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Approximation by modified Szász operators

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1. Introduction

I. J. L. DURRMEYER [5] defined the approximation process

$$D_n f(x) = \sum_{k=0}^{n} \Big(\int_0^1 b_{nk}(t) f(t)\, dt \Big) b_{nk}(x), \quad b_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

which can be used for restoring f if its moments $\int_0^1 f(t) t^k\, dt$ are given. In a recent paper M. M. DERRIENNIC [3] proved several results concerning these operators that have certain analogues with the corresponding results for Bernstein polynomials from which the operators D_n originate.

Now we shall similarly modify the Szász—Mirakian operators [7, 8]

$$S_n(f;\ x) = \sum_{k=0}^{\infty} f(k/n) p_{n,k}(x), \quad p_{n,k}(x) = e^{-nx}(nx)^k/k! \quad (x \geqq 0)$$

and prove exact estimates and saturation results for the modified operator. Actually, we have two modifications in mind:

$$L_n(f;\ x) = f(0) p_{n,0}(x) + n \sum_{k=1}^{\infty} \Big(\int_0^{\infty} f(t) p_{n,k-1}(t)\, dt \Big) p_{n,k}(x)$$

and

$$\mathrm{L}_n(f;\ x) = n \sum_{k=0}^{\infty} \Big(\int_0^{\infty} f(t) p_{n,k}(t)\, dt \Big) p_{n,k}(x).$$

Clearly L_n is the perfect analogue of D_n but it will turn out that L_n has much nicer properties than L_n; the main difference between them is that L_n reproduces every linear function while L_n reproduces only the constant ones.

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Since we are interested in uniform approximation and the transforms $L_n(f)$ and $\mathbb{L}_n(f)$ are continuous on $[0, \infty)$ (if they exist at all), in what follows we assume the continuity of f on $[0, \infty)$. Besides, when treating uniform approximation we shall always assume the boundedness of f as well.

We shall prove global results, i.e. the whole interval $[0, \infty)$ will be considered, but because of the strong localization valid both for L_n and $\mathbb{L}_n$, these also solve the corresponding local problems.

In what follows let $\|\cdot\|$ denote the supremum norm and $\varphi(x)=\sqrt{x}$.

2. Weighted estimates

I. Results. Let

$$\Delta_h^2(f;\ x) = f(x-h)-2f(x)+f(x+h) \quad (x \geqq h)$$

be the usual symmetric second difference of f and

$$\omega^2(f;\ \delta) = \sup_{0\leqq h\leqq \delta,\, x\geqq h} |\Delta_h^2(f;\ x)|$$

the modulus of smoothness of f.

Theorem 1. *For every function $f\in C[0, \infty)$ we have*

$$|L_n(f;\ x)-f(x)| \leqq 11\omega^2(f;\ \sqrt{x/n}). \tag{2.1}$$

Corollary. *Let $f\in C[0, \infty)$ be bounded. Then there exists a non-negative and continuous function ψ: $[0, \infty)\to R_+$, $\psi(0)=0$, such that*

$$|L_n(f;\ x)-f(x)| \leqq K\psi(x/n) \quad (x \geqq 0, \quad n = 1, 2, \dots)$$

holds if and only if f is uniformly continuous on $[0, \infty)$.

The proof of Theorem 1 follows that of [1, Theorem 8] and it gives somewhat more: if $\{L_n\}$ is an arbitrary sequence of positive operators reproducing the linear functions and $(1/2)L_n((t-x)^2; x)=\mu_n^2(x)$, then

$$|L_n(f;\ x)-f(x)| \leqq 11\omega^2(f;\ \mu_n(x)) \quad (x \geqq 0).$$

The saturation result is as follows:

Theorem 2. *Let $f\in C[0, \infty)$, $f(x)=O(e^{Ax})$ $(A>0)$. If $L_n(f; x)-f(x)=o_x(x/n)$ $(x\geqq 0,\ n\to\infty)$, then f is a linear function; furthermore*

$$|L_n(f;\ x)-f(x)| \leqq Kx/n \quad (x \geqq 0,\ n = 1, 2, \dots)$$

holds if and only if f has a derivative belonging to Lip 1, *where*

$$\text{Lip } 1=\{f:\ |f(x+h)-f(x)|\leqq K_f h,\ x\geqq 0,\ h>0\}.$$

The next result solves the so-called non-optimal approximation problem. Let

$$\mathrm{Lip}^2\alpha = \{f\in C[0,\infty):\ \omega^2(f;\ \delta) \leqq K_f\delta^\alpha,\quad \delta>0\}.$$

With this notation we have:

Theorem 3. *Let $f\in C[0,\infty)$ be bounded. Then with $0<\alpha<1$,*

$$|L_n(f;\ x)-f(x)| \leqq K(x/n)^\alpha \quad (x\geqq 0,\ n=1,2,\ldots)$$

holds if and only if $f\in\mathrm{Lip}^2\,2\alpha$.

