On the restricted convergence and $(C, 1, 1)$-summability of double orthogonal series

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Dedicated to Professor László Leindler on his 50th birthday

1. Introduction. Let $(X, \mathscr{F}, \mu)$ be an arbitrary positive measure space and $\{\varphi_{ik}(x): i, k=1, 2, ...\}$ an orthonormal system (in abbreviation: ONS) on X. We shall consider the double orthogonal series

$$(1.1) \qquad \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \varphi_{ik}(x),$$

where $\{a_{ik}: i, k=1, 2, ...\}$ is a double sequence of real numbers (coefficients), for which

$$(1.2) \qquad \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty.$$

By the extended Riesz—Fischer theorem there exists a function $f(x) \in L^2 = L^2(X, \mathscr{F}, \mu)$ such that (1.1) is the Fourier series of $f(x)$ with respect to $\{\varphi_{ik}(x)\}$ and the rectangular partial sums

$$s_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \varphi_{ik}(x) \quad (m, n = 1, 2, ...)$$

converge to $f(x)$ in the L^2-metric:

$$\int [s_{mn}(x) - f(x)]^2 \, d\mu(x) \to 0 \quad \text{as} \quad \min(m, n) \to \infty.$$

Here and in the sequel, the integrals are taken over the whole space X.

Beside $s_{mn}(x)$ we consider the first arithmetic means, the so-called $(C, 1, 1)$-means $\sigma_{mn}(x)$ of series (1.1) defined by

$$\sigma_{mn}(x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} s_{ik}(x) =$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{i-1}{m}\right) \left(1 - \frac{k-1}{n}\right) a_{ik} \varphi_{ik}(x) \quad (m, n = 1, 2, ...).$$

2. Unrestricted convergence. It is well-known that condition (1.2) itself does not ensure the pointwise convergence of $s_{mn}(x)$ or $\sigma_{mn}(x)$. The extension of the famous Rademacher—Menšov theorem proved by a number of authors (see [1], [7], etc.) reads as follows.

Theorem A. *If*

(2.1) $$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \, [\log{(i+1)}]^2 \, [\log{(k+1)}]^2 < \infty,$$

then

$$s_{mn}(x) \to f(x) \quad a.e. \quad as \quad \min{(m, n)} \to \infty$$

and there exists a function $F(x) \in L^2$ such that

$$\sup_{m, n \geq 1} |s_{mn}(x)| \leq F(x) \quad a.e.$$

In this paper the logarithms are to the base 2.

The following theorem (see, e.g. [8]) gives information on the order of magnitude of $s_{mn}(x)$ in the more general setting of (1.2).

Theorem B. *Under condition* (1.2),

(2.2) $$s_{mn}(x) = o_x\{\log{(m+1)} \log{(n+1)}\} \quad a.e. \quad as \quad \max{(m, n)} \to \infty$$

and there exists a function $F(x) \in L^2$ such that

$$\sup_{m, n \geq 1} \frac{|s_{mn}(x)|}{\log{(m+1)} \log{(n+1)}} \leq F(x) \quad a.e.$$

Similarly to the case of the single orthogonal series, the convergence properties improve when the first arithmetic means $\sigma_{mn}(x)$ are considered instead of the rectangular partial sums $s_{mn}(x)$. The following extension of the summation theorem of Menšov and Kaczmarz was proved in [10]. We note that it was stated earlier in [5] and [4], but the proofs are not complete in them.

Theorem C. *If*

(2.3) $$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \, [\log\log{(i+3)}]^2 \, [\log\log{(k+3)}]^2 < \infty,$$

then

$$\sigma_{mn}(x) \to f(x) \quad a.e. \quad as \quad \min{(m, n)} \to \infty$$

and there exists a function $F(x) \in L^2$ such that

$$\sup_{m, n \geq 1} |\sigma_{mn}(x)| \leq F(x) \quad a.e.$$

The order of magnitude of $\sigma_{mn}(x)$, under condition (1.2), is also better in general than that of $s_{mn}(x)$. In contrast to the theory of single orthogonal series, the performance of an Abel transformation is avoided in the proof given in [11].

