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A generalization of a theorem of G. Freud on the differentiability of functions

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0. Introduction

Let f be a function defined on R^1 and let m be a positive integer. Then f has a Taylor polynomial of order m at $x=a$ if and only if there is a number C_m such that

$$\Delta_h^m f(x) = C_m h^m + ((x-a)^m + h^m)\varepsilon(x, h), \tag{0.1}$$

where $\varepsilon(x, h) \to 0$, as $x \to a$ and $h \to 0$.

This result was announced by G. FREUD in [3]. It is our purpose to prove an L^p-version of (0.1) for functions in R^n, $1 \leqq p \leqq \infty$, and differentiability of order $l>0$. This involves finding the L^p-form of (0.1). (See Section 1 for the exact definitions.) The methods of proof use approximation theorems of Jackson type due to H. WHITNEY [10] and JU. A. BRUDNYI [1]. As an application we consider the problem to characterize differentiability in terms of L^p-differentiability together with certain additional conditions. Such problems have been studied by the author in [4]—[6]. See also B.-M. STOCKE [8].

Section 1 contains our notation and the definitions. Our results are then stated in Section 2 and proved in Sections 4 and 5. The lemmas needed in the proofs are given in Section 3.

1. Notation and definitions

1.1. We use standard notation for points $x=(x_1, ..., x_n)$ and real or complex valued functions $f(x)$ in R^n. For $E \subset R^n$ we denote Lebesgue outer measure and Lebesgue measure by $|E|^*$ and $|E|$ respectively. Integration with respect to Lebesgue measure is written $\int_E f(x)\,dx$ and $L^p(E)$ denotes the usual Lebesgue classes, $1 \leqq p \leqq$

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$\leqq\infty$. All functions considered are measurable. We define $\|f\|_{L^p(E)}=\left(\int_E |f(x)|^p dx\right)^{1/p}$, $1\leqq p\leqq\infty$, with the usual modification when $p=\infty$. When $E=R^n$ we just write $\|f\|_p$.

We let $I=I(a, s)$ denote a cube in R^n with its sides parallel to the axes, its centre in a and having diameter s. The ball with centre at a and radius r is denoted by $B(a,r)$. The density of a measurable set E at x_0 is defined by $\lim_{r\to 0} |E\cap B(x_0, r)|\cdot$ $\cdot|B(x_0, r)|^{-1}$, if the limit exists. We let $c(\alpha, \beta, \ldots)$ denote constants, which may be different at each occurrence, depending on $\alpha, \beta, \ldots$. Constants only depending on n are denoted by c. Polynomials P are written $P(x)=\sum C_\alpha x^\alpha$, where the summation is over multi indices α.

1.2. Let $l>0$. A function f is said to be (ordinary) differentiable of order l at $x=a$ in R^n if there is a (Taylor) polynomial $P(x)$ of degree at most l such that $R(x, a)=f(x)-P(x)=o(|x-a|^l)$, as $x\to a$. We say that f is L^p-differentiable at $x=a$ of order l if

$$\left(|B(a, r)|^{-1} \int_{B(a,r)} |R(x, a)|^p\, dx\right)^{1/p} = o(r^l), \quad r \to 0,$$

for a suitable polynomial P. Here we make the usual modification when $p=\infty$. The polynomials P are unique in all cases.

The differences $\Delta_h^m f(x)$ are defined inductively by $\Delta_h^0 f(x)=f(x)$ and $\Delta_h^{m+1} f(x)=$ $=\Delta_h^m f(x+h)-\Delta_h^m f(x)$. It is easily verified that if $P(x)=\sum_{|\alpha|\leqq m} C_\alpha x^\alpha$, then $\Delta_h^m P(x)=$ $=m!\sum_{|\alpha|=m} C_\alpha\cdot h^\alpha$, for all x. For more properties of $\Delta_h^m f$, see [5] and [9, p. 102].

1.3. In this section we define a smoothness property for functions called C_s^j, using j-th order differences. We considered a slightly different property, also denoted by C_s^j, in [5]. For a comparison of these properties, see the second remark following Definition 1.1.

