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On approximation by Euler means of orthogonal series

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1. Introduction

Let $\{\varphi_n(x)\}$ be an orthonormal system on the interval $[0, 1]$. We consider orthogonal series

(1) $$\sum_{n=0}^{\infty} c_n \varphi_n(x), \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty$$

and their Euler means of order q, $0<q<1$, ((E, q)-means)

(2) $$t_n(x) = \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} s_k(x) \quad (n = 0, 1, \ldots),$$

where $s_k(x) = \sum_{\nu=0}^{k} c_\nu \varphi_\nu(x)$. Our main interest is directed to the rate of convergence of strong (E, q)-means

(3) $$\tau_n^{(\gamma)}(x) = \Big\{\sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} |s_k(x) - f(x)|^\gamma\Big\}^{1/\gamma} \quad (n = 0, 1, \ldots)$$

on $[0, 1]$ with $0<q<1$ and $\gamma>0$; $f(x)$ is connected with the given series (1) by the Riesz—Fischer theorem.

To this end we assume

(4) $$\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) < \infty,$$

where $\{\lambda(n)\}$ should be a nondecreasing sequence of positive real numbers tending to infinity which satisfies for a suitable δ, $0<\delta<1$, the following condition

(5) $$\lambda(n) \leqq C_\delta \lambda([\delta n]) \quad (n = 1, 2, \ldots).$$

Let the class Λ consist of all such sequences $\{\lambda(n)\}$. For (E, q)-means, V. I. KOLJADA [3] proved

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Theorem A. (a) *Let* $\{\lambda(n)\}\in\Lambda$ *and* $0<q<1$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)\log^2 n<\infty$, *then*

$$t_n(x)-f(x) = o_x(1/\lambda(n))$$

holds true almost everywhere (a.e.).

(b) *Let* $\{\lambda(n)\}\in\Lambda$. *If the sequence* $\{\mu(n)\}$ *of positive numbers satisfies the condition* $\lambda(n)=o(\mu(n))$, *then there exists an orthogonal series* $\sum_{n=0}^{\infty} c_n\varphi_n(x)$ *with* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)\log^2 n<\infty$ *and*

$$\overline{\lim_{n\to\infty}}\, \mu(n)\,|t_n(x)-f(x)| = \infty.$$

Here we prove for strong (E, q)-means

Theorem 1. *Let* $\{\lambda(n)\}\in\Lambda$ *and* $0<q<1$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)<\infty$, *then* $t_n(x)-f(x)=o_x(\lambda^{-1}(n))$ *a.e. implies*

$$\tau_n^{(\gamma)}(x) = o_x(1/\lambda(n)) \quad a.e.$$

for every $\gamma>0$.

We show that the conditions in Theorem 1 are not redundant:

Remark 1. For any method (E, q), $0<q<1$, there exists an orthogonal series $\sum_{n=0}^{\infty} c_n\psi_n(x)$ with $\sum_{n=0}^{\infty} c_n^2\lambda^2(n)<\infty$, $\{\lambda(n)\}\in\Lambda$, such that

$$\tau_n^{(\gamma)}(x) \neq o_x(1/\lambda(n)) \quad \text{a.e.} \quad (\gamma \geqq 1).$$

Remark 2. For any method (E, q), $0<q<1$, there exists an orthogonal series $\sum_{n=0}^{\infty} c_n\psi_n(x)$, $\sum_{n=0}^{\infty} c_n^2<\infty$, with $|t_n(x)-f(x)|=o_x(1/\lambda(n))$ a.e. $(\{\lambda(n)\}\in\Lambda)$ and

$$\tau_n^{(\gamma)}(x) \neq o_x(1/\lambda(n)) \quad \text{a.e.} \quad (\gamma > 0).$$

The proof of the remarks will be found in Section 4.

