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Representation of functionals via summability methods. II

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Dedicated to Professor L. Leindler on his 50th birthday

1. Introduction

This article is a direct continuation of the paper [4]. There we showed that if K is a metrizable compact space and $C(K)$ is the sup-normed Banach-space of all real valued continuous functions on K, then to every $L\in C^*(K)$ there are sequences $\{c_k\}\in l^\infty$ and $\{x_k\}\subseteqq K$ such that for every $f\in C(K)$

$$(1.1)\qquad Lf=\lim_{n\to\infty}(1/n)(c_1 f(x_1)+\dots+c_n f(x_n))$$

holds. We proved also that every positive linear functional L with norm 1 (shortly *PL*1 functional) has the form

$$(1.2)\qquad Lf=\lim_{n\to\infty}(1/n)(f(x_1)+\dots+f(x_n))$$

with a suitable sequence $\{x_k\}\subseteqq K$.

Extensions to subadditive functionals by replacing lim with limsup were also treated. Using the language of [4] we call a functional on a certain space which has the form (1.1) or (1.2) a weighted $(C, 1)$-functional or a $(C, 1)$-functional, respectively.

Here, in Section 2, we show that these results can be extended to $Q[0, 1]$, the space of functions having discontinuities only of the first kind, and that $Q[0, 1]$ is maximal, in a certain sense, among spaces having this representability property. In Section 3 we determine those functionals of $R[0, 1]$, the space of Riemann-integrable functions, which have the form (1.1) and Section 5 contains an application of this result to density measures: we give all finitely additive measures which can be obtained as the density of a certain sequence in $\mathbf{R}^n$. Finally, in Section 4 we solve the problem: by which summability methods can we replace the arithmetical mean method (i.e. the $(C, 1)$-method) in (1.1) and (1.2)?

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2. The space $Q[0, 1]$

Let $Q[0, 1]$ be the sup-normed real Banach space of bounded functions defined on $[0, 1]$ having discontinuities only of the first kind, i.e. $f\in Q[0, 1]$ if and only if

$$f(x+0) = \lim_{y\to x+0} f(y), \quad f(x-0) = \lim_{y\to x-0} f(y),$$
$$f(1+0) \overset{\text{def}}{=} f(1), \quad f(0-0) \overset{\text{def}}{=} f(0)$$

exist at every point $x\in[0, 1]$. It is an easy task to prove that $Q[0, 1]$ is exactly the uniform closure of the set of step functions. KALTENBORN [1] determined the dual space $Q^*[0, 1]$ by the aid of a certain generalized Stieltjes-integral.

Now we shall show that on $Q[0, 1]$ every $PL1$ functional is a $(C, 1)$-functional and that there is no larger "natural" space with this property.

Theorem 1. *On* $Q[0, 1]$ *every* $PL1$ *functional* L *has the form* (1.2) *with a suitable sequence* $\{x_k\}$.

This yields at once

Corollary 1. *Every* $L\in Q^*[0, 1]$ *has the form* (1.1).

Note that $Q[0, 1]$ is far from being separable.

Now let B be a sup-normed space of bounded functions defined on $[0, 1]$ which is closed under substitution of continuously differentiable homeomorphisms of $[0, 1]$, i.e. if $\varphi\colon [0, 1]\to[0, 1]$ is a strictly increasing continuously differentiable function with $\varphi(0)=0$, $\varphi(1)=1$ and $f\in B$ then $f\circ\varphi\in B$. Such spaces are $C[0, 1]$; $Q[0, 1]$; $R[0, 1]$ — the set of all Riemann-integrable functions; the space of left continuous functions, etc. We shall show that $Q[0, 1]$ is maximal among such spaces having the $(C, 1)$-representability property.

Theorem 2. *Let B be as above. If $Q\subset B$ then for some $x_0\in[0, 1]$ no extension of the functional*

$$L_{x_0}f = (f(x_0-0)+f(x_0+0))/2 \quad (f\in Q[0, 1])$$

to B is a weighted $(C, 1)$-functional.

Proof of Theorem 1. Let $L\in Q^*[0, 1]$ be a $PL1$ functional and set

$$H_1 = \{x \mid L\chi_{\{x\}} > 0\},$$
$$H_2 = \Big\{x \mid \lim_{\varepsilon\to 0+0} L\chi_{(x, x+\varepsilon)} > 0\Big\}, \quad H_3 = \Big\{x \mid \lim_{\varepsilon\to 0+0} L\chi_{(x-\varepsilon, x)} > 0\Big\}.$$

(χ_A denotes the characteristic function of the set A.) Since L is bounded, these sets are countable say, $H_1=\{y_k^{(1)}\}$, $H_2 = \{y_k^{(2)}\}$, $H_3 = \{y_2^{(3)}\}$, and if

$$\tau_k^{(1)} = L\chi_{\{y_k^{(1)}\}}, \quad \tau_k^{(2)} = \lim_{\varepsilon\to 0} L\chi_{(y_k^{(2)}, y_k^{(2)}+\varepsilon)}, \quad \tau_k^{(3)} = \lim_{\varepsilon\to\infty} L\chi_{(y_k^{(3)}-\varepsilon, y_k^{(3)})},$$

then for the numbers

$$\mu_1 = \sum_k \tau_k^{(1)}, \quad \mu_2 = \sum_k \tau_k^{(2)}, \quad \mu_3 = \sum_k \tau_k^{(3)}$$

we have $\mu_1+\mu_2+\mu_3 \leqq \|L\| = 1$. An easy consideration shows that for

$$L_1 f = (1/\mu_1) \sum_k \tau_k^{(1)} f(y_k^{(1)}), \quad L_2 f = (1/\mu_2) \sum_k \tau_k^{(2)} f(y_k^{(2)}+0),$$

$$L_3 f = (1/\mu_3) \sum_k \tau_k^{(3)} f(y_k^{(3)}-0) \quad (f \in Q[0, 1])$$

the functional $L^* = L - \mu_1 L_1 - \mu_2 L_2 - \mu_3 L_3$ is a positive functional with norm $\mu_4 \overset{\text{def}}{=} 1 - \mu_1 - \mu_2 - \mu_3$. Let

$$L_4 f = (1/\mu_4) L^* f \quad (f \in Q[0, 1]).$$

By our construction

$$L_4 \chi_{\{x\}} = \lim_{\varepsilon \to 0} L_4 \chi_{(x \pm \varepsilon, x)} = 0 \quad (x \in [0, 1]),$$

therefore the function

$$\alpha(x) = L_4 \chi_{[0, x]} \tag{2.1}$$

is a continuous and increasing function. Exactly as in the proof of [4, Corollary 3] it can be proved that if $\{z_k\} \subseteq [0, 1]$ is an arbitrary dense sequence then there exists a sequence $\{x_i\}$ such that with the notation

$$\sigma_n(\{x_i\}, f) = (1/n)(f(x_1) + \ldots + f(x_n))$$

we have

$$\alpha(z_k) = \lim_{n \to \infty} \sigma_n(\{x_i\}, \chi_{[0, z_k]})$$

for every k. By the monotonicity and continuity of α,

$$\alpha(x) = \lim_{n \to \infty} \sigma_n(\{x_i\}, \chi_{[0, x]}) \tag{2.2}$$

also holds for every $x \in [0, 1]$, and since the set of step functions is dense in $Q[0, 1]$ we can conclude by (2.1) and (2.2) that

$$L_n f = \lim_{n \to \infty} \sigma_n(\{x_i\}, f)$$

for every $f \in Q[0, 1]$, i.e. L_4 is a $(C, 1)$-functional.

Since it is easy to verify that L_1, L_2 and L_3 are also $(C, 1)$-functionals and since $L = \mu_1 L_1 + \ldots + \mu_4 L_4$, $\mu_1 + \ldots + \mu_4 = 1$, the theorem follows by a familiar argument (cf. [4]).

Also, the proof of Corollary 1 is standard (cf. [4]).

Proof of Theorem 2. If $Q \subset B$ then there exists a function f which does not have e.g. right hand limit at a certain point x_0. Let $\tilde{L}$ be any extension of L_{x_0}

to B, and let us suppose on the contrary that $\tilde{L}$ is represented in the sense of (1.1) by the sequences $\{c_k\}$, $\{x_k\}$. The idea is to construct a function in B by the aid of f for which the limit in (1.1) does not exist.