Definition 1.1. Let j be a positive integer and let $0\leqq s\leqq j+1$. Let f be a function defined in a neighbourhood of $x=a$ in R^n. We say that f has property C_s^j at $x=a$ if there exist numbers C_α, $|\alpha|=j$, such that for every $\varepsilon>0$ there are t and δ, $0<t<\min(\varepsilon, 1)$ and $\delta>0$, such that

$$\sup_{|h|\leqq t|x-a|} \left|\Delta_h^j f(x)-j!\sum_{|\alpha|=j} C_\alpha h^\alpha\right| \leqq \varepsilon|x-a|^s, \tag{1.1}$$

for all x, $0<|x-a|<\delta$. We take $C_\alpha=0$, $|\alpha|=j$, if $s\leqq j$.

Replacing (1.1) by

$$\left(|B(0, t|x-a|)|^{-1} \int_{B(0,t|x-a|)} \left|\Delta_h^j f(x)-j!\sum_{|\alpha|=j} C_\alpha h^\alpha\right|^p dh\right)^{1/p} \leqq \varepsilon|x-a|^s, \tag{1.2}$$

$1\leqq p\leqq\infty$, gives an equivalent definition of property C_s^j. For the proof, see [5, Lemma 5.2].

The numbers C_α in (1.1) and (1.2) are unique, when $s>j$. In the case when $s\leqq j$, (1.1) and (1.2) are independent of the choice of C_α in the sense that they hold with all $C_\alpha=0$ if and only if they hold for some arbitrary C_α, $|\alpha|=j$.

Remark. We get an equivalent definition of property C_s^j if we in (1.1) or (1.2) replace $\Delta_h^j f$ by the binary differences $B_h^j f$, see [5, p. 51]. In proving this it is no loss of generality to assume that $C_\alpha=0$, $|\alpha|=j$, in (1.1) and (1.2) and hence the proof of [5, Lemma 5.4] applies. We leave the details to the reader.

Remark. The present definition of property C_s^j differs from the one in [5, p. 53] only when $s>j$. For example in the case of [5, Theorem 3.2], the two definitions agree. It can be proved that there is an alternative formulation of Theorem 3.2 in [5] based on (a)—(c) in [5, pp. 53—54] and using property C_s^j. Finally we note that property C_s^1 in the same as property B_s in [4, p. 9], when $0<s\leqq 1$.

2. Main results

2.1. Our first theorem is a characterization of L^p-differentiability which generalizes G. Freud's result in [3].

Theorem 2.1. *Let $f(x)$ be a function defined in a neighbourhood of $x=a$ in R^n. Let $l>0$, $1\leqq p\leqq\infty$ and let $m\leqq l<m+1$, where m is a non-negative integer. Then f is L^p-differentiable at $x=a$ if and only if there are numbers C_α, $|\alpha|=m$, such that*

$$\sup_{|h|\leqq t}\Big(|B(a,t)|^{-1}\int_{B(a,t)}\big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\big|^p\,dx\Big)^{1/p}=o(t^l),\quad as\quad t\to 0. \tag{2.1}$$

We make the usual modification in (2.1) when $p=\infty$.

As we mentioned in the introduction, Theorem 2.1 also holds for the case of differentiability in the ordinary sense with (2.1) replaced by

$$\sup_{|h|\leqq s}\ \sup_{|x-a|\leqq s}\big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\big|=o(s^l), \tag{2.2}$$

as $s\to 0$. Essentially the same proof applies. We omit the details.

Remark. In the case $0<l<1$, we have $m=0$ and $\Delta_h^0 f(x)=f(x)$. Then (2.1) is just the definition of L^p-differentiability. In the case when $l\geqq 1$ and (2.1) holds, there exist numbers C_α, $|\alpha|\leqq m-1$, such that the L^p-differential of f at $x=a$ is given by $P(x)=\sum_{|\alpha|\leqq m}C_\alpha(x-a)^\alpha$.

2.2. Next we use ideas from [4]—[6] and combine Theorem 2.1 with (2.2) to prove results on the connection between differentiability and L^p-differentiability.

Theorem 2.2. *Let $l>0$ and $1\leqq p\leqq\infty$. Let m be an integer such that $m\leqq l<m+1$, when $l\geqq 1$ and $m=1$, when $0<l<1$. Then f is differentiable at $x=a$ of order l if and only if f has an L^p-differential of order l at $x=a$ and f has property C_l^m at $x=a$.*

The following variant of Theorem 2.2 can be proved with the same methods.