For L_n the situation is much more complex. We mention only that if $\omega^1(f;\delta)$ denotes the ordinary modulus of continuity of f, then

$$|\mathsf{L}_n(f;\ x)-f(x)| \leqq K\omega^1(f;\ \sqrt{(x+1/n)/n}) \quad (x>0,\ n=1,2,\ldots)$$

follows from a result of SHISHA and MOND [4, p. 28] and by well known properties of ω^1 this implies that

$$(2.2)\qquad (1/(1+\sqrt{x}))|\mathsf{L}_n(f;\ x)-f(x)| \leqq K\omega^1(f;\ 1/\sqrt{n}) \quad (x>0,\quad n=1,2,\ldots).$$

It can be shown that in general (2.2) cannot be improved since neither the weight $\{1+\sqrt{x}\}^{-1}$ nor the rate $\omega^1(f;\ 1/\sqrt{n})$ can be replaced by a smaller quantity. An analogue of Theorem 1 holds for L_n as well. However it is far less obvious how the analogues of Theorems 2—3 look like in the case of L_n.

II. Proofs. Proof of Theorem 1. We follow the argument of [1, Theorem 8]. Using the relations

$$\int_0^\infty p_{n,k}(x)\,dx = 1/n,\quad (k/n)p_{n,k}(x) = xp_{n,k-1}(x)\quad (n=1,2,\ldots,k\geqq 0)$$

one can easily verify that $L_n(g;x)\equiv g(x)$ for every linear function g, furthermore

$$(2.3)\qquad L_n((t-x)^2, x) = x/n \quad (x>0,\quad n=1,2,\ldots).$$

First let us suppose that f is twice continuously differentiable on $[0,\infty)$. By Taylor's formula

$$(2.4)\qquad \begin{aligned} f(t) &= f(x)+(t-x)f'(x)+\int_x^t\int_x^s f''(u)\,du\,ds = \\ &= f(x)+(t-x)f'(x)+\int_x^t (t-\tau)f''(\tau)\,d\tau. \end{aligned}$$

Here

$$\left|\int_x^t\int_x^s f''(u)\,du\,ds\right| \leqq \|f''\|(t-x)^2/2,$$

hence

(2.5) $$|L_n(f;\ x)-f(x)| \leqq \|f''\| L_n((t-x)^2/2;\ x) = \|f''\|\, x/n.$$

Now let

(2.6) $$f_h(x) = (2/h)^2 \int_0^{h/2}\int_0^{h/2} [2f(x+s+t)-f(x+2(s+t))]\, ds\, dt.$$

By [1, (21)], we have $\|f-f_h\| \leqq \omega^2(f; h)$, $\|f_h''\| \leqq 9h^{-2}\omega^2(f; h)$ and so using (2.5) for f_h, we obtain

$$|L_n(f;\ x)-f(x)| \leqq |L_n(f-f_h;\ x)|+|(f-f_h)(x)|+|L_n(f_h;\ x)-f_h(x)| =$$
$$\leqq 2\omega^2(f;\ h)+9h^{-2}\omega^2(f, h)x/n,$$

and putting here $h=\sqrt{x/n}$ we obtain (2.1).

Proof of Corollary. We have to prove only the necessity part. Let $x \geqq 1$ and $n=M[x]$. Exactly as in the proof of Theorem 4 below we have $|L_n'(f; x)| \leqq \leqq \|f\|\sqrt{n/x}$ and so if we assume

$$|L_n(f;\ x)-f(x)| \leqq K\psi(\sqrt{x/n})$$

we obtain that

$$|f(x)-f(x+h)| \leqq 2K\psi(\sqrt{x+h}/\sqrt{M[x]})+|L_n(f;\ x+h)-L_n(f;\ x)| \leqq$$
$$\leqq 2K\psi(3/\sqrt{M})+h\|f\|\sqrt{M}$$

which can be made arbitrarily small by choosing first a large M and then sufficiently small h.