We shall only sketch the proof. By linearity we may suppose that there are sequences $1=u_1>v_1>u_2>v_2>\ldots>x_0$ converging to x_0 with

$$\lim_{k\to\infty} f(u_k) = 1, \quad \lim_{k\to\infty} f(v_k) = 0.$$

For the sake of convenience we shall use the notation

$$\sigma_n(g) = (1/n)\big(c_1 g(x_1)+\ldots+c_n g(x_n)\big)$$

in the rest of the proof.

Since for every $\varepsilon>0$ $(\varepsilon<1-x_0)$ we have $\tilde{L}\chi_{(x_0,x_0+\varepsilon)}=1/2$, $\tilde{L}\chi_{(x_0+\varepsilon,1]}=0$, the sequences $\{n_j\}$, $\{\varepsilon_j\}$, $\{x_i^{(j)}\}_{i=1}^{k_j}$, $\{c_i^{(j)}\}_{i=1}^{k_j}$ and $\eta_j\to 0$ can be determined successively according to the requirements:

$$\sigma_{n_1}(\chi_{(x_0,1)}) = 1/2+\eta_1, \quad |\eta_1| < 1/2, \quad \varepsilon_1 = \min_{\substack{1\le k\le n_1\\ x_k>x_0}} (x_k-x_0),$$

$$\{x_i^{(1)}\}_{i=1}^{k_1} = \{x_k \mid 1 \le k \le n_1,\ x_k > x_0\},$$

and let $c_i^{(1)}$ $(1\le i\le k_1)$ be the corresponding constants (i.e. if $x_i^{(1)}=x_\nu$ then let $c_i^{(1)}=c_\nu$);

$$\sigma_{n_2}(\chi_{(x_0,\varepsilon_1)}) = 1/2+\eta_2, \quad |\eta_2| < 1/4, \quad \varepsilon_2 = \min_{\substack{1\le k\le n_2\\ x_k>x_0}} (x_k-x_0),$$

$$\{x_i^{(2)}\}_{i=1}^{k_2} = \{x_k \mid 1 \le k \le n_2,\quad x_0 < x_k < x_0+\varepsilon_1\}$$

and $\{c_i^{(2)}\}_{i=1}^{k_2}$ the set of the corresponding constants, and so on. We may assume as well that $(k_1+\ldots+k_i)/n_{i+1}\to 0$ as $i\to\infty$.

Now let

$$\varphi_1, \varphi_2\colon \bigcup_{j=1}^{\infty} \{x_i^{(j)}\}_{i=1}^{k_j} \to \{u_k\}_{k=1}^{\infty} \cup \{v_k\}_{k=1}^{\infty}$$

be $1-1$, monotonically increasing mappings with the properties:

$$\begin{aligned} &\varphi_1(x_i^{(2j-1)}) = \varphi_2(x_i^{(2j-1)}) \in \{u_k\}_{k=1}^{\infty}, \quad 1 \le i \le k_{2j-1}, \\ &\varphi_1(x_i^{(2j)}) \in \{u_k\}_{k=1}^{\infty}, \quad \varphi_2\{x_i^{(2j)}\} \in \{v_k\}_{k=1}^{\infty}, \quad 1 \le i \le k_{2j}, \end{aligned} \qquad j = 1, 2, \ldots,$$

$$\big(\varphi_\tau(x_i^{(j)})-x_0\big)/(x_i^{(j)}-x_0) = o(1) \quad (\tau = 1, 2)$$

as $j\to\infty$ uniformly in $1\le i\le k_j$. φ_1 and φ_2 can be extended to continuously differentiable homeomorphisms of $[0, 1]$ with $\varphi_1'(x_0)\equiv\varphi_2'(x_0)$ and $\varphi_1(x)\equiv\varphi_2(x)$ for $x\in[0, x_0]$.

The construction gives

$$\sigma_{n_{2j+1}}(f\circ\varphi_1-f\circ\varphi_2)=\sigma_{n_{2j+1}}(\chi_{(x_0,1)}f\circ\varphi_1-\chi_{(x_0,1)}f\circ\varphi_2)=$$

$$=o(1)+1/n_{2j+1}\sum_{i=1}^{k_{2j+1}}c_i^{(2j+1)}\big(f(\varphi_1(x_i^{(2j+1)}))-f(\varphi_2(x_i^{(2j+1)}))\big)=o(1)+0=o(1)$$

and

$$\sigma_{n_{2j}}(f\circ\varphi_1-f\circ\varphi_2)=o(1)+\big(1+o(1)\big)\sigma_{n_{2j}}\big(\chi_{(x_0,x_0+\varepsilon_{2j-1})}(f\circ\varphi_1-1)\big)-$$

$$-\big(1+o(1)\big)\sigma_{n_{2j}}(\chi_{(x_0,x_0+\varepsilon_{2j-1})}f\circ\varphi_2)+\big(1+o(1)\big)\sigma_{n_{2j}}(\chi_{(x_0,x_0+\varepsilon_{2j-1})})=$$

$$=o(1)+o(1)+o(1)+\big(1+o(1)\big)\big(1/2+o(1)\big),$$

i.e. either for $f\circ\varphi_1\in B$ or for $f\circ\varphi_2\in B$ the limit on the right of (1.1) does not exist, which contradicts our assumption concerning the sequences $\{c_k\}$, $\{x_k\}$.

3. The space $\mathscr{R}[0, 1]$

Let $\mathscr{R}[0, 1]$ denote the space of Riemann-integrable bounded functions defined on $[0, 1]$. We equip $\mathscr{R}[0, 1]$ with the sup norm. By Theorem 2 $\mathscr{R}[0, 1]$ has bounded linear functionals which are not weighted $(C, 1)$-functionals. In the present section we characterize the (weighted) $(C, 1)$-functionals of $\mathscr{R}[0, 1]$. An application to density measures will be given in the last section.

Theorem 3. *A functional $L\in\mathscr{R}^*[0, 1]$ is a weighted $(C, 1)$-functional (i.e. it has form (1.1)) if and only if L is of the form*

$$Lf=\sum_{i=1}^{\infty}\mu_i f(\tau_i)+\int_0^1 f(t)g(t)\,dt\quad (f\in\mathscr{R}[0, 1]),$$

where $\tau_i\in[0, 1]$ $(1\leqq i)$, $\sum_{i=1}^{\infty}|\mu_i|<\infty$ *and* $g\in L^1[0, 1]$.

Corollary 2. *A PL1 functional $L\in\mathscr{R}^*[0, 1]$ is a $(C, 1)$-functional (i.e. it has form (1.2)) if and only if there are* $\tau_i\in[0, 1]$, $\mu_i\geqq 0$ $(1\leqq i)$, $g\in L^1[0, 1]$, $g\geqq 0$ *such that*

$$\int_0^1 g(t)\,dt=1,\quad \sum_{i=1}^{\infty}\mu_i\leqq 1,$$

and for every $f\in\mathscr{R}[0, 1]$

$$Lf=\sum_{i=1}^{\infty}\mu_i f(\tau_i)+\Big(1-\sum_{i=1}^{\infty}\mu_i\Big)\int_0^1 f(t)g(t)\,dt.$$

Proof. First we prove the necessity part of Theorem 3. Let us call a point $x\in[0, 1]$ a singular point of L if $L\chi_{\{x\}}\neq 0$, and a functional having the form

$$Lf=\sum_{i=1}^{\infty}\mu_i f(\tau_i),\quad \sum|\mu_i|<\infty,$$

will be called a discrete functional. First we show

Lemma 1. *For every $L\in\mathscr{R}^*[0, 1]$ the set of singular points is countable and $L=L_1+L_2$ where L_1 is a discrete functional and L_2 is without singular points.*

Proof. Since for arbitrary points $x_1, \dots, x_n$ we have

$$\Big|\sum_{i=1}^{n}\pm L\chi_{\{x_i\}}\Big|=\Big|L\sum_{i=1}^{n}\pm\chi_{\{x_i\}}\Big|\leqq\|L\|,$$

there are at most countably many singular points of L. Let they be $\tau_1, \tau_2, \dots$. The previous inequality shows that the numbers $\mu_i=L_{\{\tau_i\}}$ satisfy $\sum_{i=1}^{\infty}|\mu_i|\leqq\leqq\|L\|$. Now

$$L_1 f=\sum_{i=1}^{\infty}\mu_i f(\tau_i)\quad\text{and}\quad L_2=L-L_1$$

clearly satisfy the requirements of the lemma.