Theorem 2.3. *Let $0<l<m$, where m is an integer, and let $1\leqq p\leqq\infty$. Then a function f is differentiable at $x=a$ of order l if and only if f is L^p-differentiable at $x=a$ of order l and f has property C_l^m at $x=a$.*

Theorems 2.2 and 2.3 should be compared to the corresponding results in [4] and [5]. Theorem 2.3 generalizes Theorem 3.2 in [5] and Theorem 2.2 is a generalization of the alternative formulation of Theorem 3.2 in [5] which was mentioned in the second remark following Definition 1.1. Examples show that property C_l^m alone does not imply differentiability of order l in Theorems 2.2 and 2.3.

Our results in [4], [5] and the present paper can be summarized as follows. We want to characterize differentiability of a function $f(x)$ at $x=a$ by L^p-differentiability together with certain additional conditions. These additional conditions can roughly be described as follows (differentiability of order l, $1\leqq m\leqq l<m+1$, where m is an integer):

(i) ([4]) f_m has property C_l^1 at $x=a$, where $f_m(x)=f(x)-\sum_{1\leqq|\alpha|\leqq m} C_\alpha(x-a)^\alpha$,

(ii) ([5]) f has properties C_s^j at $x=a$, for $1\leqq j\leqq m+1$, and suitable s,

(iii) (present paper) f has property C_l^m (or C_l^{m+1}) at $x=a$.

The case (i) was applied to Bessel potentials of L^p-functions in [4]. The advantage of (iii) compared to (ii) is that it has just one single condition.

2.3. It was proved in [4, Lemma 5.2] that property C_l^1, $0<l\leqq 1$, at $x=a$ implies that $|f(x)-b|\leqq M\cdot|x-a|^l$, for suitable numbers b and M, when x is close to a. Our next theorem generalizes this result. Roughly speaking, we prove that property C_l^m is of Lipschitz type.

Theorem 2.4. *Let m be a positive integer and $s>m-1$, and let f be a function. Assume that there are $\varepsilon>0$, $\delta>0$ and $0<t<1$ such that $0<|x-a|\leqq\delta$ implies*

$$\sup_{|h|\leqq t|x-a|} |\Delta_h^m f(x)| \leqq \varepsilon|x-a|^s. \tag{2.3}$$

Then there is a unique polynomial $P(x)$ of degree at most $(m-1)$ such that

$$|f(x)-P(x)|\leqq M|x-a|^s \tag{2.4}$$

for $0<|x-a|\leqq 4\delta/5$, where M is a suitable number depending on n, m, s and t.

Corollary. *Let f have property C_l^m at $x=a$, where l and m are as in Theorem 2.2. Then there is a unique polynomial P of degree less than l such that*

$$|f(x)-P(x)| \leqq M|x-a|^l$$

for $0<|x-a|<\delta$, where $\delta>0$ and M are suitable constants.

The corollary follows easily from Definition 1.1 and Theorem 2.4.

3. Some lemmas

It is well known that if a function $f(x)$ satisfies $\limsup_{h\to 0} |\Delta_h^m f(x)|<\infty$, for every x in a measurable set E, then f is bounded in a neighbourhood of a.e. $x\in E$, cf. [7, p. 249]. We need the following L^p-form of that result.

Lemma 3.1. *Let m be a positive integer and $1\leqq p\leqq\infty$. Assume that for every x_0 in a measurable set $E\subset R^n$, there is $r_0=r_0(x_0)>0$ such that*

$$\int_{|h|\leqq r_0} dh \int_{|x-x_0|\leqq r_0} |\Delta_h^m f(x)|^p\, dx <\infty,$$

if $1\leqq p<\infty$, and

$$\operatorname*{ess\,sup}_{|x-x_0|\leqq r_0} \operatorname*{ess\,sup}_{|h|\leqq r_0} |\Delta_h^m f(x)| <\infty,$$

if $p=\infty$. Then for a.e. $x_0\in E$ there is $r=r(x_0)>0$ such that f belongs to $L^p(B)$, where $B=B(x_0,r)$.