The condition $t_n(x)-f(x)=o_x((\lambda(n))^{-1})$ in our theorem may be substituted by a condition concerning the rate of approximation of certain partial sums, as follows from the next theorem. For this purpose we consider sequences of natural numbers $\{m_i\}$ with a gap condition

$$\alpha\sqrt{m_i} < m_{i+1}-m_i < \beta\sqrt{m_{i+1}} \quad (0<\alpha<\beta<\infty). \tag{6}$$

Theorem 2. *Let the sequences* $\{m_i\}$ *and* $\{m_i^*\}$ *satisfy a gap condition* (6) *and let* $\{\lambda(n)\}\in\Lambda$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)<\infty$, *then for partial sums and* (E, q)*-means,* $0<q<1$*; of the series* (1)

(a) $s_{m_i}(x)-f(x)=o_x(1/\lambda(m_i))$ *a.e. implies* $s_{m_i^*}(x)-f(x)=o_x(1/\lambda(m_i^*))$ *a.e.;*
(b) $t_{m_i}(x)-f(x)=o_x(1/\lambda(m_i))$ *a.e. implies* $t_n(x)-f(x)=o_x(1/\lambda(n))$ *a.e.;*
(c) $s_{m_i}(x)-f(x)=o_x(1/\lambda(m_i))$ *a.e. holds if and only if* $t_n(x)-f(x)=o_x(1/\lambda(n))$ *a.e.*

As an immediate consequence of the last theorem we get a result comparing (E, q)-means with different orders, proved by E. MARTIN [5] for a subclass of Λ:

Corollary. *Let* $\{\lambda(n)\}\in\Lambda$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)<\infty$, *then for* (E, q)*-means* $t_n^{(q)}(x)$ *and* (E,p)-means $t_n^{(p)}(x)$, $0<p, q<1$, *of the series* (1) *it holds:*

$$t_n^{(q)}(x)-f(x) = o_x(1/\lambda(n)) \quad \text{a.e. implies} \quad t_n^{(p)}(x)-f(x) = o_x(1/\lambda(n)) \quad \text{a.e.}$$

2. Lemmas

For the proof of our theorems we require some auxiliary results. In the following we assume a method (E, q) with a fixed order q, $0<q<1$, and we put $\bar{n}=\min\{k\in\mathbf{N}\colon n\leqq qk\}$. Obviously $[q\bar{n}]=n$ holds. Now it is possible to define the sequences $\{n_i\}$ and $\{\bar{n}_i\}$ by the following relations

$$n_0 = 1, \quad n_{i+1} = n_i+[\sqrt{\bar{n}_i}] \quad (i = 0, 1, \dots). \tag{7}$$

We put for brevity

$$e_{n\nu} = \binom{n}{\nu} q^\nu(1-q)^{n-\nu} \quad (\nu \leqq n)$$

and consider the differences

$$t_n(x)-s_{[qn]}(x) = \sum_{k=0}^{n} d_{nk}c_k\varphi_k(x)$$

with

$$d_{nk} = \begin{cases} -\sum_{\nu=0}^{k-1} e_{n\nu} & (1\leqq k\leqq qn),\\ \sum_{\nu=k}^{n} e_{n\nu} & (qn<k\leqq n),\\ 0 & (k=0 \quad\text{or}\quad k>n). \end{cases}$$

The following estimates can be found in L. KANTOROWITSCH [1] (Lemma 1), resp. H. SCHWINN [7] (p. 20).

Lemma 1. *Let* $0<q'<q<1$. *Then we have*

(a) $e_{nk} \leqq d_{n(k+1)} \leqq A(s)/n^s \quad (0<k\leqq q'n;\ s>0)$;

(b) $\sum_{i:[q'\bar{n}_i]\leqq k\leqq \bar{n}_i} d_{\bar{n}_i k}^2 \leqq B \quad (k=1, 2, \dots)$.

With regard to some structural properties of Λ, V. I. KOLJADA [3] (Lemma 1) proved

Lemma 2. *Let* $\{\lambda(n)\}\in\Lambda$.
(a) *There exists a constant* $p>0$ *with* $\lambda(n)=O(n^p)$.
(b) *For every* δ', $0<\delta'<1$, *it yields* $\lambda(n)\leqq C_{\delta'}\lambda([\delta' n])$.

Lemma 3. *Let the coefficients of the series* (1) *satisfy condition* (4) *with* $\{\lambda(n)\}\in\Lambda$, *and with respect to* (E,q), $0<q<1$, *let the sequences* $\{n_i\}$, $\{\bar n_i\}$ *be defined as in* (7). *Then*

$$t_{\bar n_i}(x)-s_{n_i}(x)=o_x(1/\lambda(\bar n_i)) \quad a.e.$$

Proof. This lemma is evident if we show

$$\sum:=\sum_{i=1}^{\infty}\int_0^1\lambda^2(\bar n_i)(t_{\bar n_i}(x)-s_{n_i}(x))^2\,dx=\sum_{i=1}^{\infty}\lambda^2(\bar n_i)\sum_{k=1}^{\bar n_i}d^2_{\bar n_i k}c_k^2<\infty.$$