Theorem D. *Under condition* (1.2),

$$(2.4) \qquad \sigma_{mn}(x) = o_x \left\{ \log \log (m+3) \log \log (n+3) \right\} \quad a.e. \ as \quad \max (m, n) \to \infty$$

and there exists a function $F(x) \in L^2$ *such that*

$$\sup_{m, n \geq 1} \frac{|\sigma_{mn}(x)|}{\log \log (m+3) \log \log (n+3)} \leq F(x) \quad a.e.$$

3. Restricted convergence. In the statements of Theorems A and C both m and n tend to ∞ independently of each other.

We say that m and n tend restrictedly to ∞ if $\min (m, n) \to \infty$ in such a way that the ratios m/n and n/m remain bounded, i.e., there exists a real number $\theta \geq 1$ such that $\theta^{-1} \leq n/m \leq \theta$ while both m and n tend to ∞. We say that $s_{mn}(x)$ or $\sigma_{mn}(x)$ restrictedly converges to $f(x)$ a.e. if $s_{mn}(x)$ or $\sigma_{mn}(x)$ tends to $f(x)$ a.e., respectively, whenever m and n tend restrictedly to ∞. In the case of $\sigma_{mn}(x)$, we may say that series (1.1) is restrictedly $(C, 1, 1)$-summable to $f(x)$ a.e.

The first remarkable fact is that the a.e. restricted convergence of $s_{mn}(x)$ cannot be ensured in general by any weaker condition than (2.1). This means that, in terms of coefficient tests, there is no difference between the a.e. unrestricted convergence and the a.e. restricted convergence of the rectangular partial sums of double orthogonal series.

Theorem E. *For every nonincreasing sequence* $\{\varepsilon(m): m = 1, 2, \ldots\}$ *of positive numbers tending to* 0 *as* $m \to \infty$, *there exist a double ONS* $\{\varphi_{ik}(x)\}$ *on the unit square* $I^2 = [0, 1] \times [0, 1]$ *and a double sequence* $\{a_{ik}\}$ *of coefficients such that*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \varepsilon (\min (i, k)) \left[\log (\max (i, k) + 1) \right]^4 < \infty$$

and

$$\limsup_{m, n \to \infty : 1/2 \leq n/m \leq 2} |s_{mn}(x)| = \infty \quad a.e. \ on \quad I^2.$$

The order of magnitude of $s_{mn}(x)$, under condition (1.2), exhibits the same phenomenon. Relation (2.2) is also the best possible when m and n restrictedly tend to ∞.

Theorem F. *For every* $\{\varepsilon(m)\}$ *occurring in Theorem E, there exist a double ONS* $\{\varphi_{ik}(x)\}$ *on* I^2 *and a double sequence* $\{a_{ik}\}$ *of coefficients such that condition* (1.2) *is satisfied and*

$$\limsup_{m, n \to \infty : 1/2 \leq n/m \leq 2} \frac{|s_{mn}(x)|}{\varepsilon (\min (m, n)) \left[\log (\max (m, n) + 1) \right]^2} = \infty \quad a.e. \ on \quad I^2.$$

Both Theorem E and Theorem F were actually proved in [12] (though the fulfilment of the condition $1/2 \le n/m \le 2$ is not stated explicitly there).

Now, the main results of the present paper say that the situation is quite different for the first arithmetic means $\sigma_{mn}(x)$. The a.e. restricted convergence of $\sigma_{mn}(x)$ can be ensured under a weaker condition than (2.3).

Theorem 1. *If*

$$(3.1) \qquad \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \left[\log\log\left(\max(i,k)+3\right)\right]^2 < \infty,$$

then $\sigma_{mn}(x)$ restrictedly converges to $f(x)$ a.e. and for every $\theta \ge 1$ there exists a function $F_\theta(x) \in L^2$ such that

$$(3.2) \qquad \sup_{m,n \ge 1: \theta^{-1} \le n/m \le \theta} |\sigma_{mn}(x)| \le F_\theta(x) \quad a.e.$$

Assuming only (1.2), the order of magnitude of $\sigma_{mn}(x)$ becomes smaller in comparison with (2.4) in the case when m and n tend restrictedly to ∞.