Proof of Lemma 3.1. Let $1\leqq p<\infty$. It is no loss of generality to assume that f is finite in E. Define for $i=1,2,\dots$

$$E_i = \Big\{x_0\in E;\ \int_{|h|\leqq 1/i} dh \int_{|x-x_0|\leqq 1/i} |\Delta_h^m f(x)|^p\, dx \leqq i\Big\},$$

then $E=\bigcup_1^\infty E_i$. In contrast to the case in [7], the sets E_i are here measurable. There exists a closed subset P of E, with $|E\setminus P|$ arbitrarily small, such that the restriction of f to P is continuous on P. Define $B_i=\{x;\ E_i$ has density one at $x\}$, $C=\{x;\ P$ has density one at $x\}$ and $A_i=B_i\cap C$, $i=1,2,\dots$. Let $A=\bigcup_1^\infty A_i$, then

$$E\setminus A\subset\Big(\bigcup_1^\infty (E_i\setminus B_i)\Big)\cup(E\setminus P)\cup(P\setminus C).$$

We get $|E\setminus A|\leqq|E\setminus P|$, since $E_i\setminus B_i$ and $P\setminus C$ have measure zero. It is our purpose to prove that f is locally in L^p near every point in A. Let i be fixed and let $x_0\in A_i$.

We now have the following identity

$$(-1)^m(f(x)-f(x_0)) = \sum_{s=0}^{m-1}(-1)^s\binom{m}{s}(f(x_0)-f(z+sh))+\Delta_h^m f(z),$$

where $z+mh=x$. Let x be fixed, $0<|x-x_0|<r$. We use [5, Lemma 4.3] to split the last term in the above identity as follows

$$(-1)^m(f(x)-f(x_0)) = \sum_{s=0}^{m-1}(-1)^s\binom{m}{s}(f(x_0)-f(z+sh))+$$
$$+\sum_{i=1}^{m-1}(-1)^i\binom{m}{i}\Delta^m_{(i/m)(k-h)}f(z+ih)+(-1)^m\Delta^m_{k-h}f(x)-$$
$$-\sum_{j=1}^{m}(-1)^j\binom{m}{j}\Delta^m_{h+(j/m)(k-h)}f(z).$$

Let $|z-x_0|\leqq M|x-x_0|$, where M is to be chosen below, then $|h|\leqq(M+1)r$. Taking the L^p-norm over $|k|\leqq r$ we get

$$(3.1)\qquad |f(x)-f(x_0)| \leqq \sum_{s=0}^{m-1}\binom{m}{s}|f(x_0)-f(z+sh)|+$$
$$+c(n, M)\sum_{j=0}^{m-1}\Big(|B(0, r)|^{-1}\int_{|w|\leqq(M+2)r}|\Delta_w^m f(z+jh)|^p\,dw\Big)^{1/p}+$$
$$+c(n, m, M)\Big(|B(0, r)|^{-1}\int_{|w|\leqq(M+2)r}|\Delta_w^m f(x)|^p\,dw\Big)^{1/p} = I_1(z)+I_2(z)+I_3(x).$$

There are $r>0$ and $M=M(r)$ such that the density at x_0 of the set

$$S = \{z;\ (j/m)x+(1-j/m)z\in P,\ \text{ for }\ j = 1, 2, \dots, m-1\ \text{ and }\ |z-x_0|\leqq M|x-x_0|\}$$

is arbitrarily close to one, for all $0<|x-x_0|<r$, cf. [7, p. 267]. Further,

$$\int_{B(x_0,r)} I_2(z)^p\,dz \leqq c(n, M)r^{-n}i^p$$

if $0<r\leqq c(n, m)\cdot(1/i)$. Hence, if r is small enough, we can for every x, $0<|x-x_0|<r$, find $z\in S$ such that $I_1(z)$ is arbitrarily small and $I_2(z)\leqq c(n, M, r, p)$. Integrating the p-th power of (3.1) w.r.t. x over the set $B(x_0, r)$ yields

$$\int_{B(x_0,r)}|f(x)-f(x_0)|^p\,dx \leqq c(n, M, r, p)$$

for some $r>0$.

We have proved that f is locally in L^p at every point of A. The conclusion of the lemma follows since $|E\setminus A|$ can be made arbitrarily small. The case $p=\infty$ is treated analogously and we omit the details.