With the aid of Lemma 2 (a) we have $\lambda(n)=O(n^p)$. Choosing $s=p+1$ in Lemma 1 (a), we get $(0<q'<q<1)$

$$\sum\nolimits_1:=\sum_{i=1}^{\infty}\lambda^2(\bar n_i)\sum_{k=1}^{[q'\bar n_i]}d^2_{\bar n_i k}c_k^2=O(1)\sum_{i=1}^{\infty}\frac{1}{\bar n_i^2}\sum_{k=1}^{\infty}c_k^2<\infty.$$

Finally, with Lemma 1 (b) and Lemma 2 (b), we get

$$\sum\nolimits_2=\sum_{i=1}^{\infty}\lambda^2(\bar n_i)\sum_{k=[q'\bar n_i]+1}^{\bar n_i}d^2_{\bar n_i k}c_k^2\leqq C_{q'}^2\sum_{i=1}^{\infty}\sum_{k=[q'\bar n_i]+1}^{\bar n_i}d^2_{\bar n_i k}c_k^2\lambda^2(k)\leqq C_{q'}^2B\sum_{k=1}^{\infty}c_k^2\lambda^2(k)<\infty$$

which proves together with $\sum_1<\infty$ the convergence of $\sum$.

Using the sequence $\{n_i\}$ defined in (7) we construct the following sequence of functions

$$\sigma_n^*(x)=(1/(n+n_{i+1}-2n_i+1))\sum_{k=n_i}^{n}(s_k(x)-s_{n_i}(x)) \quad (n_i\leqq n<n_{i+1};\ i=0,1,\ldots).$$

The next lemma which is an analogue of a result of G. Sunouchi [9] can be found in [8]:

Lemma 4. *For any* $\gamma>0$,

$$\int_0^1\Big\{(1/2(n_{i+1}-n_i))\sum_{n=n_i+1}^{n_{i+1}-1}|s_n(x)-s_{n_i}(x)-\sigma_n^*(x)|^\gamma\Big\}^{2/\gamma}dx\leqq$$

$$\leqq A(\gamma)\sum_{n=n_i+1}^{n_{i+1}-1}c_n^2 \quad (i=0,1,\ldots).$$

Lemma 5. *Condition* (4) *with* $\{\lambda(n)\}\in\Lambda$ *implies*

$$\sigma_n^*(x)=o_x(1/\lambda(n)) \quad a.e.:$$

Proof. We consider

$$\Delta_i(x) := \max_{n_i<n<n_{i+1}} |\sigma_n^*(x)|^2$$

and get on account of $\sigma_{n_i}^*(x)\equiv 0$ when applying the Cauchy inequality

$$\Delta_i(x) \leqq \Big\{\sum_{n=n_i+1}^{n_{i+1}-1} |\sigma_n^*(x)-\sigma_{n-1}^*(x)|\Big\}^2 \leqq (n_{i+1}-n_i) \sum_{n=n_i+1}^{n_{i+1}-1} (\sigma_n^*(x)-\sigma_{n-1}^*(x))^2.$$

Using the relations $\lambda(n_{i+1})\leqq C^*\lambda(n_i)$ (cf. (5), Lemma 2 (b)) and $n_{i+1}-n_i<n+$ $+n_{i+1}-2n_i\leqq 2(n_{i+1}-n_i)$ if $n_i<n<n_{i+1}$, we get

$$\int_0^1 \lambda^2(n_{i+1})\Delta_i(x)\,dx =$$

$$= O(1)\lambda^2(n_i)(n_{i+1}-n_i) \sum_{n=n_i+1}^{n_{i+1}-1} (1/(n+n_{i+1}-2n_i)^4) \sum_{k=n_i+1}^{n} (k+n_{i+1}-2n_i)^2 c_k^2 =$$

$$= O(1) \sum_{k=n_i+1}^{n_{i+1}-1} c_k^2\lambda^2(k).$$

Thus $\int_0^1 \sum_{i=0}^{\infty} \lambda^2(n_{i+1})\Delta_i(x)\,dx<\infty$ and B. Levi's theorem leads us to

$$\lambda(n)|\sigma_n^*(x)| \leqq \lambda(n_{i+1}) \max_{n_i\leqq n<n_{i+1}} |\sigma_n^*(x)| = o_x(1) \quad \text{a.e.}$$

$(n_i\leqq n<n_{i+1})$ which proves our lemma.