Theorem 2. *Under condition (1.2), for every $\theta \ge 1$*

$$(3.3) \qquad \max_{n: \theta^{-1} \le n/m \le 0} |\sigma_{mn}(x)| = o_x\{\log\log(m+3)\} \quad a.e. \ as \quad m \to \infty$$

and there exists a function $F_\theta(x) \in L^2$ such that

$$\sup_{m,n \ge 1: \theta^{-1} \le n/m \le \theta} \frac{|\sigma_{mn}(x)|}{\log\log(m+3)} \le F_\theta(x) \quad a.e.$$

It is worth including two interesting consequences of Theorems 1 and 2. The following Theorem 3 extends a theorem of BORGEN [3] from single orthogonal series to double ones. We remark that the possibility of this extension was already indicated in [9].

Theorem 3. *If condition (1.2) is satisfied and series (1.1) is restrictedly $(C, 1, 1)$-summable to $f(x)$ a.e., then for every $\theta \ge 1$*

$$(3.4) \qquad \frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x)-f(x)]^2 \to 0 \quad a.e. \ as \quad m \to \infty.$$

If, in addition, for every $\theta \ge 1$ there exists a function $F_\theta(x) \in L^2$ such that (3.2) is satisfied, then there exists a function $G_\theta(x) \in L^2$ such that

$$\sup_{m \ge 1} \left\{ \frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x)-f(x)]^2 \right\}^{1/2} \le G_\theta(x) \quad a.e.$$

Here by $\sum_{k=\theta^{-1}i}^{\theta i}$ we mean that the summation is extended over those integers k for which $\theta^{-1} \le k/i \le \theta$.

Via the Cauchy inequality, relation (3.4) implies that

$$\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}|s_{ik}(x)-f(x)| = o_x\{1\}\quad\text{a.e. as}\quad m\to\infty.$$

Our last theorem in this Section shows that, under condition (1.2), a certain average of $s_{ik}^2(x)$ is essentially less than it would be expected on the basis of (2.2).

Theorem 4. *Under condition (1.2), for every* $\theta\geqq1$

$$(3.5)\qquad\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}s_{ik}^2(x) = o_x\{\log\log(m+3)\}^2\quad\text{a.e. as}\quad m\to\infty.$$

and there exists a function $F_\theta(x)\in L^2$ *such that*

$$\sup_{m\geqq1}\frac{1}{\log\log(m+3)}\left\{\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}s_{ik}^2(x)\right\}^{1/2}\leqq F_\theta(x)\quad a.e.$$

By (3.5) and the Cauchy inequality, we have again

$$\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}|s_{ik}(x)| = o_x\{\log\log(m+3)\}\quad\text{a.e. as}\quad m\to\infty.$$

Finally, we raise two open questions: Under what conditions can we conclude that

$$(3.6)\qquad\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}[s_{ik}(x)-f(x)]^2 = o_x\{1\}\quad\text{a.e.}$$

and

$$(3.7)\qquad\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}s_{ik}^2(x) = o_x\{\log\log(m+3)\log\log(n+3)\}^2\quad\text{a.e.}$$

as $\min(m,n)\to\infty$ (while m and n run to ∞ independently of each other)?

4. Proofs of Theorems 1 and 2. For the sake of brevity, we introduce the following notation. Given a system $\{f_p(x): p=0, 1, ...\}$ of functions in L^2 and a sequence $\{\lambda(p)\}$ of positive numbers, we write

$$f_p(x) = o_x\{\lambda(p)\}\quad\text{a.e.}$$

if

$$f_p(x)/\lambda(p)\to0\quad\text{a.e.}\quad\text{as}\quad p\to\infty$$

and there exists a function $F(x)\in L^2$ such that

$$\sup_{p\geqq0}|f_p(x)|/\lambda(p)\leqq F(x)\quad a.e.$$

First we present five lemmas.