Proof of Theorem 2. First we prove the following strong localization result: if $f(x)=O(e^{Ax})$ and $f(x)$ vanishes on an interval $(a-\varepsilon, b+\varepsilon)$ $(\varepsilon>0)$, then $L_n(f; x)= =O(n^{-2})$ uniformly on $[a, b]$. In fact one can easily see that if A is an integer, then

$$e^{At}p_{n,k}(t) \leqq Ke^{A_1k/n}p_{n-A,k}(t)$$

with a constant A_1 and so

$$|L_n(f;\ x)| \leqq Kn\sum_{k=0}^{\infty} e^{A_1k/n}p_{n,k}(x)\Big(\int_0^{a-\varepsilon}+\int_{b+\varepsilon}^{\infty}\Big)p_{n-A,k}(t)\,dt = I_1+I_2.$$

$$I_2 \leqq nK\Big(\sum_{k\leqq(b+\varepsilon/2)n} p_{n,k}(x)\int_{b+\varepsilon}^{\infty} p_{n-A,k}(t)\,dt+\sum_{k>(b+\varepsilon/2)n} p_{n,k}(x)e^{A_1k/n}\int_0^{\infty} p_{n-A,k}(t)\,dt\Big) \leqq$$
$$\leqq \frac{K}{n^2}\sum_{k\leqq(b+\varepsilon/2)n} p_{n,k}(x)+K\sum_{k>(b+\varepsilon/2)n} p_{n,k}(b)e^{A_1k/n}n \leqq$$
$$\leqq Kn^{-2}+Knp_{n,[(b+\varepsilon/2)n]}(b)\exp[A_1(b+\varepsilon/2)] \leqq Kn^{-2},$$

where to obtain the second inequality we used the estimate (see [6, p. 212])

$$\int_{|u-k|>\eta n} (e^{-ku}u^k/k!)\,du \leqq Kk^3/(\eta n)^6,$$

for the second and third ones we used the facts that $p_{n,k}(x)$ increases on the interval $(0, k/n)$ and

$$p_{n,k+1}(b)/p_{n,k}(b) \leqq (b/(b+\varepsilon/2))\exp(A_1/n) \leqq 1-\varepsilon/2b$$

for sufficiently large n and $k>(b+\varepsilon/2)n$, and to obtain the last one we used Stirling's formula.

One can get similar estimates for I_1 and thus our assertion concerning the localization of L_n has been proved. For later applications let us mention that in view of our proof, the same holds true for the operators L_n.

Using the above localization, one can see that the proofs of [4, Theorem 5.1] and [4, Theorem 5.4] hold for L_n on every finite interval $[a, b]\subseteq[0, \infty)$ and we obtain that

$$L_n(f, x)-f(x) = o_x(1)$$

implies that f is locally and hence globally linear, furthermore

$$L_n(f;\ x)-f(x) = O(x/n)$$

implies that f has a derivative which is absolutely continuous on every interval $(a, b)\subseteq[0, \infty)$. But using again our localization, the proof of [4, Lemma 5.5] yields that

$$\lim_{n\to\infty}(n/x)(L_n(f;\ x)-f(x)) = f''(x)/2$$

at every point x, where $f''(x)$ exists. So

$$L_n(f;\ x)-f(x) = O(x/n) \quad \text{implies} \quad f''(x) = O(1)$$

and this is the same as $f'\in \mathrm{Lip}\,1$.

The sufficiency of the condition $f'\in\mathrm{Lip}\,1$ follows from Theorem 1 since it implies $\omega^2(f;\delta)=O(\delta^2)$.

Proof of Theorem 3. Here, again, the sufficiency directly follows from Theorem 1.

To prove the necessity of the condition $f\in\mathrm{Lip}^2\,2\alpha$, first we verify the inequality

(2.7) $$|L_n''(f, x)| \leqq K\omega^2(f;\ \delta)(n/x+\delta^{-2}) \quad (\delta, x>0,\ n=1, 2, \ldots)$$

(cf. also [1, (23)]). Let

(2.8) $$F_{n0}(f) = f(0),\quad F_{nk}(f) = n\int_0^\infty f(t)p_{n,k-1}(t)\,dt \quad (k=1, 2, \ldots).$$