We need also another lemma.

Lemma 2. *If $L\in\mathscr{R}^*[0, 1]$ is a weighted $(C, 1)$-functional without singular points then the function $\alpha(x)=L\chi_{[0,x]}$ $(x\in[0, 1])$ is absolutely continuous.*

Proof. If $0=w_0<w_1<\dots<w_n=1$ are arbitrary points then for certain signs $+, -$ we have

$$\sum_{i=0}^{n-1}|\alpha(w_{i+1})-\alpha(w_i)|=L\big(\pm\chi_{[w_0, w_1]}+\sum_{i=1}^{n-1}\pm\chi_{(w_i, w_{i+1}]}\big)\leqq\|L\|,$$

i.e. α is of bounded variation. We show first that α is continuous.

Let us suppose on the contrary that α is not continuous at the point x. Then either $\alpha(x+0)\neq\alpha(x)$ or $\alpha(x-0)\neq\alpha(x)$, let us consider e.g. the former case. If e.g. $\alpha(x+0)>\alpha(x)$ then there are constants $\varepsilon>0$, $\delta>0$ such that for $x<y<x+\delta$ we have $\alpha(y)-\alpha(x)>\varepsilon$. Since L is a weighted $(C, 1)$-functional, there are sequences $\{c_i\}$, $\{x_i\}$ such that

$$\lim_{n\to\infty}\sigma_n(f)\overset{\text{def}}{=}\lim_{n\to\infty}(1/n)\big(c_1f(x_1)+\dots+c_nf(x_n)\big)=Lf \tag{3.1}$$

holds for every $f\in\mathscr{R}[0, 1]$.

Let $x<y_1<x+\delta$ be arbitrary. By our assumptions there is an n_1 such that $\sigma_{n_1}(\chi_{(x,y_1]})>\varepsilon$. Let $z_1, \ldots, z_{k_1}$ be those x_i's for which $x_i \in (x, y_1]$, $1 \leqq i \leqq n_1$. (For the sake of easier printing, in subscripts we shall write $\{z_i; \nu, \mu\}$ for $\{z_i\}_{i=\nu}^{\mu}$.) Then we have also $\sigma_{n_1}(\chi_{\{z_i;1,k_1\}})>\varepsilon$. L is without singular points, therefore there exists an N_1 with

$$\sigma_n(\chi_{\{z_i;1,k_1\}}) < \varepsilon/4 \quad \text{for} \quad n \geqq N_1.$$

After this let $x<y_2<x+\delta/2$ be such that

$$y_2 - x < \min_{\substack{1 \leqq i \leqq N_1 \\ x_i > x}} (x_i - x)$$

is satisfied. Again there is an $n_2 > N_1$ with

$$\sigma_{n_2}(\chi_{(x,y_2]}) > \varepsilon$$

and if $z_{k_1+1}, \ldots, z_{k_2}$ are the x_i's for which $x_i \in (x, y_2]$, $1 \leqq i \leqq n_2$ then there is an N_2 such that for $n \geqq N_2$ we have

$$\sigma_n(\chi_{\{z_i;1,k_2\}}) < \varepsilon/4.$$

Repeating this argument we obtain a sequence $\{z_i\}_{i=1}^{\infty}$ converging to x and sequences $\{n_j\}_{j=1}^{\infty}$, $\{N_j\}_{j=1}^{\infty}$ such that

$$\sigma_{n_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{n_j}(\chi_{\{z_i;k_{j-1}+1,k_j\}}) + \sigma_{n_j}(\chi_{\{z_i;1,k_{j-1}\}}) \geqq \varepsilon - \varepsilon/4 = 3\varepsilon/4$$

while

$$\sigma_{N_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{N_j}(\chi_{\{z_i;1,k_j\}}) < \varepsilon/4$$

i.e.

$$\lim_{n \to \infty} \sigma_n(\chi_{\{z_i,1,\infty\}})$$

does not exist, which is a contradiction since $\chi_{\{z_i;1,\infty\}}$ is Riemann-integrable.

The absolute continuity of $\alpha(x)$ will be proved by a similar argument. Let α be the signed Borel measure associated with $\alpha(x)$ (cf. [3, p. 173]), in the sense

$$\alpha([0, x]) = \alpha(x),$$

and let α^+, α^- and $|\alpha| = \alpha^+ + \alpha^-$ be the positive and negative parts and the total variation of α, respectively (cf. [3, pp. 134, 125]). We have to prove that α is absolutely continuous with respect to the Lebesgue measure. Suppose not. Then either α^+ or α^- is not absolutely continuous, let us consider e.g. the first case. Since α^+ and α^- have disjoint (not necessarily compact) supports and the singular part of α^+ does not vanish, the regularity of α^+ and α^- yields a closed set $H_0 \subseteqq [0, 1]$ with Lebesgue measure zero and constants ε, $\delta_0, \delta_1, \ldots > 0$, $\delta_n \to 0$, such that

$$\alpha^+(H_0) > \varepsilon, \quad \alpha^-(H_0^{\delta_n}) < \varepsilon/4^{n+1} \quad (n = 0, 1, \ldots)$$

are satisfied where

$$H_0^{\delta} = \{x \mid \operatorname{dist}(x, H_0) < \delta\}.$$

Let $\eta_0=\delta_0$. We have $\alpha(H_0^{\eta_0})>\varepsilon-\varepsilon/4$ and since $H_0^{\eta_0}$ is the union of finitely many intervals we obtain together with this also that

$$L\chi_{H_0^{\eta_0}} > \varepsilon-\varepsilon/4,$$

by which

$$\sigma_{n_0}(\chi_{H_0^{\eta_0}}) > \varepsilon-\varepsilon/4$$

for some n_0. Let $z_1, \ldots, z_{k_0}$ be those x_i's for which $1\leqq i\leqq n_0$ and $x_i\in H_0^{\eta_0}$ are satisfied. Since L has no singular point, there exists N_0 such that for every $n\geqq N_0$ we have

$$\sigma_n(\chi_{\{z_i;\,1,k_0\}}) < \varepsilon/8.$$

Since α is continuous we have $|\alpha|(\{x\})=0$ for every x and the regularity of the measure $|\alpha|$ yields that we can choose disjoint closed intervals $U_1, \ldots, U_{N_0}$ around the points $x_1, \ldots, x_{N_0}$ in such a way that $|\alpha|(\bigcup_{i=1}^{N_0} U_i)<\varepsilon/16$ is satisfied. Let $x_i\in U_i'\subseteqq U_i$ be open intervals without common endpoints with U_i and

$$H_1 = H_0/(\bigcup_{i=1}^{N_0} U_i').$$

If $\eta_1>0$ is less than δ_1 and less than the distances between the endpoints of the U_i''s and U_i's and also less than the distances of the endpoints of U_i''s from x_i's then we have

$$H_1^{\eta_1} \supseteqq H_0^{\eta_1}/[H_0\cap(\bigcup_{i=1}^{N_0} U_i')]^{\eta_1} \supseteqq H_0^{\eta_1}/\bigcup_{i=1}^{N_1} U_i,$$

$$\alpha^+(H_1^{\eta_1}) \geqq \alpha^+(H_0^{\eta_1})-\alpha^+(\bigcup_{i=1}^{N_1} U_i) \geqq \varepsilon-\varepsilon/16, \quad \alpha^-(H_1^{\eta_1}) \leqq \alpha^-(H_1^{\eta_1}) \leqq \varepsilon/16,$$

and hence

$$\alpha(H_1^{\eta_1}) = L\chi_{H_1^{\eta_1}} > \varepsilon-\varepsilon/8.$$

There exists an $n_1>N_0$ with

$$\sigma_{n_1}(\chi_{H_1^{\eta_1}}) > \varepsilon-\varepsilon/8,$$

and if $z_{k_0+1}, \ldots, z_{k_1}$ are the points x_i for which $x_i\in H_1^{\eta_1}$, $1\leqq i\leqq n_1$ then we have $z_i\notin H_0$, $z_i\neq z_j$ for $1\leqq j\leqq k_0$, $k_0+1\leqq i\leqq k_1$;

$$\sigma_{n_1}(\chi_{\{z_i;\,1,k_1\}}) = \sigma_{n_1}(\chi_{\{z_i;\,k_0+1,k_1\}})+\sigma_{n_1}(\chi_{\{z_i;\,1,k_0\}}) \geqq \varepsilon-\varepsilon/8-\varepsilon/8$$

and

$$\sigma_n(\chi_{\{z_i;\,1,k_1\}}) < \varepsilon/8 \quad \text{for} \quad n\geqq N_1$$

for some N_1. If $U_1^*, \ldots, U_{N_1}^*$ are disjoint closed intervals around $x_1, \ldots, x_{N_1}$ with $|\alpha|(\bigcup_{i=1}^{N_1} U_i^*)<\varepsilon/32$ and $x_i\in U_i''\subseteqq U_i^*$, U_i'' open, $H_2=H_1\setminus(\bigcup_{i=1}^{N_1} U_i'')$ then exactly as above we obtain for small $\eta_2>0$,

$$\alpha(H_2^{\eta_2}) = L\chi_{H_2^{\eta_2}} > \varepsilon-\varepsilon/8-\varepsilon/16.$$