Lemma 1. *Under condition* (3.1),

(4.1) $$s_{2^p, 2^p}(x) - f(x) = o_x\{1\} \quad a.e.$$

This is an immediate consequence of the following Theorem G proved in [7]. Let $Q_0 \subset Q_1 \subset Q_2 \subset \ldots$ be an arbitrary sequence of finite regions in $N^2 = \{(i, k): i, k = 1, 2, \ldots\}$ such that $\bigcup_{p=0}^{\infty} Q_p = N^2$. Set

$$s_p(Q; x) = \sum_{(i,k) \in Q_p} a_{ik} \varphi_{ik}(x) \quad (p = 0, 1, \ldots).$$

Theorem G. *If*

(4.2) $$\sum_{p=0}^{\infty} \left(\sum_{(i,k) \in Q_p \setminus Q_{p-1}} a_{ik}^2 \right) [\log(p+2)]^2 < \infty \quad (Q_{-1} = \emptyset),$$

then

(4.3) $$s_p(Q; x) - f(x) = o_x\{1\} \quad a.e.$$

Now, it is not hard to verify that in the special case when $Q_p = \{(i, k): i, k = 1, 2, \ldots, 2^p\}$ $(p = 0, 1, \ldots)$ the conditions (3.1) and (4.2) are equivalent, while the statements (4.1) and (4.3) coincide.

Lemma 2. *Under condition* (1.2),

(4.4) $$s_{2^p, 2^p}(x) - \sigma_{2^p, 2^p}(x) = o_x\{1\} \quad a.e.$$

Proof. Using the representation

$$s_{2^p, 2^p}(x) - \sigma_{2^p, 2^p}(x) = \sum_{i=1}^{2^p} \sum_{k=1}^{2^p} \left(\frac{i-1}{2^p} + \frac{k-1}{2^p} - \frac{(i-1)(k-1)}{2^{2p}} \right) a_{ik} \varphi_{ik}(x),$$

we can simply estimate as follows

$$\int [s_{2^p, 2^p}(x) - \sigma_{2^p, 2^p}(x)]^2 \, d\mu(x) \leq$$

$$\leq 3 \sum_{i=1}^{2^p} \sum_{k=1}^{2^p} \left(\frac{(i-1)^2}{2^{2p}} + \frac{(k-1)^2}{2^{2p}} + \frac{(i-1)^2(k-1)^2}{2^{4p}} \right) a_{ik}^2 = 3(I_1 + I_2 + I_3), \quad \text{say.}$$

By (1.2),

$$I_1 = \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} (i-1)^2 a_{ik}^2 \sum_{p: 2^p \geq \max(i,k)} \frac{1}{2^{2p}} < \infty.$$

A similar inequality holds true for I_2. Finally, $I_3 \leq I_1$. The application of B. Levi's theorem completes the proof of (4.4).

Lemma 3. *Under condition* (1.2), *for every* $\theta \geq 1$

(4.5) $$M_{p,\theta}^{(3)}(x) = \max_{\theta^{-1} 2^p < m \leq \theta 2^p + 1} |\sigma_{m, 2^p}(x) - \sigma_{2^p, 2^p}(x)| = o_x\{1\} \quad a.e.$$

The symmetric counterpart of Lemma 3 is the following: Under condition (1.2), for every $\theta \geqq 1$

(4.6) $\qquad M_{p,\theta}^{(4)}(x) = \max_{\theta^{-1}2^p < n \leqq \theta 2^p + 1} |\sigma_{2^p,n}(x) - \sigma_{2^p,2^p}(x)| = o_x\{1\}$ a.e.