Since

$$p_{n,k}'(x) = n(p_{n,k-1}(x)-p_{n,k}(x)) = (k/x-n)p_{n,k}(x),$$

simple computation gives that with $\gamma_{n,k}=(k/n-x)^2-k/n^2$,

$$L_n''(f;\ x)=(n/x)^2\sum_{k=0}^{\infty}\gamma_{n,k}(x)F_{n,k}(f)p_{n,k}(x)=\tag{2.9}$$
$$=n^2\sum_{k=0}^{\infty}\big(F_{n,k}(f)-2F_{n,k+1}(f)+F_{n,k+2}(f)\big)p_{n,k}(x)$$

and so

$$|L_n''(f;\ x)|\leqq(n/x)^2\sum_{k=0}^{\infty}|\gamma_{n,k}(x)||F_{n,k}(f)-F_{n,k}(f_\delta)|p_{n,k}(x)+$$
$$+n^2\sum_{k=0}^{\infty}|F_{n,k}(f_\delta)-2F_{n,k+1}(f_\delta)+F_{n,k+2}(f_\delta)|p_{n,k}(x)=I_1+I_2,$$

where the function f_δ is the same as defined in (2.6).

Since

$$|F_{n,k}(f)-F_{n,k}(f_\delta)|\leqq F_{n,k}(|f-f_\delta|)\leqq\|f-f_\delta\|\leqq\omega^2(f;\delta)$$

we have by [1, (27)]

$$I_1\leqq\omega^2(f;\ \delta)(n/x)^2\sum_{k=0}^{\infty}|\gamma_{n,k}(x)|p_{n,k}(x)\leqq K\omega^2(f;\ \delta)\,n/x.$$

In I_2 we use the Taylor expansion (2.4) for f_δ. Let first $k\geqq 1$. Then integrations by parts give that

$$n^2|F_{n,k}(f_\delta)-2F_{n,k+1}(f_\delta)+F_{n,k+2}(f_\delta)|=$$
$$=n\left|\int_0^\infty n^2p_{n,k-1}(t)\left[1-\frac{2nt}{k}+\frac{(nt)^2}{k(k+1)}\right]\Big(\int_x^t\int_x^s f_\delta''(u)\,du\,ds\Big)dt\right|=$$
$$=n\left|\int_0^\infty(p_{n,k+1}(t))''\Big(\int_x^t\int_x^s f_\delta''(u)\,du\,ds\Big)dt\right|=n\left|\int_0^\infty p_{n,k+1}(t)f_\delta''(t)\,dt\right|\leqq$$
$$\leqq\|f_\delta''\|\leqq 9\delta^{-2}\omega^2(f;\ \delta).$$

A direct but rather long computation verifies the identity

$$f_\delta(0)-2n\int_0^\infty e^{-nt}f_\delta(t)\,dt+n\int_0^\infty e^{-nt}(nt)f_\delta(t)\,dt=\int_0^\infty te^{-nt}f_\delta''(t)\,dt,\tag{2.10}$$

by which

$$n^2|F_{n,0}(f_\delta)-2F_{n,1}(f_\delta)+F_{n,2}(f_\delta)|\leqq\|f_\delta''\|\,n\int_0^\infty(nt)e^{-nt}\,dt=\|f_\delta''\|\leqq 9\delta^{-2}\omega^2(f;\ \delta).$$

Collecting our estimates we obtain (2.7).

Now using (2.7) and $|f(x)-L_n(f;x)|=O\big((x/n)^\alpha\big)$ we get

$$|\Delta_h^2(f;x)| \leqq |f(x)-L_n(f;x)|+2|f(x+h)-L_n(f;x+h)|+$$

$$+|f(x+2h)-L_n(f;x+2h)|+\Big|\int_0^h\int_0^h L_n''(f,x+s+t)\,ds\,dt\Big| \leqq$$

$$\leqq K\big((x/n)^\alpha+h^2(n/x+\delta^{-2})\omega^2(f;\delta)\big).$$

Putting here $\delta=\sqrt{x/n}$ we obtain

$$\omega^2(f;h) \leqq K(\delta^{2\alpha}+(h/\delta)^2\omega^2(f;\delta)) \quad (h,\delta>0)$$

and it is well-known (see [2, Lemma]) that this and the boundedness of f imply $\omega^2(f;h)=O(h^{2\alpha})$, i.e. $f\in\mathrm{Lip}^2\,2\alpha$.

Our proofs are complete.