Repeating this argument we obtain sequences $\{z_i\}_{i=1}^{\infty}$, $\{H_j\}_{j=1}^{\infty}$, $\{n_j\}_{j=1}^{\infty}$, $\{N_j\}_{j=1}^{\infty}$, $\{k_j\}_{j=1}^{\infty}$ such that $H_{j+1}\subseteqq H_j$ closed, $z_i\notin H_j$ for $i\leqq k_{j-1}$, $z_i\neq z_j$ for $i\neq j$, the sequence $\{z_i\}$ may have limit points only in $H=\bigcap_{j=1}^{\infty} H_j$ and

$$\sigma_{n_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{n_j}(\chi_{\{z_i;k_{j-1}+1,k_j\}})-\sigma_{n_j}(\chi_{\{z_i;1,k_{j-1}\}}) \geqq \tag{3.2}$$
$$\geqq (\varepsilon-\varepsilon/8-\varepsilon/16-\dots)-\varepsilon/8 \geqq \varepsilon/2$$

but

$$\sigma_{N_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{N_j}(\chi_{\{z_i;1,k_j\}}) < \varepsilon/8. \tag{3.3}$$

Since

$$\chi_{\{z_i;1,\infty\}} = \chi_{H\cup\{z_i;1,\infty\}}-\chi_H$$

and $H, H\cup\{z_i\}_{i=1}^{\infty}$ are closed and have Lebesgue measure zero, we obtain that $\chi_{\{z_i;1,\infty\}}$ is Riemann-integrable and (3.2)—(3.3) contradict our assumption concerning the convergence of $\{\sigma_n(g)\}$ for every $g\in\mathscr{R}[0,1]$. This contradiction proves Lemma 2.

Let us return to the proof of Theorem 3. By Lemma 1 $L=L_1+L_2$ where L_1 is discrete and L_2 has no singular point. An easy argument gives that every discrete functional is a weighted $(C, 1)$-functional, so if L is assumed to be weighted $(C, 1)$-functional then L_2 is also a weighted $(C, 1)$-functional. By Lemma 2 the function $\alpha(x)=L_2\chi_{[0,x]}$ is absolutely continuous, let $g(x)=\alpha'(x)$ (a.e.). Then $g\in L^1[0,1]$ and

$$L_2\chi_{[0,x]} = \alpha(x) = \int_0^x g(t)\,dt\,.$$

by which

$$L_2 h = \int_0^1 hg \tag{3.4}$$

for every step-function h. Let $f\in\mathscr{R}^*[0,1]$ be arbitrary, and let

$$L_2^* h = L_2(hf) \quad (h\in\mathscr{R}[0,1]).$$

It is obvious that $L_2^*\in\mathscr{R}^*[0,1]$ and together with L_2, L_2^* is a weighted $(C, 1)$-functional without singular points. By Lemma 2 the function

$$\alpha^*(x) = L_2^*\chi_{[0,x]} = L_2(f\chi_{[0,x]})$$

is absolutely continuous and hence to every $\varepsilon>0$ there exists a $\delta>0$ such that if H is disjoint union of finitely many intervals and m denotes the Lebesgue measure, then $m(H)<\delta$ implies

$$|L_2(f\chi_H)| < \varepsilon. \tag{3.5}$$

We may assume that

$$(3.6)\qquad |L_2\chi_H| < \varepsilon, \quad \int_H |g| < \varepsilon \quad \text{for} \quad m(H) < \delta$$

are also satisfied. Since f is Riemann-integrable there are step-functions φ and Φ such that

$$\varphi \leqq f \leqq \Phi, \quad |\varphi|, |\Phi| \leqq \sup |f|, \quad \int_0^1 (\Phi - \varphi) < \varepsilon\delta.$$

Thus, if

$$H = \{x \mid \Phi(x) - \varphi(x) > \varepsilon\}$$

then $m(H) < \delta$. Let H be the disjoint union of the closed, half-closed or open intervals $\{u_1, v_1\}, \dots, \{u_n, v_n\}$ and let $w_i \in \{u_i, v_i\}$. We may assume that φ is constant on each interval $\{u_i, v_i\}$. By (3.4)—(3.6)

$$\Big|L_2 f - \int_0^1 fg\Big| \leqq \big|L_2((f-\varphi)\chi_{[0,1]\setminus H})\big| + |L_2(f\chi_H)| +$$

$$+\Big|\sum_{i=1}^n \varphi(w_i)\big(\alpha(v_i) - \alpha(u_i)\big)\Big| + \Big|L_2\varphi - \int_0^1 \varphi g\Big| + \Big|\Big|\int_{[0,1]\setminus H} (f-\varphi)g\Big| + \Big|\int_H (f-\varphi)g\Big| \leqq$$

$$\leqq \varepsilon\|L_2\| + \varepsilon + \sup|\varphi| \sum_{i=1}^n |\alpha(v_i) - \alpha(u_i)| + 0 + \varepsilon\|g\|_{L^1} + 2\sup|f| \int_H |g| \leqq$$

$$\leqq \varepsilon(\|L_2\| + \|g\|_{L^1} + 1) + 2\varepsilon \sup|f| + 2\varepsilon \sup|f| \leqq K\varepsilon$$

with a K independent of ε, by which the equality

$$L_2 f = \int_0^1 fg \quad (f \in \mathscr{R}[0, 1])$$

is verified, and the necessity of our condition is proved.

The necessity of the condition in Corollary 2 follows easily from the above consideration, all what we have to mention is that, by the positivity of L and by $\|L\| = 1$, we have $\mu_i \geqq 0$, $\sum_{i=1}^\infty \mu_i \leqq 1$, and in the case $\sum_{i=1}^\infty \mu_i < 1$ the derivative of $\alpha(x) = L_2\chi_{[0,x]}$ is positive because L_2 is also a positive functional (notice that for every n and $f \geqq 0$

$$Lf \geqq L\Big(\sum_{i=1}^n f(\tau_i)\chi_{\{\tau_i\}}\Big) = \sum_{i=1}^n \mu_i f(\tau_i)\Big).$$

After these let us turn to the sufficiency part of our proof. Obviously it is sufficient to prove this for Corollary 2, and since a functional of the form

$$Lf = \sum_{i=1}^\infty \mu_i f(\tau_i), \quad \mu_i \geqq 0, \quad \sum_{i=1}^\infty \mu_i = 1$$

is easily seen to be a $(C, 1)$-functional our task has reduced to the verification of the following: if

$$Lf = \int_0^1 fg, \quad (f \in \mathcal{R}[0, 1])$$

where $g \in L^1[0, 1]$, $g \geqq 0$ and $\int_0^1 g(t)\,dt = 1$ then L is a $(C, 1)$-functional.