Proof of Lemma 3. It is clear that

(4.7)
$$M_{p,\theta}^{(3)}(x) \leqq \max_{\theta^{-1}2^p < m \leqq 2^p} |\sigma_{m,2^p}(x) - \sigma_{2^p,2^p}(x)| +$$
$$+ \max_{2^p < m \leqq \theta 2^p + 1} |\sigma_{m,2^p}(x) - \sigma_{2^p,2^p}(x)| = M_{p,\theta}^{(5)}(x) + M_{p,\theta}^{(6)}(x), \quad \text{say.}$$

(If $\theta = 1$, then $M_{p,\theta}^{(5)}(x) \equiv 0$.) For example, we prove that

(4.8) $\qquad\qquad\qquad M_{p,\theta}^{(6)}(x) = o_x\{1\}$ a.e.

We begin with the obvious estimate

$$M_{p,\theta}^{(6)}(x) \leqq \sum_{m=2^p+1}^{\theta 2^p+1} |\sigma_{m,2^p}(x) - \sigma_{m-1,2^p}(x)|,$$

whence, via the Cauchy inequality,

$$[M_{p,\theta}^{(6)}(x)]^2 \leqq (2\theta - 1) \sum_{m=2^p+1}^{\theta 2^p+1} m[\sigma_{m,2^p}(x) - \sigma_{m-1,2^p}(x)]^2.$$

Using the representation

$$\sigma_{m,2^p}(x) - \sigma_{m-1,2^p}(x) = \sum_{i=1}^{m}\sum_{k=1}^{2^p} \frac{i-1}{m(m-1)} \left(1 - \frac{k-1}{2^p}\right) a_{ik}\, \varphi_{ik}(x),$$

we can easily see that

$$\sum_{p=0}^{\infty} \int [M_{p,\theta}^{(6)}(x)]^2 \, d\mu(x) \leqq$$

$$\leqq (2\theta - 1) \sum_{p=0}^{\infty}\sum_{m=2^p+1}^{\theta 2^p+1} m \sum_{i=1}^{m}\sum_{k=1}^{2^p} \frac{(i-1)^2}{m^2(m-1)^2} \left(1 - \frac{k-1}{2^p}\right)^2 a_{ik}^2 \leqq$$

$$\leqq (2\theta - 1)^2 \sum_{p=0}^{\infty} \frac{1}{2^{2p}} \sum_{i=2}^{\theta 2^p+1}\sum_{k=1}^{2^p} (i-1)^2 a_{ik}^2 =$$

$$\leqq (2\theta - 1)^2 \sum_{i=2}^{\infty}\sum_{k=1}^{\infty} (i-1)^2 a_{ik}^2 \sum_{p:2^p+1 \geqq \max(i/\theta,\, 2k)} \frac{1}{2^{2p}} < \infty.$$

Applying B. Levi's theorem, we get (4.8).

Similarly, we can prove that

(4.9) $\qquad\qquad\qquad M_{p,\theta}^{(5)}(x) = o_x\{1\}$ a.e.

The combination of (4.7), (4.8) and (4.9) provides (4.5) to be proved.

15*

Lemma 4. *Under condition* (1.2), *for every* $\theta \geqq 1$

$$M_{p,\theta}^{(7)}(x) =$$

(4.10)
$$= \max_{2^p < m \leqq 2^p+1} \max_{\theta-1 2^p < n \leqq \theta 2^p+1} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^p}(x)| = o_x\{1\} \quad a.e.$$

Proof. It is enough again to prove that

(4.11)
$$M_{p,\theta}^{(8)}(x) = \max_{2^p < m \leqq 2^p+1} \max_{2^p < n \leqq \theta 2^p+1} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^p}(x)| = o_x\{1\} \quad a.e.$$

We use the trivial estimate

$$M_{p,\theta}^{(8)}(x) \leqq \sum_{m=2^p+1}^{2^p+1} \sum_{n=2^p+1}^{\theta 2^p+1} |\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x)|,$$

whence, by the Cauchy inequality,

$$[M_{p,\theta}^{(8)}(x)]^2 \leqq (2\theta-1) \sum_{m=2^p+1}^{2^p+1} \sum_{n=2^p+1}^{\theta 2^p+1} mn[\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x)]^2.$$