3. Uniform approximation

I. Results. Let C_B be the set of bounded and continuous functions defined on $[0,\infty)$. A very natural question is the following: For which functions f do the transforms $L_n(f)$ $(\mathsf{L}_n(f))$ approximate f *uniformly* on the whole interval $[0,\infty)$? The answer is given by

Theorem 4. *If $f\in C_B$, then $L_n(f)-f=o(1)$ $(n\to\infty)$ is satisfied uniformly on $[0,\infty)$ iff $f(x^2)$ is uniformly continuous on $[0,\infty)$.*

The saturation result sounds as

Theorem 5. *Let $f\in C_B$. Then $L_n(f)-f=o(1/n)$ $(n\to\infty)$ uniformly on $[0,\infty)$ iff f is constant, furthermore $L_n(f)-f=O(1/n)$ iff f has a locally absolutely continuous derivative with $|xf''(x)|\leqq K_f$ $(x>0)$.*

Finally concerning the non-optimal approximation we have

Theorem 6. *Let $f\in C_B$ and $0<\alpha<1$. Then $L_n(f)-f=O(n^{-\alpha})$ holds uniformly on $[0,\infty)$ iff*

$$x^\alpha|\Delta_h^2(f;x)| \leqq Kh^{2\alpha} \quad (x>h,\ h>0) \tag{3.1}$$

is satisfied.

The analogues of Theorems 4 and 5 for L_n are:

Theorem 7. *If $f\in C_B$, then $\mathsf{L}_n(f)-f=o(1)$ $(n\to\infty)$ holds uniformly on $[0,\infty)$ iff $f(x^2)$ is uniformly continuous on $[0,\infty)$.*

Theorem 8. *Let* $f\in C_B$. *Then* $\mathrm{L}_n(f)-f=o(1/n)$ $(n\to\infty)$ *iff* f *is constant, furthermore* $\mathrm{L}_n(f)-f=O(1/n)$ *iff* f *has a continuous derivative with* $xf'(x)\in \mathrm{Lip}\,1$.

We can see that $\{L_n\}$ and $\{\mathrm{L}_n\}$ do not differ from the point of view of uniform approximation but they do differ from the point of view of global saturation, e.g. if $f\in C_B$ is twice continuously differentiable on $(0, \infty)$ and

$$f(x)=\begin{cases} x\log x & x\in[0,1] \\ 0 & x>2,\end{cases}$$

then f belongs to the saturation class of $\{L_n\}$ but not to that of $\{\mathrm{L}_n\}$ (it is easy to see that the former includes the latter).

In the proofs of Theorems 4—6 we shall use the general results of [9]. Theorems 7 and 8 do not come so easily, they require special considerations.

II. Proofs. Proof of Theorem 4. Since L_n reproduces the linear functions and satisfies (2.3) we can apply [9, Theorem 2] and the remark made after it, according to which $L_n(f)-f=o(1)$ and the uniform continuity of

$$f\circ g^{-1}(x)=f(x^2)\quad \Big(g(x)=(1/4)\int_0^x(1/\varphi(t))\,dt,\quad \varphi(x)=\sqrt{x}\Big)$$

are equivalent provided we can verify that

$$L_n'(f;\,x)\leqq K_n/\varphi(x)\quad \big(x>0,\quad \varphi(x)=\sqrt{x}\big).$$

With the notation (2.8) we get by an application of the Schwartz inequality

$$|L_n'(f;\,x)|=(n/x)\Big|\sum_{k=0}^{\infty}(k/n-x)F_{n,k}(f)p_{n,k}(x)\Big|\leqq(n/x)\,\|f\|\sum_{k=0}^{\infty}|k/n-x|\,p_{n,k}(x)\leqq$$

$$\leqq(n/x)\,\|f\|\Big(\sum_{k=0}^{\infty}(k/n-x)^2p_{n,k}(x)\Big)^{1/2}=\|f\|\sqrt{n}/\sqrt{x},$$

where at the last step we used the fact that $\sum_{k=0}^{\infty}(x-k/n)^2p_{n,k}(x)=x/n$. This proves the theorem.

Proof of Theorem 5. The first part of the statement of the theorem follows from Theorem 2 and the boundedness of f. Let $\varepsilon>0$ and

$$h_{x,\varepsilon}(t)=\begin{cases}(t-x)^2 & \text{for}\quad |t-x|\geqq\varepsilon,\\ 0 & \text{for}\quad |t-x|<\varepsilon.\end{cases}$$

By [9, Proposition 1] the second part will also follow if

$$L_n(h_{x,\varepsilon};\,x)=o_{x,\varepsilon}(1/n)\quad (n\to\infty)$$

is satisfied for every $x>0$ and $\varepsilon>0$. Since a stronger result was proved in the proof of Theorem 2, the proof is over.

