1+c_2(\dot{g}(t)/g^2(t))]\,d(|y|)-c_3\psi(t)|y|\}$$

on the set $S(\alpha,\eta)\times R_+$, where c_1, c_2, c_3 are positive constants, $\psi\colon R_+\to R_+$ is a continuous function such that $\int_0^t \psi(s)\,ds/t$ is bounded on R_+. Consequently, if η is sufficiently small, then for arbitrary continuous functions $u,v\colon R_+\to S(\alpha,\eta)$ we have

$$\lim_{\sigma\to\infty}\int_{t_0}^{t_0+\sigma}\dot{W}_{(3.3)}(u(t),v(t),t)\,dt=-\infty$$

uniformly with respect to $t_0\in R_+$, which implies property P.

The theorem is proved.

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