On the basis of the representation

$$\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{(i-1)(k-1)}{m(m-1)n(n-1)} a_{ik} \varphi_{ik}(x),$$

we can conclude that

$$\sum_{p=0}^{\infty} \int [M_{p,\theta}^{(8)}(x)]^2 d\mu(x) \leqq$$

$$\leqq (2\theta-1) \sum_{p=0}^{\infty} \sum_{m=2^p+1}^{2^p+1} \sum_{n=2^p+1}^{\theta 2^p+1} mn \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{(i-1)^2(k-1)^2}{m^2(m-1)^2 n^2(n-1)^2} a_{ik}^2 \leqq$$

$$\leqq (2\theta-1)^2 \sum_{p=0}^{\infty} \frac{1}{2^{4p}} \sum_{i=1}^{2^p+1} \sum_{k=1}^{\theta 2^p+1} (i-1)^2(k-1)^2 a_{ik}^2 =$$

$$= (2\theta-1)^2 \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} (i-1)^2(k-1)^2 a_{ik}^2 \sum_{p: 2^p+1 \geqq \max(i,k/\theta)} \frac{1}{2^{4p}} < \infty.$$

Applying B. Levi's theorem, we get (4.11).

Since the relation

$$\max_{2^p < m \leqq 2^p+1} \max_{\theta-1 2^p < n \leqq 2^p} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^q}(x)| = o_x\{1\} \quad a.e.$$

can be similarly proved, this completes the proof of Lemma 4.

Proof of Theorem 1. We can estimate in the following way: for $2^p < m \leqq$ $\leqq 2^{p+1}$ and $\theta^{-1} \leqq n/m \leqq \theta$ $(p=0, 1, ...)$ we have

$$|\sigma_{mn}(x) - f(x)| \leqq |s_{2^p, 2^p}(x) - f(x)| + |\sigma_{2^p, 2^p}(x) - s_{2^p, 2^p}(x)| +$$
$$+ M_{p,1}^{(3)}(x) + M_{p,\theta}^{(4)}(x) + M_{p,\theta}^{(7)}(x).$$

Now, we have to collect (4.1), (4.4), (4.5), (4.6) and (4.10) in order to obtain the statement of Theorem 1.

Proof of Theorem 2. It is quite similar to that of Theorem 1. Relying on Lemmas 2, 3 and 4, it is enough to prove the next

Lemma 5. *Under condition* (1.2),

$$s_{2^p, 2^p}(x) = \sigma_x \{\log (p+2)\} \quad a.e.$$

Proof of Lemma 5. We will prove the following more general proposition: Whatsoever the monotonic sequence $\{Q_p: p=0, 1,\}$ of finite regions in $\mathbf{N}^2$ is, under condition (1.2) we have

(4.12) $$s_p(Q; x) = o_x \{\log (p+2)\} \quad a.e.$$

(cf. the notation before Theorem G above).

To this effect, let us set

$$A_r = \Big(\sum_{(i,k) \in Q_r \setminus Q_{r-1}} a_{ik}^2 \Big)^{1/2} \quad (r = 0, 1, ...; Q_{-1} = \emptyset)$$

and

$$\Phi_r(x) = \begin{cases} \dfrac{1}{A_r} \displaystyle\sum_{(i,k) \in Q_r \setminus Q_{r-1}} a_{ik} \varphi_{ik}(x) & \text{if } A_r \neq 0, \\ \dfrac{1}{|Q_r \setminus Q_{r-1}|^{1/2}} \displaystyle\sum_{(i,k) \in Q_r \setminus Q_{r-1}} \varphi_{ik}(x) & \text{if } A_r = 0, \end{cases}$$

where by $|Q_r \setminus Q_{r-1}|$ we denote the number of the lattice points of $\mathbf{N}^2$ contained in $Q_r \setminus Q_{r-1}$.