Lemma 3.2. *Let* $1 \leqq p \leqq \infty$ *and assume that* (2.1) *holds. Then* f *is locally in* L^p *at* a.

Proof of Lemma 3.2. Let $1 \leqq p < \infty$. By (2.1) there are positive numbers δ and M such that $|h| \leqq \delta$ and $|x_0 - a| \leqq \delta$ imply

$$\int_{B(x_0,\delta)} |\Delta_h^m f(x)|^p \, dx \leqq M^p.$$

Then by Lemma 3.1 there are x_0, with $|x_0 - a| < \delta/2$, and positive numbers r and N such that $\int_{B(x_0,r)} |f(x)|^p dx = N^p < \infty$. There is nothing to prove unless $r \leqq \delta$. Consider the identity

$$\Delta_h^m f(x) = \sum_{i=0}^{m-1} (-1)^{m-i} \binom{m}{i} f(x+ih) + f(x+mh). \tag{3.2}$$

Let $|h| = r/m$. We integrate the p-th power of (3.2) w.r.t. x over $B(x_0, s)$, where $s = r - (m-1) \cdot |h| = r/m$. This gives

$$\int_{B(x_0+mh,s)} |f(y)|^p \, dy \leqq (M + 2^m N)^p.$$

Since a certain portion of $B(x_0+mh, s)$ lies outside $B(x_0, r)$, a simple geometrical argument shows that $\int_{B(x_0,r_1)} |f(y)|^p dy$ is finite, where $r_1 = r + |h|/2$. If $r_1 > \delta$ we are done. Otherwise, repeating this procedure a finite number of times, we find $r_0 > \delta$ such that $\int_{B(a,\delta/2)} |f(x)|^p dx \leqq \int_{B(x_0,r_0)} |f(x)|^p dx < \infty$. The case $p = \infty$ is handled analogously. This completes the proof of Lemma 3.2.

Our next lemma is a slightly generalized form of a lemma due to De Giorgi, see [2, p. 140]. We omit the proof.

Lemma 3.3. *Let* $I_1 = I(a, u)$ *and* $I_2 = I(b, v)$ *be two cubes such that* $I_1 \subset I_2$. *Let* j *be a non-negative integer and* $1 \leqq p \leqq \infty$. *Then*

$$\sup_{I_2} |D^\alpha P(x)| \leqq c(n, j)(v/u)^j v^{-|\alpha|} \left(|I_1|^{-1} \int_{I_1} |P(x)|^p dx\right)^{1/p}$$

for all polynomials P *of degree* $\leqq j$. *The lemma also holds for balls instead of cubes.*

We make the usual modification in the right hand side of the inequality when $p = \infty$.

4. Proof of Theorems 2.1 and 2.2

4.1. Proof of Theorem 2.1. The proof of the necessity follows easily by integrating the identity

$$\Delta_h^m f(x) - m! \sum_{|\alpha|=m} C_\alpha h^\alpha = \Delta_h^m (f-P)(x),$$

where $P(x) = \sum_{|\alpha|\leqq m} C_\alpha (x-a)^\alpha$. The details are left to the reader.

Suppose that (2.1) holds. We may assume that $C_\alpha = 0$ for $|\alpha| = m$, and that $f(x) = 0$ outside some neighbourhood of $x = a$. Then by Lemma 3.2 we may also assume that f is in L^p.

Let $I_s = I(a, s)$ be a cube. Define $I_{s,h} = \{x;\ x+kh \in I_s,$ for $k = 0, 1, \dots, m\}$, then $I_{s,h} \subset I_s$. Now by [1, Theorem 1'] there is, for every $s > 0$, a polynomial P_s of degree at most $(m-1)$ such that

$$\|f - P_s\|_{L^p(I_s)} \leqq c(n, m) \sup_{|h|\leqq s} \|\Delta_h^m f\|_{L^p(I_{s,h})}.$$

Then by our assumption (2.1) we get

$$\|f - P_s\|_{L^p(I_s)} \leqq s^l v(s), \tag{4.1}$$

where $v(s)$ is a non-decreasing function tending to zero as $s \to 0$.