Proof of Theorem 6. Let

$$\omega_\varphi(f;\ \delta) = \sup_{0\leqq h\leqq\delta,\, x>0} |\Delta^2_{h\varphi(x)}(f;\ x)| \quad (\varphi(x) = \sqrt{x})$$

be the modified modulus of smoothness of f. Putting $h=\delta\sqrt{x}$ into (3.1) we can see that (3.1) is equivalent of $\omega_\varphi(f;\delta)=O(\delta^{2\alpha})$. Now by [9, Proposition 2, Corollary] the theorem will be proved if we verify

$$|xL_n''(f;\ x)| \leqq Kn\|f\| \quad (f\in C_B), \tag{3.2}$$

and for any $f\in C_B$ having an absolutely continuous derivative

$$|xL_n''(f;\ x)| \leqq K\|\varphi^2 f''\|. \tag{3.3}$$

The proof of (3.2) is easy. Using (2.9) we have

$$|xL_n''(f;\ x)| \leqq (n^2/x) \sum_{k=0}^{\infty} |F_{n,k}(f)|\,|\gamma_{n,k}(x)|\, p_{n,k}(x) \leqq$$

$$\leqq n\|f\| \Big((n/x) \sum_{k=0}^{\infty} (k/n-x)^2 p_{n,k}(x) + (n/x) \sum_{k=0}^{\infty} (k/n^2) p_{n,k}(x)\Big) = 2n\|f\|.$$

In the proof of (3.3) we use again (2.9) so that

$$|xL_n''(f;\ x)| \leqq n^2 x \sum_{k=0}^{\infty} |F_{n,k}(f) - 2F_{n,k+1}(f) + F_{n,k+2}(f)|\, p_{n,k}(x).$$

By (2.10)

$$\begin{aligned} &n^2 x\,|F_{n,0}(f) - 2F_{n,1}(f) + F_{n,2}(f)|\, p_{n,0}(x) \leqq \\ &\leqq \|\varphi^2 f''\| \Big(n \int_0^\infty e^{-nt}\, dt\Big)\, nxe^{-nx} \leqq \|\varphi^2 f''\|. \end{aligned} \tag{3.4}$$

For $k\geqq 1$ let us apply Taylor's formula (2.4) by which

$$x\,|F_{n,k}(f) - 2F_{n,k+1}(f) + F_{n,k+2}(f)| = \tag{3.5}$$

$$= \left| nx \int_0^\infty p_{n,k}(t) \Big(1 - \frac{2nt}{k+1} + \frac{(nt)^2}{(k+1)(k+2)}\Big) \int_0^t (t-\tau) f''(\tau)\, d\tau\, dt \right|.$$

Here

$$\left|1 - \frac{2nt}{k+1} + \frac{(nt)^2}{(k+1)(k+2)} - \Big(1 - \frac{nt}{k+1}\Big)^2\right| \leqq \frac{(nt)^2}{(k+1)^2(k+2)}$$

and by

$$\Big|\int_x^t ((t-\tau)/\tau)\, d\tau\Big| \leqq \Big|\int_x^t ((t-x)/x)\, d\tau\Big| \leqq (t-x)^2/x$$

we also have

$$\left|\int_x^t (t-\tau)f''(\tau)\,d\tau\right| \leqq \|\varphi^2 f''\|(t-x)^2/x.$$

Hence we can continue (3.5) as

$$\leqq \|\varphi^2 f''\| n \int_0^\infty p_{n,k}(t)\left(\left(1-\frac{nt}{k+1}\right)^2+\frac{(nt)^2}{(k+1)^2(k+2)}\right)(t-x)^2\,dt \leqq$$

$$\leqq \|\varphi^2 f''\| n \int_0^\infty p_{n,k}(t)\left[\left(1-\frac{2nt}{k+1}+\frac{(nt)^2}{(k+1)(k+2)}\right)+\frac{2(nt)^2}{(k+1)^2(k+2)}\right](t-x)^2\,dt =$$

$$= \|\varphi^2 f''\|(E_k-2E_{k+1}+E_{k+2})+2E_{k+2}/(k+1),$$

where

$$E_k = n\int_0^\infty e^{-nt}\big((nt)^k/k!\big)(t-x)^2\,dt = (k+1)(k+2)/n^2-2(k+1)x/n+x^2.$$