It is clear that $\{\Phi_r(x): r=0, 1, ...\}$ is an ONS, and by (1.2)

$$\sum_{r=0}^{\infty} A_r^2 = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty.$$

By the classical Rademacher estimate (see, e.g. [2, p. 83]),

$$\sum_{r=0}^{p} A_r \Phi_r(x) = o_x \{\log (p+2)\} \quad a.e.$$

But this is equivalent to (4.12) since

$$s_p(Q; x) = \sum_{r=0}^{p} A_r \Phi_r(x) \quad (p = 0, 1, ...).$$

5. Proofs of Theorems 3 and 4. We begin with the following

Lemma 6. *Under condition* (1.2), *for every* $\theta \geq 1$

$$(5.1) \qquad A_{m,\theta}(x) = \left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1}i}^{\theta i} [s_{ik}(x) - \sigma_{ik}(x)]^2 \right\}^{1/2} = o_x\{1\} \quad \text{a.e.}$$

Proof. Our first aim is to show that the function $F_\theta(x)$ defined by

$$F_\theta(x) = \left\{ \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=\theta^{-1}m}^{\theta m} [s_{mn}(x) - \sigma_{mn}(x)]^2 \right\}^{1/2}$$

belongs to L^2. To this end, we use the representation

$$s_{mn}(x) - \sigma_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \left(\frac{i-1}{m} + \frac{k-1}{n} - \frac{(i-1)(k-1)}{mn} \right) a_{ik} \varphi_{ik}(x)$$

and estimate the termwise integrated series from above as follows

$$\int F_\theta^2(x)\, d\mu(x) \leq 3 \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=\theta^{-1}m}^{\theta m} \sum_{i=1}^{m} \sum_{k=1}^{n} \left(\frac{(i-1)^2}{m^2} + \frac{(k-1)^2}{n^2} + \frac{(i-1)^2(k-1)^2}{m^3 n^2} \right) a_{ik}^2 =$$

$$= 3(I_4 + I_5 + I_6), \quad \text{say.}$$

Performing elementary steps, by (1.2)

$$I_4 \leq (\theta - \theta^{-1} + 1) \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{i=2}^{m} \sum_{k=1}^{\theta m} (i-1)^2 a_{ik}^2 =$$

$$= (\theta - \theta^{-1} + 1) \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} (i-1)^2 a_{ik}^2 \sum_{m:\, m \geq \max(i, k/\theta)} \frac{1}{m^3} < \infty.$$

Similarly,

$$I_5 \leq \theta^2 (\theta - \theta^{-1} + 1) \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} (k-1)^2 a_{ik}^2 \sum_{m:\, m \geq \max(i, k/\theta)} \frac{1}{m^3} < \infty.$$

And, finally, $I_6 \leq I_4$.

Now, if we apply the well-known Kronecker lemma (see, e.g. [2, p. 72]) we come to (5.1).

Proof of Theorem 3. By assumption, for every $\theta \geq 1$, $\sigma_{mn}(x)$ converges to $f(x)$ a.e. as $m, n \to \infty$ and $\theta^{-1} \leq n/m \leq \theta$. Consequently,

$$(5.2) \qquad B_{m,\theta}(x) = \left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1}i}^{\theta i} [\sigma_{ik}(x) - f(x)]^2 \right\}^{1/2} \to 0 \quad \text{a.e.}$$

(Here we cannot guarantee the existence of a function $F_\theta(x) \in L^2$ such that $B_{m,\theta}(x) \leq \leq F_\theta(x)$ a.e. for every $m = 1, 2, \dots$.)

If we take into consideration the triangle inequality

(5.3) $$\left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1}i}^{\theta i} [s_{ik}(x)-f(x)]^2 \right\}^{1/2} \leqq A_{m,\theta}(x) + B_{m,\theta}(x),$$

then (5.1) and (5.2) imply (3.4) to be proved. The additional statement in Theorem 3 also easily follows from (5.3).