Now let P_k be the polynomial P_s, when $s = 2^{-k}$, $k = 1, 2, \dots$. We intend to prove that the sequence $\{P_k\}_1^\infty$ converges uniformly on compact sets to a polynomial P with the desired properties. More exactly, we prove that there is a polynomial P of degree at most $(m-1)$ such that

$$|D^\alpha P_k(a) - D^\alpha P(a)| = o(2^{-k(l-|\alpha|)}), \quad \text{as } k \to \infty, \tag{4.2}$$

for $|\alpha| \leqq m-1$. It then follows from Taylor's formula that for $s = 2^{-k}$ we have $\sup_{x\in I_s} |P_k(x) - P(x)| = o(2^{-kl})$, as $k \to \infty$. Combining this with (4.1) we get the conclusion of the theorem with the polynomial P defined above.

It remains to prove (4.2). Now (4.1) and Lemma 3.3 give that for $|\alpha| \leqq m-1$,

$$|D^\alpha P_{k+1}(a) - D^\alpha P_k(a)| \leqq c(n, m)\Big(|I_{k+1}|^{-1} \int_{I_{k+1}} |P_{k+1}(x) - P_k(x)|^p\, dx\Big)^{1/p} \leqq$$

$$\leqq c(n, m) 2^{k(|\alpha|-l)} v(2^{-k}).$$

Then for $i > k$ we get by summation

$$|D^\alpha P_i(a) - D^\alpha P_k(a)| \leqq \sum_k^{i-1} |D^\alpha P_{j+1}(a) - D^\alpha P_j(a)| \leqq c(n, m, l) 2^{k(|\alpha|-l)} v(2^{-k}).$$

Hence the sequences $\{D^\alpha P_k(a)\}_1^\infty$, $|\alpha|\leqq m-1$, converge and there is a polynomial P such that

$$|D^\alpha P_k(a)-D^\alpha P(a)| = o(2^{k(|\alpha|-l)}),$$

as $k\to\infty$. This proves (4.2) and thereby completes the proof of Theorem 2.1.

4.2. Proof of Theorem 2.2. We omit the necessity part of the proof since it is straightforward, cf. the corresponding results in [4] and [5]. Assume that f is L^p-differentiable of order l and that f has property C_l^m at $x=a$, for l and m as in the theorem. We first assume that $1\leqq m\leqq l<m+1$. Let $P(x)=\sum_{|\alpha|\leqq m} C_\alpha(x-a)^\alpha$ be the L^p-differential of f at $x=a$. Then the constants C_α, $|\alpha|=m$, in $P(x)$ are the same as the constants C_α in (2.1) by the remark following Theorem 2.1. We claim that it is no loss of generality to assume that we have the same constants C_α, $|\alpha|=m$, in (1.1) and (2.1).

When $l=m$ our claim follows from the fact that in this case (1.1) is independent of the choice of constants C_α. Now let $m<l<m+1$ and denote these constants in (1.1) and (2.1) by C_α and C'_α respectively. Then

$$\Big|m!\sum_{|\alpha|=m}(C_\alpha-C'_\alpha)h^\alpha\Big| \leqq \Big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\Big|+\Big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C'_\alpha h^\alpha\Big|.$$

Let $0<\varepsilon<1$ be arbitrary and choose t and δ as in Definition 1.1. Let $0<s<$ $<\min(t,\delta/2)$ and $|h|\leqq st$. We integrate the above inequality to the p-th power w.r.t. x over the set $E_s=\{x;\ s\leqq|x-a|\leqq 2s\}$. Then using (1.1) and (2.1) we get

$$\sup_{|h|\leqq st}\Big|m!\sum_{|\alpha|=m}(C_\alpha-C'_\alpha)h^\alpha\Big| \leqq o(s^l)+2^l\varepsilon s^l, \quad \text{as} \quad s\to 0.$$

It follows easily that $C_\alpha=C'_\alpha$, $|\alpha|=m$, by letting s tend to zero, since $l>m$. Thereby our claim is proved.

Now consider the identity

$$f(x)-P(x) = (-1)^m\Big(\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\Big)-$$
$$-\sum_{k=1}^m\binom{m}{k}(-1)^k\big(f(x+hk)-P(x+kh)\big).$$

Let $\varepsilon>0$ be arbitrary. Choose t and δ as in Definition 1.1 and let $0<|x-a|<\delta$. Then integrating this identity to the p-th power w.r.t. h over the set $|h|\leqq t|x-a|$ and using (1.1) and the definition of the L^p-derivative yields

$$|f(x)-P(x)| \leqq \varepsilon|x-a|^l+o(|x-a|^l), \quad \text{as} \quad x\to a.$$

This proves the theorem when $l\geqq 1$. The case $0<l<1$ is much simpler and its proof is left to the reader.