Now

$$E_k-2E_{k+1}+E_{k+2} = 2/n^2, \quad 2E_{k+2}/(k+1) \leqq 3\big((k+4)/n^2-2x/n+x^2/(k+1)\big)$$

and so

$$n^2x\sum_{k=1}^\infty |F_{n,k}(f)-2F_{n,k+1}(f)+F_{n,k+2}(f)|\,p_{n,k}(x) \leqq$$

$$\leqq \|\varphi^2 f''\| n^2\Big(\sum_{k=1}^\infty (2/n^2)\,p_{n,k}(x)+3n^2\sum_{k=1}^\infty \big((k+4)/n^2\big)p_{n,k}(x)-$$

$$-2(x/n)\sum_{k=1}^\infty p_{n,k}(x)+x^2\sum_{k=1}^\infty p_{n,k}(x)/(k+1)\Big)\Big) \leqq$$

$$\leqq \|\varphi^2 f''\|\Big(14+3n^2\Big(-x/n+2xe^{-nx}/n+(x/n)\sum_{k=1}^\infty p_{n,k+1}(x)\Big)\Big) \leqq$$

$$\leqq \|\varphi^2 f''\|(14+6nxe^{-nx}) \leqq 20\|\varphi^2 f''\|$$

and this together with (3.4) prove (3.3), and thus the proof is complete.

Proof of Theorem 7. First let us suppose that $f(x^2)$ is uniformly continuous on $[0, \infty)$. It is easy to see that then $f(x\pm\delta\sqrt{x})-f(x)\to 0$ as $\delta\to 0$ uniformly in $x\geqq 0$ and so to a given $\varepsilon>0$ there is a $\delta>0$ such that $|f(x\pm h\sqrt{x})-f(x)|\leqq\varepsilon$ whenever $0<h\leqq\delta$. This implies that

$$f(x)-\varepsilon-2\|f\|\delta^{-2}(t-x)^2/x \leqq f(t) \leqq f(x)+\varepsilon+2\|f\|\delta^{-2}(t-x)^2/x$$

and hence for $x\geqq 1$

$$|L_n(f;\, x)-f(x)| \leqq \varepsilon+2\|f\|\delta^{-2}L_n\big((t-x)^2, x\big)/x =$$

$$= \varepsilon+2\|f\|\delta^{-2}(2x+2/n)/nx \leqq \varepsilon+8\|f\|/\delta^2 n < 2\varepsilon$$

provided $n > 8\delta^{-2}\varepsilon^{-1}$. Since $L_n(f; x) - f(x) = o(1)$ $(n \to \infty)$, $0 \leqq x \leqq 1$, follows from the analogue of the localization theorem proved in the proof of Theorem 2 and from (2.2), the sufficiency part of the theorem has been verified.

Now let us suppose that $L_n(f; x) - f(x) = o(1)$ $(n \to \infty)$ uniformly on $[0, \infty)$. Exactly as in the proof of Theorem 4 one can show that $|L_n'(f; x)| \leqq K_n/\sqrt{x}$ and so

$$|f(x+\delta\sqrt{x}) - f(x)| \leqq 2\,\|L_N(f) - f\| + K_N \left| \int_x^{x+\delta\sqrt{x}} \tau^{-1/2}\, d\tau \right| < 2\varepsilon$$

if N is sufficiently large and δ is small. Thus $f(x+\delta\sqrt{x}) - f(x)$ tends to zero uniformly on $[0, \infty)$ as $\delta \to 0$ and this again is equivalent to the uniform continuity of $f(x^2)$. Thus the proof is complete.

Proof of Theorem 8. We shall constantly use the identities

$$L_n(1;\, x) = 1, \quad L_n(t, x) = x + 1/n \quad \text{and} \quad L_n\big((t-x)^2, x\big) = 2x/n + 2/n^2.$$

We separately prove the sufficiency and necessity of the given conditions.

(1) Sufficiency. Let us suppose that $xf'(x) \in \mathrm{Lip}\, 1$. This yields the absolute continuity of f' and the boundedness of $g(x) = (xf'(x))'$. We may suppose that $|g| \leqq 1$. Since then

$$f(x) = \int_0^x (1/\tau) \int_0^\tau g(u)\, du\, d\tau + c \log x + d,$$

with some constants c and d, and $f \in C_B$ implies $c = 0$, it follows that $|f'| \leqq 1$ and so $|xf''(x)| = |g(x) - f'(x)| \leqq 2$, i.e. $|f''(x)| \leqq 2/x$. For

$$h_x(t) = \int_x^t \int_x^\tau (du/u)\, d\tau,$$

we have

$$h_x(t) = t \log\big(1 + (t-x)/x\big) - t + x \leqq t(t-x)/x - (t-x) = (t-x)^2/x, \quad t \geqq 0$$