Dependent on the order q of (E, q) and with the aid of sequence $\{\bar{n}_i\}$ (cf. (7)) we introduce a sequence $\{\bar{\lambda}(n)\}$ by the definition

$$\bar{\lambda}(n) = \lambda(\bar{n}_i) \quad (n_i \leqq n < n_{i+1};\ i = 0, 1, \ldots).$$

Then the method (E, q) transforms $\{(\lambda(n))^{-1}\}$, resp. $\{(\bar{\lambda}(n))^{-1}\}$ with the following properties:

Lemma 6. *Let* $\{\lambda(n)\}\in\Lambda$ *and* $\gamma>0$. *Then*

(a) $\sum_{k=0}^{n} e_{nk}(1/\lambda^\gamma(k)) = O(1/\lambda^\gamma(n))$,

(b) $\sum_{k=0}^{n} e_{nk}(1/\bar{\lambda}^\gamma(k)) = O(1/\bar{\lambda}^\gamma(n))$,

(c) $e_{n0} = o(1/\lambda^\gamma(n))$.

Proof. (a) On account of Lemma 2 (a) $\lambda(n)=O(n^p)$; choosing $s=p\gamma$ in Lemma 1 (a) we get with (5) and with $0<q'<q$

$$\lambda^\gamma(n) \sum_{k=1}^{n} e_{nk}(1/\lambda^\gamma(k)) \leqq \lambda^\gamma(n)\Big\{\lambda^{-\gamma}(1)\, d_{n,[q'n]+1}+\lambda^{-\gamma}([q'n]) \sum_{k=[q'n]+1}^{n} e_{nk}\Big\} = O(1).$$

(b) By the relation $\lambda(n)\leqq\bar{\lambda}(n)=O(\lambda(n))$ this case is equivalent to (a). (c) follows similarly.

3. Proof of the theorems

Proof of Theorem 1. With the aid of sequence $\{n_i\}$ (cf. (7)) depending on the method $(E, q)=(e_{nk})$ we consider with $j(k)=i$ if $n_i \leqq k < n_{i+1}$ the following estimation

$$\{\lambda(n)\tau_n^{(\gamma)}(x)\}^\gamma \leqq C(\gamma)\{\lambda^\gamma(n)e_{n0}|s_0(x)-f(x)|^\gamma +$$

$$+\lambda^\gamma(n)\sum_{k=1}^{n} e_{nk}|s_k(x)-s_{n_{j(k)}}(x)-\sigma_k^*(x)|^\gamma+\lambda^\gamma(n)\sum_{k=1}^{n} e_{nk}|\sigma_k^*(x)|^\gamma+$$

$$+\lambda^\gamma(n)\sum_{k=1}^{n} e_{nk}|s_{n_{j(k)}}(x)-f(x)|^\gamma\} =: C(\gamma)\{\tau_n^{(\mathrm{I})}(x)+\tau_n^{(\mathrm{II})}(x)+\tau_n^{(\mathrm{III})}(x)+\tau_n^{(\mathrm{IV})}(x)\}.$$

With the aid of Lemma 6 (c), Lemma 5 and Lemma 6 (a) we get a.e.

$$\tau_n^{(\mathrm{I})}(x) = o_x(1); \quad \tau_n^{(\mathrm{III})}(x) = o_x(1). \tag{8}$$

With regard to the assumptions of the theorem and Lemma 3 at first $|s_{n_i}(x)-f(x)| \leqq \leqq |s_{n_i}(x)-t_{\bar{n}_i}(x)|+|t_{\bar{n}_i}(x)-f(x)|=o_x((\lambda(\bar{n}_i))^{-1})$ is true and with Lemma 6 (b)

$$\tau_n^{(\mathrm{IV})}(x) = o_x(1) \quad \text{a.e..} \tag{9}$$

To prove $\tau_n^{(\mathrm{II})}(x)=o_x(1)$ a.e., in the following for the sake of brevity, we put

$$\delta_k(x) = s_k(x)-s_{n_{j(k)}}(x)-\sigma_k^*(x).$$

We notice at first that it suffices to consider exponents $\gamma \geqq 2$; if $\gamma<2$ it follows according to Hölder's inequality that

$$\Big\{\sum_{k=1}^{n} e_{nk}|\delta_k(x)|^\gamma\Big\}^{1/\gamma} \leqq \Big\{\sum_{k=1}^{n} e_{nk}\Big\}^{(1-\gamma/2)(1/\gamma)}\Big\{\sum_{k=1}^{n} e_{nk}|\delta_k(x)|^2\Big\}^{1/2}. \tag{10}$$