Exactly as in the proof of [4, Corollary 3] one can give a sequence $\{x_k\}$ such that

$$\lim_{n\to\infty} \sigma_n(\{x_k\}, \chi_{[0, z_j]}) = \int_0^{z_j} g(t)\,dt \quad \left(\sigma_n(\{x_k\}, g) = (1/n) \sum_{k=1}^{n} g(x_k)\right)$$

is satisfied for a sequence $\{z_j\}$ dense in $[0, 1]$. By monotonicity and by the continuity of $\int_0^z g(t)\,dt$ we obtain the same relation for every $z \in [0, 1]$ and hence

$$\lim_{n\to\infty} \sigma(\{x_k\}, h) = \int_0^1 hg$$

for every step function h. If $f \in \mathcal{R}[0, 1]$, $\varepsilon > 0$, are arbitrary then there are step functions φ, Φ with the properties:

$$\varphi \leqq f \leqq \Phi, \quad |\varphi|,\ |\Phi| \leqq \sup |f|,$$

$m(H) < \varepsilon$, where $H = \{x \mid \Phi(x) - \varphi(x) \geqq \varepsilon\}$ (see above) and these yield

$$L\varphi = \int_0^1 \varphi g = \lim_{n\to\infty} \sigma_n(\{x_k\}, \varphi) \leqq \liminf_{n\to\infty} \sigma_n(\{x_k\}, f) \leqq$$

$$\leqq \limsup_{n\to\infty} \sigma_n(\{x_k\}, f) \leqq \lim_{n\to\infty} \sigma_n(\{x_k\}, \Phi) = \int_0^1 \Phi g = L\Phi,$$

$$L\varphi \leqq Lf \leqq L\Phi, \quad L\Phi - L\varphi = \int_0^1 (\Phi - \varphi) g \leqq \varepsilon \int_0^1 g + 2 \sup |f| \int_H |g|.$$

Since here the right hand side can be made arbitrary small by appropriate choice of ε, these formulas prove the convergence

$$\lim_{n\to\infty} \sigma_n(\{x_k\}, f) = Lf$$

for every $f \in \mathcal{R}[0, 1]$ and the theorem is proved.

4. Other summability methods

In this section we characterize those matrix summability methods which can be substituted in [4, Corollary 3] for the $(C, 1)$-method.

Let thus $T=(t_{n,k})_{n,k=1}^{\infty}$ be an infinite matrix. We say that T sums the sequence $\{s_k\}$ to the limit $T\text{-}\lim_k s_k$ if

$$T\text{-}\lim_k s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} s_k$$

holds. If for every convergent $\{s_k\}$ we have

$$T\text{-}\lim_k s_k = \lim_{k\to\infty} s_k.$$

then T is said to be regular. By the well known Toeplitz theorem T is regular if and only if

(i) $\lim_{n\to\infty} t_{nk}=0$ for every k,

(ii) $\sum_{k=1}^{\infty} |t_{nk}|=O(1)$,

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk}=1$

hold.

By analogy to $(C, 1)$-functionals let us call a functional $L\in C[0, 1]$ a T-functional if there exists a sequence $\{x_k\}_{k=1}^{\infty}\subseteq[0, 1]$ such that

$$Lf = T\text{-}\lim_k f(x_k)$$

holds for every $f\in C[0, 1]$. In order to avoid unnecessary technical difficulties we assume T to be non-negative. Our matrices $T=(t_{nk})$ will have the property that

$$S(T) := \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk}$$

exists. We say that T is decomposed into the matrices $T_1, \ldots, T_n, \ldots$ (in abbreviation $T=\bigcup_n T_n$) if the columns of each T_n are columns of T, they follow each other in T_n in the same order as in T, and each column of T belongs to exactly one of the matrices T_n. Now let us call T completely regular if T can be decomposed into the matrices T_1, T_2 such that $S(T_1)=S(T_2)=S(T)/2$ is satisfied, furthermore T_1 and T_2 can be decomposed into T_{11}, T_{12} and T_{21}, T_{22}, respectively such that $S(T_{11})= =\ldots=S(T_{22})=S(T)/4$ is satisfied, $T_{11}, \ldots, T_{22}$ can further be decomposed into $T_{111}, \ldots$ and so on. E.g. the complete regularity of the $(C, 1)$ matrix is a trivial fact.

We shall prove

Theorem 4. *Let T be a non-negative matrix. The following assertions are equivalent:*

(i) *every $PL1$ functional on $C[0, 1]$ is a T-functional,*

(ii) *there exists a sequence $\{x_k\} \subseteq [0, 1]$ such that*

$$T\text{-}\lim_k f(x_k) = \int_0^1 f(t)\,dt$$

for every $f \in C[0, 1]$,

(iii) *T is regular and completely regular.*

Corollary 3. *Any of* (i)—(iii) *implies that to every $L \in C^*[0, 1]$ there are bounded sequences $\{c_k\}_{k=1}^\infty$ and $\{x_k\}_{k=1}^\infty \subseteq [0, 1]$ such that*

$$T\text{-}\lim_k c_k f(x_k) = Lf \tag{4.1}$$

holds for every $f \in C[0, 1]$.

Remarks 1. In (ii) the functional $f \to \int f$ can be exchanged for every functional $f \to \int f\,d\mu$ with continuous μ, but it cannot be exchanged for one with discontinuous μ.

2. In Theorem 4 we characterized the matrices by which every $PL1$ functional can be represented rather than those by which every $L \in C^*[0, 1]$ can be represented in the form (4.1). Clearly, if (4.1) holds and if we multiply the columns of T by certain numbers and at the same time we divide the c_k's by the same numbers then the T' and $\{c_k'\}$ obtained still satisfy (4.1); therefore the characterization of the T's with the (4.1) representability property is rather hopeless.

Proof. (i)⇒(ii) is obvious. First we show that (iii) implies (i). To this end we need the following definition and lemma. For $x \in [0, 1]$ and $\varepsilon > 0$ let

$$f_{x,\varepsilon}(t) = \begin{cases} 0 & \text{if } |t-x| > \varepsilon \\ 1-|t-x|/\varepsilon & \text{if } |t-x| \leqq \varepsilon \end{cases} \quad t \in [0, 1].$$

We say that x is a singular point of the $PL1$ functional L if

$$\mu_x = \liminf_{\varepsilon \to 0+0} Lf_{x,\varepsilon} = \lim_{\varepsilon \to 0+0} Lf_{x,\varepsilon} > 0.$$

(Note that this notion differs from that used in the proof of Theorem 3). A functional L having the form $Lf = \sum \mu_i f(\tau_i)$, $\sum \mu_i = 1$, $i \geqq 0$, will be called discrete.

Lemma 3. *Let $L \in C^*[0, 1]$ be a $PL1$ functional.*

(α) *The set of the singular points of L is countable and $\sum_x \mu_x \leqq 1$.*

(β) *$L = \mu L_1 + (1-\mu) L_2$ where $0 \leqq \mu \leqq 1$, L_1 and L_2 are $PL1$ functionals, L_1 is discrete and L_2 does not have singular points.*

(γ) *If the PL1 functional L has no singular point then to every* $\eta>0$ *there exists an* $\varepsilon>0$ *such that* $Lf_{x,\varepsilon}<\eta$ *for every* $x\in[0, 1]$.

Proof. If $\tau_1, \ldots, \tau_k\in[0, 1]$ are distinct points and $\varepsilon_1, \ldots, \varepsilon_k$ are so small that

$$\sum_{i=1}^{k} f_{\tau_i, \varepsilon_i} \leqq 1$$

is satisfied then

$$0 \leqq \sum_{i=1}^{k} Lf_{\tau_i, \varepsilon_i} \leqq 1$$

which proves (α).

Let $\tau_1, \tau_2, \ldots$ be the singular points of L and let

$$L_1 f = (1/\mu) \sum_{i=1}^{\infty} \mu_{\tau_i} f(\tau_i), \quad \mu = \sum_{i=1}^{\infty} \mu_{\tau_i} \quad (f\in C[0, 1]).$$

If $\mu<1$ then $L_2=\big(1/(1-\mu)\big)(L-\mu L_1)$ is without singular points. For every $f\geqq 0$, $\delta>0$ and $k\geqq 1$ there are $\varepsilon_1, \ldots, \varepsilon_k>0$ with

$$f \geqq (1-\delta) \sum_{i=1}^{k} f(\tau_i) f_{\tau_i, \varepsilon_i},$$

by which

$$Lf \geqq (1-\delta) \sum_{i=1}^{k} f(\tau_i) Lf_{\tau_i, \varepsilon_i} \geqq (1-\delta) \sum_{i=1}^{k} f(\tau_i) \mu_{\tau_i}.$$

Since here $\delta>0$ and $k\geqq 1$ are arbitrary we can deduce that L_2 is again a $PL1$ functional which proves (β).

Finally, if (γ) were not true then there would be an $\eta>0$ and a sequence $x_1, \ldots, x_n, \ldots$ with $Lf_{x_n, 1/n}\geqq\eta$. If x is a cluster point of $\{x_n\}$ then to every $\varepsilon>0$ there would be an n with $(1/2)f_{x_n, 1/n}\leqq f_{x,\varepsilon}$ by which $Lf_{x,\varepsilon}\geqq(1/2)\eta$ ($\varepsilon>0$) contradicting the assumption that L does not have any singular point.