and $h_x''(t) = 1/t$, and the latter implies (see above) that the functions $\gamma_\pm = 2h_x \pm f$ are convex $(\gamma_\pm'' \geqq 0)$. Now if γ is convex and differentiable, then, because of the inequality $\gamma(t) - \gamma(x) \geqq \gamma'(x)(t-x)$, we have

$$L_n(\gamma;\, x) - \gamma(x) \geqq \gamma'(x) L_n(t-x, x) = \gamma'(x)(1/n).$$

Putting here $\gamma = \gamma_\pm$ and taking into account that

$$|\gamma_\pm'(x)| = \left| \int_x^t (du/u)|_{t=x} \pm f'(x) \right| = |\pm f'(x)| \leqq 1$$

we get

$$|L_n(f;\ x)-f(x)| \leqq L_n(h_x;\ x)-h_x(x)+1/n \leqq$$
$$\leqq x^{-1}L_n\big((t-x)^2, x\big)+1/n = 2(x+1/n)/nx+1/n \leqq 5/n,$$

provided $x \geqq 1/n$.

If $0<x\leqq 1/n$, then we argue as follows. Since $f(t)=f(x)+\int\limits_x^t f'(\tau)\,d\tau$, we have

$$|L_n'(f;\ x)| = \Big|\sum_{k=0}^{\infty} n\big(F_{n,k+1}(f)-F_{n,k+2}(f)\big)p_{n,k}(x)\Big| =$$
$$= \Big|\sum_{k=0}^{\infty} n\int\limits_0^{\infty} ne^{-nt}((nt)^k/k!)(1-nt/(k+1))\int\limits_x^t f'(\tau)\,d\tau\,dt\Big| =$$
$$= \Big|\sum_{k=0}^{\infty} n\int\limits_0^{\infty} p_{n,k+1}'(t)\int\limits_x^t f'(\tau)\,d\tau\,dt\Big| = \Big|\sum_{k=0}^{\infty} n\int\limits_0^{\infty} p_{n,k+1}(t)f'(t)\,dt\Big| \leqq \|f'\| \leqq 1$$

and so, according to what we have proved above

$$|L_n(f;\ x)-f(x)| \leqq 2|x-1/n|+|L_n(f;\ 1/n)-f(1/n)| \leqq 7/n \quad (0<x\leqq 1/n)$$

and the sufficiency part of the theorem has been established.

(2) Necessity. Let $W_1(t)=1/t$, $W_2(t)\equiv 1$. Using Taylor's formula (2.4) and the strong localization proved in Theorem 2 (clearly the proof also works for L_n) one can easily prove that if $f\in C_B$ is twice continuously differentiable on $[0, \infty)$ then

$$\lim_{n\to\infty} n\big(L_n(f;\ x)-f(x)\big) = \lim_{n\to\infty}\Big(nL_n(f'(x)(t-x);\ x)+nL_n\Big(\int\limits_x^t (t-\tau)f''(\tau)\,d\tau;\ x\Big) =$$
$$= f'(x)+xf''(x) = \big(xf'(x)\big)' = [(1/W_2)(f'/W_1)'](x)$$

and so [4, Theorem 5.7] gives (use also the localization) that

$$L_n(f;\ x)-f(x) = o(1/n)$$

implies that f has the form $f(x)=c\log x+d$ and so $(f\in C_B)$ f is constant. This proves the first half of the theorem.

Now suppose $L_n(f, x)-f(x)=O(1/n)$ uniformly on $[0, \infty)$. We set

$$g(x) = \overline{\lim_{n\to\infty}}\, n\big(L_n(f;\ x)-f(x)\big).$$

If $[a, b]\subseteq(0, \infty)$ is arbitrary, then $g\in L^1(a, b)$. Using once more the localization result for L_n we get from [4, Theorem 5.8] that on (a, b) f has the form

$$f(x) = d+c\log x+\int\limits_a^x (1/t)\int\limits_a^t g(s)\,ds\,dt,$$

i.e. f' is absolutely continuous on (a, b), and since (a, b) is arbitrary, it follows that f' is locally absolutely continuous on $(0, \infty)$. By [4, Lemma 5.9] and our localization result

$$\overline{\lim_{n\to\infty}}\, n(L_n(f;\ x)-f(x)) = (tf'(t))'\big|_{t=x}$$

at every x, where the right hand side makes sense, and so $L_n(f; x)-f(x)=O(1/n)$ yields $(x(f'(x))'=O(1)$ and this completes the proof of the theorem.

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