Since $\sum_{k=0}^{n} e_{nk}=1$, $\tau_n^{(\mathrm{II})}(x)=o_x(1)$ with $\gamma=2$ implies this relation for $0<\gamma<2$, too. In the next step we divide $\tau_n^{(\mathrm{II})}(x)$ and consider now with $\gamma \geqq 2$, $0<q'<q$,

$$\tau_n'(x) := \lambda^\gamma(n)\sum_{k=1}^{[q'n]} e_{nk}|\delta_k(x)|^\gamma.$$

Lemma 1 (a) and Lemma 2 (a) lead us to

$$\tau_n'(x) = O(n^{p\gamma})(A(s)/n^{s-1})\sum_{i=0}^{\overline{j(n)}}(1/(n_{i+1}-n_i))\sum_{k=n_i+1}^{n_{i+1}-1}|\delta_k(x)|^\gamma$$

and with Lemma 4 using $\{\sum a_n\}^{1/p} \leqq \sum a_n^{1/p}$ $(a_n \geqq 0; p \geqq 1)$ we get with $s=(p+1)\gamma+1$

$$\int_0^1 \{\tau_n'(x)\}^{2/\gamma}\,dx = O(1/n^2)\sum_{i=0}^{j(n)} \int_0^1 \Big\{(1/2(n_{i+1}-n_i))\sum_{k=n_i+1}^{n_{i+1}-1} |\delta_k(x)|^\gamma\Big\}^{2/\gamma}\,dx =$$

$$= O(1/n^2)\sum_{k=0}^{\infty} c_k^2.$$

This shows that $\int_0^1 \sum_{n=1}^{\infty} \{\tau_n'(x)\}^{2/\gamma}\,dx = O(1)\sum_{k=0}^{\infty} c_k^2 < \infty$, i.e.

(11) $$\tau_n'(x) = o_x(1) \quad \text{a.e..}$$

In the last step of the proof it remains to consider

$$\tau_n''(x) = \lambda^\gamma(n) \sum_{k=[q'n]+1}^{n} e_{nk}|\delta_k(x)|^\gamma.$$

With the aid of (5) and estimation $e_{nk}=O(n^{-1/2})$ (cf. A. Rényi [6], p. 127) it holds

$$\tau_n''(x) = O(1) \sum_{k=[q'n]+1}^{n} (\lambda^\gamma(k)/\sqrt{k})|\delta_k(x)|^\gamma.$$

Using Lemma 4 and the facts that $\{n_i\}$ satisfies a gap condition (6) and $n_{i+1}=O(n_i)$, we obtain by $\gamma \geqq 2$

$$\int_0^1 \Big\{\sum_{k=2}^{\infty} (\lambda^\gamma(k)/\sqrt{k})|\delta_k(x)|^\gamma\Big\}^{2/\gamma}\,dx \leqq \int_0^1 \sum_{i=0}^{\infty}\Big\{\sum_{k=n_i+1}^{n_{i+1}-1} (\lambda^\gamma(k)/\sqrt{k})|\delta_k(x)|^\gamma\Big\}^{2/\gamma}\,dx =$$

$$= O(1)\sum_{i=0}^{\infty} \lambda^2(n_i) \int_0^1 \Big\{(1/2(n_{i+1}-n_i))\sum_{k=n_i+1}^{n_{i+1}-1} |\delta_k(x)|^\gamma\Big\}^{2/\gamma}\,dx =$$

$$= O(1)\sum_{i=0}^{\infty} \lambda^2(n_i) \sum_{k=n_i+1}^{n_{i+1}} c_k^2 = O(1)\sum_{k=2}^{\infty} c_k^2\lambda^2(k) < \infty.$$

In the same way as I. J. Maddox [4] did we conclude that $\tau_n''(x)=o_x(1)$ a.e. which finally shows together with (11) that $\tau_n^{(\mathrm{II})}(x)=o_x(1)$ a.e.. Considering in addition (8) and (9), the proof of Theorem 1 is complete.