Now in the proof of (iii)$\Rightarrow$(i) we prove first that every $PL1$ functional L without singular points is a T-functional. An easy argument gives that T can be converted into a triangle-matrix $T^*=(t^*_{nk})$ (i.e. $t^*_{nk}=0$ for $k>n$) which is also regular and completely regular and the limits $T\text{-}\lim_k s_k$ and $T^*\text{-}\lim_k s_k$ exist at the same time and they are equal for every bounded sequence $\{s_k\}$ (first make T to be row-finite and then repeat the rows of T sufficiently many times). Thus, from the point of view of our problem T and T^* are equivalent, so we may assume without loss of generality T to be a triangle-matrix.

Also one can show easily that the complete regularity of T implies the following: if $0\leqq s\leqq 1$ then T can be decomposed into T_1 and T_2 so that $S(T_1)=s$, $S(T_2)=1-s$ are satisfied, furthermore, to every $0\leqq r_1\leqq s$ and $0\leqq r_2\leqq 1-s$ the obtained T_1 and

T_2 can be decomposed into T_{11}, T_{12}, T_{21} and T_{22} so that $S(T_{11})=r_1$, $S(T_{12})=s-r_1$, $S(T_{21})=r_2$, $s(T_{22})=(1-s)-r_2$ are satisfied, etc. We shall call such decompositions completely regular.

Let us consider the functions

$$g_k^{(m)}(x)=\begin{cases}1 & \text{if} \quad k/2^m \leqq x \leqq 1,\\ 0 & \text{if} \quad 0\leqq x\leqq (k-1)/2^m, \quad m=1,2,\ldots, \quad 1\leqq k\leqq 2^m,\\ \text{linear on} & [(k-1)/2^m, k/2^m],\end{cases}$$

$$g_0^{(m)}(x)\equiv 1.$$

Let $q_k^{(m)}=Lg_k^{(m)}$ and $q_k^{(m)}=:\sum_{j=k}^{2^m} p_j^{(m)}$ $(0\leqq k\leqq 2^m,\ m=1,2,\ldots)$. By positivity we have $p_k^{(m)}\geqq 0$ and $\sum_{k=0}^{2^m} p_k^{(m)}=1$.

Let $T=T_0^{(1)}\cup T_1^{(1)}\cup T_2^{(1)}$ be a completely regular decomposition of T such that

$$S(T_0^{(1)})=p_0^{(1)},\quad S(T_1^{(1)})=p_1^{(1)},\quad S(T_2^{(1)})=p_2^{(1)}$$

are satisfied, and let

$$x_n^{(1,0)}=\begin{cases}1 & \text{if} \quad n\in \operatorname{ind} T_2^{(1)},\\ 1/2 & \text{if} \quad n\in \operatorname{ind} T_1^{(1)},\\ 0 & \text{if} \quad n\in \operatorname{ind} T_0^{(1)}\end{cases}$$

where ind T' denotes the set of those natural numbers j for which the j-th column of T belongs to T'. It is clear, that there exists a number $N^{(1,0)}$ such that for $s\geqq N^{(1,0)}$ we have

$$|t_s(\{x_n^{(1,0)}\}, g_i^{(1)})-Lg_i^{(1)}|<1/2 \quad (i=0,1,2),$$

where the notation

$$t_s(\{x_k\}, g):=\sum_{k=1}^{s} t_{sk}\, g(x_k)$$

is used.

For a given m and $0\leqq k\leqq 2^m$ let us consider the functions

$$g_0^{(m+1)},\ g_1^{(m+1)},\ \ldots,\ g_{2k}^{(m+1)},\ g_{k+1}^{(m)},\ g_{k+2}^{(m)},\ \ldots,\ g_{2^m}^{(m)}. \tag{4.2}$$

In the following φ will denote any of these functions. Let $p_{2k}^{*(m+1)}$ be defined by

$$q_{2k}^{(m+1)}=q_{k+1}^{(m)}+p_{2k}^{*(m+1)} \quad \text{if} \quad 2k<2^{m+1} \quad \text{and} \quad p_{2^{m+1}}^{*(m+1)}=p_{2^{m+1}}^{(m+1)}.$$

We suppose that for the pair (m,k) we have already defined the completely regular decomposition of T into the matrices

$$T_0^{(m+1)},\ T_1^{(m+1)},\ \ldots,\ T_{2k-1}^{(m+1)},\ T_{2k}^{*(m+1)},\ T_{k+1}^{(m)},\ \ldots,\ T_{2^m}^{(m)},$$

the sequence $\{x_n^{(m,k)}\}_{n=0}^{\infty}$ and the number $N^{(m,k)}$ so that

$$S(T_0^{(m+1)}) = p_0^{(m+1)}, \dots, S(T_{2k-1}^{(m+1)}) = p_{2k-1}^{(m+1)}, S(T_{2k}^{*(m+1)}) = p_{2k}^{*(m+1)},$$
$$S(T_{k+1}^{(m)}) = p_{k+1}^{(m)}, \dots, S(T_{2^m}^{(m)}) = p_{2^m}^{(m)},$$

for $n > N^{(m,k)}$ we have

$$x_n^{(m,k)} = \begin{cases} i/2^{m+1} & \text{if } n \in \operatorname{ind} T_i^{(m+1)} \quad (i = 0, \dots, 2k-1), \\ 2k/2^{m+1} & \text{if } n \in \operatorname{ind} T_{2k}^{*(m+1)}, \\ i/2^m & \text{if } n \in \operatorname{ind} T_i^{(m)} \quad (i = k+1, \dots, 2), \end{cases}$$

and for $s \geqq N^{(m,k)}$

$$|t_s(\{x_n^{(m,k)}\}, \varphi) - L\varphi| < 1/2^m$$

are satisfied for every φ from (4.2). We want to go over to the pair $(m, k+1)$ (if $k=2^m$ then to the pair $(m+1, 0)$; this case can be treated similarly as the following one).

The regularity of T implies that if we cancel those columns of the matrices $T_{2k}^{*(m+1)}$, $T_{k+1}^{(m)}$ which belong to the first $N^{(m,k)}$ columns of T then the obtained matrices $W_{2k}^{*(m+1)}$, $W_{k+1}^{(m)}$ are still completely regular. Let us unite $W_{2k}^{*(m+1)}$ and $W_k^{(m)}$ into the matrix $V_k^{(m)}$ (the columns in $V_k^{(m)}$ follow each other in the same order as in T), and then decompose $V_k^{(m)}$ into the completely regular matrices $T_{2k}^{(m+1)}, T_{2k+1}^{(m+1)}, T_{2k+2}^{*(m+1)}$ so that

$$S(T_{2k}^{(m+1)}) = p_{2k}^{(m+1)}, \quad S(T_{2k+1}^{(m+1)}) = p_{2k+1}^{(m+1)}, \quad S(T_{2k+2}^{*(m+1)}) = p_{2k+2}^{*(m+1)}$$

be satisfied. This is possible because

$$p_{2k+2}^{*(m+1)} + q_{k+2}^{(m)} + p_{2k+1}^{(m+1)} + p_{2k}^{(m+1)} = q_{2k}^{(m+1)} = p_{2k}^{*(m+1)} + p_{k+1}^{(m)} + q_{k+2}^{(m)}.$$

Now we set

$$x_n^{(m,k+1)} = \begin{cases} 2k/2^{m+1} & \text{if } n \in \operatorname{ind} T_{2k}^{(m+1)}, \\ (2k+1)/2^{m+1} & \text{if } n \in \operatorname{ind} T_{2k+1}^{(m+1)}, \\ (2k+2)/2^{m+1} & \text{if } n \in \operatorname{ind} T_{2k+2}^{*(m+1)}, \\ x_n^{(m,k)} & \text{otherwise.} \end{cases}$$