5. Proof of Theorem 2.4

The proof is a combination of the methods of proof in Theorem 2.2 and [4, Lemma 5.1]. Therefore we will not go into so many details.

Let x_0 be such that $|x_0=a|=4\delta/5$ and let L denote the line segment between a and x_0. Define a sequence $\{x_i\}_1^\infty$ of points on L such that $|x_i-a|=r_i=r_0(1-t/4)^i$, $i=1, 2, \dots,$ where $r_0=|x_0-a|$. Then $|x_i-x_{i+1}|=(t/4)|x_i-a|$. Define $B_i= =B(x_i, (t/4)|x_i-a|)$, $i=0, 1, \dots$. If x and $x+mh$ belong to B_i, then $|h|\leqq t|x-a|$ and hence (2.3) and the definition of B_i give

$$|\Delta_h^m f(x)| \leqq \varepsilon|x-a|^s \leqq \varepsilon\cdot 2^s|x_i-a|^s.$$

Now by Brudnyi [1, p. 158] there is a polynomial P_i of degree at most $m-1$ such that

$$|f(x)-P_i(x)| \leqq c(n, m, s)\varepsilon|x_i-a|^s, \tag{4.3}$$

for all $x\in B_i$, $i=0, 1, \dots$. We claim that the sequence $\{P_i(x)\}_0^\infty$ of polynomials converges uniformly on compact sets to a polynomial $P(x)$ with the desired properties.

We first note that for $x\in B_i\cap B_{i+1}$, $i=0, 1, \dots,$ we have $|P_i(x)-P_{i+1}(x)|\leqq \leqq c(n, m, s)\varepsilon|x_i-a|^s$. Define $E_i=B(a, 2|x_i-a|)$, $i=0, 1, \dots$. Now $B_i\cap B_{i+1}$ contains a ball B_i' of radius $(t/8)|x_i-a|$ and $B_i'\subset E_i$, $i=0, 1, \dots$. We apply Lemma 3.3 to the balls B_i' and E_i and we get

$$|D^\alpha P_i(a)-D^\alpha P_{i+1}(a)| \leqq c(n, m, s, t)\varepsilon|x_i-a|^{s-|\alpha|}$$

for $|\alpha|\leqq m-1$ and $i=0, 1, \dots$.

By the triangle inequality and summation of the last inequalities we get, since $s>m-1$,

$$|D^\alpha P_i(a)-D^\alpha P_j(a)| \leqq c(n, m, s, t)\varepsilon|x_i-a|^{s-|\alpha|}$$

for $j>i$. It now follows that there is a polynomial P of degree at most $(m-1)$ such that $|D^\alpha P(a)-D^\alpha P_i(a)|\leqq c(n, m, s, t)\varepsilon|x_i-a|^{s-|\alpha|}$ for $|\alpha|\leqq m-1$ and $i=0, 1, \dots$. This proves the first part of our claim. Taylor's formula and (4.3) yield the estimate

$$|f(x)-P(x)| \leqq c(n, m, s, t)\varepsilon|x-a|^s \tag{4.4}$$

for $x\in\bigcup_1^\infty B_i$.

We now repeat the procedure above with other points x_0', $|x_0'-a|=4\delta/5$, such that the corresponding sets $\bigcup_1^\infty B_i'$ cover the set $\{x;\ 0<|x-a|\leqq 4\delta/5\}$. This can be done in N steps, where N only depends on n and t. If x_0 and x_0' are such that the sets $\bigcup_1^\infty B_i$ and $\bigcup_1^\infty B_i'$ overlap, then the polynomials constructed above must be identical, because of (4.4). It follows that (4.4) holds for all x, $0<|x-a|\leqq 4\delta/5$. This settles the remaining part of our claim.

Finally, the uniqueness of the polynomial P follows easily from (2.4), since $s>m-1$. This completes the proof of Theorem 2.4.

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