Proof of Theorem 4. This time we rely on the following inequality:

(5.4) $$\left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1}i}^{\theta i} s_{ik}^2(x) \right\}^{1/2} \leqq A_{m,\theta}(x) + C_{m,\theta}(x),$$

where

$$C_{m,\theta}(x) = \left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1}i}^{\theta i} \sigma_{ik}^2(x) \right\}^{1/2}.$$

By Theorem 2,

$$C_{m,\theta}(x) = o_x\{\log \log (m+3)\} \quad \text{a.e.}$$

Referring again to (5.1), (5.4) implies both statements of Theorem 4.

6. On the sharpness of Theorems 1 and 2. FEDULOV [5] showed that Theorem C is the best possible in the following sense. Let $\{\varepsilon(m): m=1, 2, \ldots\}$ be a nonincreasing sequence of positive numbers tending to 0 as $m \to \infty$. Then there exist a double ONS $\{\varphi_{ik}(x)\}$ on the unit square I^2 and a double sequence $\{a_{ik}\}$ of coefficients such that

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \varepsilon(\min(i, k)) [\log \log (i+3)]^2 [\log \log (k+3)]^2 < \infty$$

and

$$\limsup_{m,n \to \infty} |\sigma_{mn}(x)| = \infty \quad \text{a.e. on } I^2.$$

(In [5] the formulation is somewhat different from ours.)

Theorem D is also exact in general. It was pointed out in [11] that, given any sequence $\{\varepsilon(m)\}$ with the properties indicated just above, there exist a double ONS $\{\varphi_{ik}(x)\}$ on I^2 and a double sequence $\{a_{ik}\}$ of coefficients such that condition (1.2) is satisfied and

$$\limsup_{m,n \to \infty} \frac{|\sigma_{mn}(x)|}{\varepsilon(\min(m, n)) \log \log (m+3) \log \log (n+3)} = \infty \quad \text{a.e. on } I^2.$$

Now we are going to add the following supplement. Theorems 1 and 2 are the best possible even in the very special case $\theta = 1$ (i.e. $m = n$). Indeed, given a sequence $\{\varepsilon(m)\}$ with the above properties, there exist a double ONS $\{\varphi_{ik}(x)\}$ on the unit

interval I and a sequence $\{a_{ik}: a_{ik}=0$ for $i \neq k\}$ such that

$$\sum_{i=1}^{\infty} a_{ii}^2 \varepsilon(i) [\log\log(i+3)]^2 < \infty$$

and

$$\limsup_{m \to \infty} |\sigma_{mm}(x)| = \infty \quad \text{a.e. on} \quad I.$$

Similarly, there exist possible another double ONS $\{\varphi_{ik}(x)\}$ on I and a double sequence $\{a_{ik}: a_{ik}=0$ for $i \neq k\}$ such that condition (1.2) is satisfied and

$$\limsup_{m \to \infty} \frac{|\sigma_{mm}(x)|}{\varepsilon(m) \log\log(m+3)} = \infty \quad \text{a.e. on} \quad I.$$

The last two counterexamples can be constructed with the help of the "one-dimensional" counterexamples of MENŠOV [6] and TANDORI [13, Theorem 8], respectively. The only important modification is that now we need an infinite number of "indifferent" orthonormal functions at our disposal in order to place them for $\varphi_{ik}(x)$ with $i \neq k$ $(i, k=1, 2, \dots)$ (and these functions do not play any role later on because for different i and k all the coefficients a_{ik} are chosen to equal 0). On the other hand, the orthonormal functions themselves occurring in the corresponding counterexamples of Menšov and Tandori are used in the capacity of $\varphi_{ii}(x)$ $(i=1, 2, \dots)$. Since the latter functions are step ones, this contruction can be carried out without any difficulty. We do not enter into further details.

On closing we remark that the results of the present paper can be extended, without any essential modification, to the case of d-multiple orthogonal series as well $(d=3, 4, \dots)$.

Note added in proof. Questions (3.6) and (3.7) are studied in another paper of mine: On the strong summability of double orthogonal series, *Michigan Math. J.,* **31** (1984), 241—255.

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