It follows easily that for $0 \leqq r \leqq 2k+2$

$$\lim_{s\to\infty} t_s(\{x_n^{(m,k+1)}\}, g_r^{(m+1)}) = \sum_{j=k+2}^{2^m} S(T_j^{(m)}) + S(T_{2k+2}^{*(m+1)}) + \sum_{j=r}^{2k+1} S(T_j^{(m+1)}) =$$
$$= \sum_{j=k+2}^{2^m} p_j^{(m)} + p_{2k+2}^{*(m+1)} + \sum_{j=r}^{2k+1} p_j^{(m+1)} = q_r^{(m+1)} = Lg_r^{(m+1)}$$

and similarly for $k+2 \leqq r \leqq 2^m$

$$\lim_{s\to\infty} t_s(\{x_n^{(m,k+1)}\}, g_r^{(m)}) = Lg_r^{(m)}.$$

Therefore, there exists a constant $N^{(m,k+1)} > N^{(m,k)}$ such that for $s \geqq N^{(m,k+1)}$ we have

$$|t_s(\{x_n^{(m,k+1)}\}, \psi) - L\psi| < 1/2^m$$

where ψ denotes any of the functions

$$g_r^{(m+1)}, \quad 0 \leqq r \leqq 2k+2, \quad g_r^{(m)}, \quad k+2 \leqq r \leqq 2^m.$$

(If we adjoin the omitted columns to $T_{2k+2}^{*(m+1)}$ we obtain again a completely regular decomposition of T and the prescribed properties hold for the pair $(m, k+1)$.) Thus, for all m and $0 \leqq k \leqq 2^m$ we can define the sequences $\{x_n^{(m,k)}\}_n$ which have also the property that for $0 \leqq n \leqq N^{(m,k)}$ and for $m' > m$ or $m' = m$ and $k' > k$, $x_n^{(m,k)}$ coincides with $x_n^{(m',k')}$, i.e. $x_n^{(m,k)} = x_n^{(m',k')}$. Hence the limit

$$x_n = \lim_{N^{(m,k)} \to \infty} x_n^{(m,k)}$$

exists for every n and $x_n \in [0, 1]$. We show that

$$T\text{-}\lim_n g_k^{(m)}(x_n) = Lg_k^{(m)}$$

for every m and k which already implies

$$T\text{-}\lim_n f(x_n) = Lf$$

for every $f \in C[0, 1]$ because the linear combinations of the $g_k^{(m)}$'s constitute a dense set in $C[0, 1]$.

Let

$$I_1^{(m,k)} = \{n \mid n \in \operatorname{ind} T_{2k}^{*(m+1)}, \ N^{(m,k)} \leqq n \leqq N^{(m,k+1)}\},$$

$$I_2^{(m,k)} = \{n \mid n \in \operatorname{ind} T_{k+1}^{(m)}, \ N^{(m,k)} \leqq n \leqq N^{(m,k+1)}\},$$

and

$$K_1^{(m,k)} = \max_{N^{(m,k)} \leqq s \leqq N^{(m,k+1)}} \sum_{j \in I_1^{(m,k)}} t_{s,j},$$

$$K_2^{(m,k)} = \max_{N^{(m,k)} \leqq s \leqq N^{(m,k+1)}} \sum_{j \in I_2^{(m,k)}} t_{s,j}.$$

We claim that $K_1^{(m,k)} \to 0$, $K_2^{(m,k)} \to 0$ as $N^{(m,k)} \to \infty$. Suppose not, e.g. $K_1^{(m,k)} \geqq \geqq \varepsilon > 0$ for infinitely many pairs (m, k). For each such (m, k) we have by our construction

$$|t_s(\{x_n^{(m,k)}\}, \ g_{2k}^{(m+1)}) - t_s(\{x_n^{(m,k)}\}, \ g_{k+1}^{(m)})| \geqq \varepsilon,$$

which together with the estimates

$$|t_s(\{x_n^{(m,k)}\}, \ g_{2k}^{(m+1)}) - Lg_{2k}^{(m+1)}| < 1/2^m,$$

$$|t_s(\{x_n^{(m,k)}\}, \ g_{k+1}^{(m)}) - Lg_{k+1}^{(m)}| < 1/2^m \quad N^{(m,k)} \leqq s \leqq N^{(m,k+1)}$$

yield for infinitely many (m, k)

$$|Lg_{2k}^{(m)} - Lg_{2k+1}^{(m)}| > \varepsilon - 2/2^m,$$

but this contradicts Lemma 3 (γ) (L is assumed to have no singular point).

Now if $|f| \leqq 1$ is *arbitrary* then for $N^{(m,k)} < s \leqq N^{(m,k+1)}$

$$|t_s(\{x_n\}, f) - t_s(\{x_n^{(m,k)}\}, f)| \leqq 2(K_1^{(m,k)} + K_2^{(m,k)})$$

and so, according to what we have just proved, to every $\varepsilon > 0$ there exists an N such that if $s \geqq N$ then

$$|t_s(\{x_n\}, \varphi) - L\varphi| < \varepsilon$$

for an arbitrary function φ from (4.2) with (m, k) satisfying $N^{(m,k)} < s \leqq N^{(m,k+1)}$. But for $m_1 < m$ every one of the $g_k^{(m_1)}$'s is a convex linear combination of such φ's by which

$$\lim_{s \to \infty} t_s((\{x_n\}, g_k^{(m)}) = Lg_k^{(m)}$$

for every m and $0 \leqq k \leqq 2^m$, and the proof is complete.

Now let L be discrete:

$$Lf = \sum_i \mu_i f(\tau_i), \quad \mu_i \geqq 0, \quad \sum_i \mu_i = 1.$$

By assumption we have matrices T_i with $T = \bigcup_i T_i$, $S(T_i) = \mu_i$, hence putting $x_n = \tau_i$ if $n \in \operatorname{ind} T_i$ we obtain

$$Lf = T\text{-}\lim_n f(x_n).$$

Finally, if $L = \mu L_1 + (1-\mu) L_2$, $0 < \mu < 1$ where L_1 is discrete and L_2 is without singular points then there are completely regular matrices T_1, T_2 such that $T = T_1 \cup T_2$, $S(T_1) = \mu$, $S(T_2) = 1 - \mu$. Above we proved that there are sequences $\{x_k^{(1)}\}$ and $\{x_k^{(2)}\}$ with

$$((1/\mu) T_1)\text{-}\lim_k f(x_k^{(1)}) = L_1 f, \quad ((1/(1-\mu) T_2)\text{-}\lim_k f(x_k^{(2)}) = L_2 f,$$

and hence putting

$$x_n = x_k^{(j)} \quad \text{if the } n\text{-th column of } T \text{ is the } k\text{-th column of } T_j;\ j = 1 \text{ or } 2,$$

we obtain

$$T\text{-}\lim_n f(x_n) = T_1\text{-}\lim_k f(x_k^{(1)}) + T_2\text{-}\lim_k f(x_k^{(2)}) = \mu L_1 f + (1-\mu) L_2 f = Lf$$

and the proof of the implication (iii)$\Rightarrow$(i) is complete.

Finally we prove that (ii) implies (iii). Let

$$\int_0^1 f = T\text{-}\lim_n f(x_n)$$

for all $f \in C[0, 1]$. Putting $f \equiv 1$ we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} = T\text{-}\lim 1 = \int_0^1 1 = 1.$$

Thus, T is regular if $\lim_{n\to\infty} t_{nk} = 0$ is also satisfied for every n, but this obviously follows from the complete regularity of T which we show in a moment.

If $I \subseteqq [0, 1]$ is an interval let $|I|$ denote its length, and let the matrix T_I be determined by

$$\operatorname{ind} T_I = \{n \mid x_n \in I\}.$$

We claim that $S(T_I) = |I|$ which already implies the complete regularity of T. For any functions $f_1, f_2 \in C[0, 1]$ satisfying $0 \leqq f_1, f_2 \leqq 1$, $f_1(x) = 0$ for $x \notin I$, $f_2(x) = 1$ for $x \in I$ we have, if l_n denotes the sum of the elements of T_I in the n-th row,

$$\int_0^1 f_1 \leqq \liminf_{n\to\infty} l_n \leqq \limsup_{n\to\infty} l_n \leqq \int_0^1 f_2$$

and since to every $\varepsilon > 0$ there are functions f_1 and f_2 of the above kind satisfying

$$|I| - \varepsilon \leqq \int_0^1 f_1 \leqq |I| \leqq \int_0^1 f_2 \leqq |I| + \varepsilon$$

we have indeed

$$S(T_I) = \lim_{n\to\infty} l_n = |I|.$$

The proof is complete.

Corollary 3 can be proved easily using Theorem 4.

5. Density measures

Let $X = \{x_k\}_{k=1}^{\infty}$ be a sequence from the n dimensional Euclidean space. We say that X has density α in the set $A \subseteqq \mathbf{R}^n$ if the number of the first n points belonging to A divided by n tends to α i.e. if

$$|\{k \mid 1 \leqq k \leqq n, x_k \in A\}|/n = (1/n) \sum_{k=1}^{n} \chi_A(x_k) \to \alpha \quad (n \to \infty).$$

Let J^n be the ring of the Jordan measurable sets of $\mathbf{R}^n$. A set function $\mu\colon J^n \to \mathbf{R}_+$ is said to be density measure if there is a sequence $X = \{x_k\}_{k=1}^{\infty}$ such that X has density $\mu(A)$ in every Jordan measurable set A. In this section we characterize these density measures. It will follow among others that they are really measures, i.e. they are countably additive and they can be extended to a Borel measure.

Theorem 5. *A* $\mu\colon J^n\to\mathbf{R}_+$ *is a density measure if and only if it has the form*

$$\mu(A) = \sum_{\tau_i\in A}\mu_i+\Big(1-\sum_{i=1}^{\infty}\mu_i\Big)\int_A g \tag{5.1}$$

with suitable $\mu_i\geqq 0, \tau_i\in\mathbf{R}^n$ $(i=1,2,\dots), g\geqq 0, g\in L^1(\mathbf{R}^n)$ *satisfying* $\sum_{i=1}^{\infty}\mu_i\leqq 1$, *and* $\int_{\mathbf{R}^n} g=1$.

In other words the density measures are the convex combinations of the discrete and (with respect to the Lebesgue measure of $\mathbf{R}^n$) absolutely continuous measures.

Corollary 4. *Every density measure is σ-additive and hence it can be extended to a Borel measure of* $\mathbf{R}^n$.

Remark 1. One could try to extend the notion of the density measure to other domains than the Jordan measurable sets but such extensions can result in that the only "density measures" are the discrete ones. This happens e.g. if we require μ to be defined for all open (and hence for all closed) sets of $\mathbf{R}^n$.

2. One could also use other summability methods, i.e. μ can be defined as

$$\mu(A) = T\text{-}\lim_k \chi_A(x_k)$$

for some $X=\{x_k\}\subseteq\mathbf{R}^n$ and a non-negative regular matrix T. A similar argument that will follow proves that Theorem 5 holds word for word for this "modified density measure".

Proof. We shall only sketch the proof. The details are very similar to those of Section 3.

I. Necessity. Let $X=\{x_k\}$ be the sequence generating μ. A standard argument yields that the number of those points x for which $\mu(\{x\})>0$ is countable. Let these be $\tau_1, \tau_2, \dots$ and let $\mu_i=\mu(\{\tau_i\})$. If N_i is that subsequence of the natural numbers N for which $k\in N_i$ iff $x_k=\tau_i$ then N_i has density μ_i in N. N_i contains a subsequence N_i' such that N_i' has density μ_i in N, too, but it is sufficiently sparse, namely if $N_i'=\{n_1^{(i)}, n_2^{(i)}, \dots\}$ then $k/n_k^{(i)}\leqq\mu_i$ is satisfied for every k. One can easily verify that if $X'=\{x_j'\}$ is that subsequence of X for which $x_k\in X'$ iff $k\notin\bigcup_{i=1}^{\infty}N_i'$ then X' satisfies the property that if $\mu^*=\sum_{i=1}^{\infty}\mu_i<1$ then X' has density

$$\mu_1(A) = \big(1/(1-\mu^*)\big)\Big(\mu(A)-\sum_{\tau_i\in A}\mu_i\Big)$$

in every set $A\in J^n$ and hence

$$\mu(A) = \sum_{\tau_i\in A}\mu_i+\Big(1-\sum_{i=1}^{\infty}\mu_i\Big)\mu_1(A).$$

If $\sum_{i=1}^{\infty} \mu_i = 1$ then, clearly,

$$\mu(A) = \sum_{r_i \in A} \mu_i$$

and our task has been reduced to show that there is a function $g \in L^1(\mathbf{R}^n)$, $g \geqq 0$, $\int_{\mathbf{R}^n} g = 1$ such that

$$\mu_1(A) = \int_A g$$

is satisfied.

Let us consider the following mapping φ: $\mathbf{R}^1 \to \mathbf{R}^n$: if $x \in \mathbf{R}^1$ has binary expansion

$$x = \ldots \alpha_{-2}\alpha_{-1}\alpha_0,\ \alpha_1\alpha_2\alpha_3\ldots$$

where infinitely many of the $\alpha_1, \alpha_2, \ldots$ vanish then putting

$$x^j = \ldots \alpha_{j-n},\ \alpha_j \alpha_{j+n} \ldots, \quad j = 1, 2, \ldots n$$

let

$$\varphi(x) = (x^1, \ldots, x^n).$$

Then $\varphi(\mathbf{R}^1) = \mathbf{R}^n$ and for each $x \in \mathbf{R}^n$, $\varphi^{-1}(x)$ consists of at most 2^n points. $\varphi([m/2^k, (m+1)/2^k])$ is a closed rectangular parallelepiped without one vertex with volume $1/2^k$ and

$$\varphi^{-1}(\{(x_1, \ldots, x_n) \mid m_1/2^k \leqq x_1 < (m_1+1)/2^k, \ldots, m_n/2^k \leqq x_n < (m_n+1)/2^k\})$$

is an interval of length $1/2^{kn}$ plus a nowhere dense set with zero Lebesgue (and hence Jordan) measure. It follows that $\varphi(A) \in J^n$ for every $A \in J^1$ and $\varphi^{-1}(A) \in J^1$ for every $A \in J^n$, furthermore, φ is a measure preserving transformation (cf. also [2, pp. 81—83]). Let x^* be an element of $\varphi^{-1}(x)$ and let $x_j^* = (x_j')^*$ i.e. $x_j^* \in \varphi^{-1}(x_j')$. We claim that $X^* = \{x_j^*\}_{j=1}^{\infty}$ has density in every Jordan measurable set of $\mathbf{R}^1$.

Let $A \in J^1$. If $x_j^* \in A$ then $x_j' \in \varphi(A)$. Conversely, if $x_j' \in \varphi(A)$ but $x_j^* \notin A$ then $x_j^* \in \varphi^{-1}(\varphi(A)) \setminus A =: B$. By what we have said above B has Jordan measure 0 and hence $\varphi(B)$ also has Jordan measure 0. But then every $B' \subseteqq \varphi(B)$ is Jordan measurable and X' has density $\mu_1(B')$ in every such B', furthermore, X' has density 0 in every point which imply, by an argument similar to that used in the proof of Lemma 2, that X' has density 0 in $\varphi(B)$. By this X^* has density 0 in $\varphi^{-1}(\varphi(B)) \supseteqq B$ and we obtain that for all j but a sequence j_n with $j_n/n \to \infty$ the conditions $x_j^* \in A$ and $x_j' \in \varphi(A)$ are equivalent, which proves our assertion.

It follows also that X^* has density $\mu^*(A) := \mu_1(\varphi(A))$ in a set $A \in J^1$. Since $\mu_1(\{\varphi(x)\}) = 0$ for every x, the consideration of Lemma 2 yields that the function

$$\alpha(x) = \mu^*((-\infty, x]) = \mu_1(\varphi(-\infty, x]) = \{\text{the density of } X^* \text{ in } (-\infty, x]\}$$

is absolutely continuous and increasing, furthermore $\alpha(-\infty)=0$, $\alpha(\infty)=1$. If

$$g(x)=\alpha'(x^*)\quad (x\in\mathbf{R}^n)$$

then

$$\mu_1(A)=\mu^*(\varphi^{-1}(A))=\int_{\varphi^{-1}(A)}\alpha'=\int_A g\quad (A\in J^n)$$

because φ is measure preserving, and the proof is complete.

II. Sufficiency. Let μ have the form (5.1). We put

$$Lf=\sum_i \mu_i f(\tau_i)+\Big(1-\sum_i \mu_i\Big)\int fg$$

for every Riemann-integrable function f. Either by the method of Section 3 or by using the above transformation φ and applying Theorem 3 one can prove that there is a sequence $X=\{x_n\}\subseteq\mathbf{R}^n$ with

$$Lf=\lim_{n\to\infty}(1/n)\big(f(x_1)+\ldots+f(x_n)\big)$$

and an application of this to the characteristic function χ_A of A yields the desired representation.

We have completed our proof.

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