Proof of Theorem 2. (a) It is easy to see that the number of the members in $\{m_i^*\}$ between two adjacent m_k and m_{k+1} is bounded, if both sequences satisfy (6). Defining $k(i)$ by $m_{k(i)} \leqq m_i^* < m_{k(i)+1}$, we get with Lemma 3

$$\sum_{i=0}^{\infty} \int_0^1 \lambda^2(m_i^*)\big(s_{m_i^*}(x) - s_{m_{k(i)}}(x)\big)^2\,dx =$$

$$= O(1)\sum_{i=0}^{\infty} \lambda^2(m_i^*) \sum_{k=m_i^*+1}^{m_{i+1}^*} c_k^2 = O(1)\sum_{k=1}^{\infty} c_k^2\lambda^2(k) < \infty,$$

i.e. $s_{m_i^*}(x)-s_{m_{k(i)}}(x)=o_x((\lambda(m_i^*))^{-1})$ a.e., which together with $s_{m_{k(i)}}(x)-f(x)= =o_x((\lambda(m_i^*))^{-1})$ a.e. proves the assertion.

(b) We have to prove

$$\mu_i(x) := \max_{m_i<n<m_{i+1}} |t_n(x)-t_{m_i}(x)| = o_x(1/\lambda(m_{i+1})) \quad \text{a.e..}$$

Similarly to [7] (p. 25) we get with the aid of the Cauchy inequality and taking into account condition (6)

$$\mu_i(x) \leqq \Big\{ \sum_{\nu=m_i+1}^{m_{i+1}} \nu(t_\nu(x)-t_{\nu-1}(x))^2\Big\}^{1/2} \Big\{ \sum_{\nu=m_i+1}^{m_{i+1}} 1/\nu\Big\}^{1/2} =$$
$$= O(1)\Big\{(m_{i+1}/\sqrt{m_i}) \sum_{\nu=m_i+1}^{m_{i+1}} (t_\nu(x)-t_{\nu-1}(x))^2\Big\}^{1/2}.$$

By virtue of the identity $t_\nu(x)-t_{\nu-1}(x)=\sum_{k=1}^{\nu}(k/\nu)e_{\nu k}c_k\varphi_k(x)$ (cf. K. Knopp, G. G. Lorentz [2]) and $m_{i+1}=O(m_i)$

$$\int_0^1 \lambda^2(m_{i+1})\mu_i^2(x)\,dx = O(\lambda^2(m_i))\sqrt{m_i} \sum_{\nu=m_i+1}^{m_{i+1}} \sum_{k=1}^{\nu} ((k/\nu)e_{n\nu})^2 c_k^2. \tag{12}$$

With an arbitrarily chosen q', $0<q'<q$, we divide the inner sums and consider the terms (cf. Lemma 2 (a), Lemma 1 (a), $s=p+3/2$)

$$\begin{aligned}\lambda^2(m_i)\sqrt{m_i} \sum_{\nu=m_i+1}^{m_{i+1}-1} \sum_{k=1}^{[q'm_i]} ((k/\nu)e_{\nu k})^2 c_k^2 =\\ = O(1)m_i^{2p}m_i m_i^{-2s}\sum_{k=1}^{\infty} c_k^2 = O(1)\frac{1}{m_i^2}\sum_{k=1}^{\infty} c_k^2.\end{aligned} \tag{13}$$

Next, using the estimation $(k/\nu)e_{\nu k}=\binom{\nu-1}{k-1}q^k(1-q)^{\nu-k}=O(\nu^{-1/2})$ (cf. A. Rényi [6], p. 127) and the conditions $m_{i+1}=O(m_i)$ and (5), we get

$$\lambda^2(m_i)\sqrt{m_i} \sum_{\nu=m_i+1}^{m_{i+1}-1} \sum_{k=[q'm_i]+1}^{\nu} ((k/\nu)e_{\nu k})^2 c_k^2 \leqq$$
$$\leqq C_{q'}^2 \sum_{k=1}^{m_{i+1}-1} c_k^2\lambda^2(k) \sum_{\nu=\max(m_i,k)}^{m_{i+1}-1} \binom{\nu-1}{k-1} q^k(1-q)^{\nu-k}$$

and this together with (13) yields (cf. (12))

$$\int_0^1 \sum_{i=1}^{\infty} \lambda^2(m_{i+1})\mu_i^2(x)\,dx = O(1)\sum_{i=1}^{\infty}(1/m_i^2)\sum_{k=1}^{\infty} c_k^2 +$$
$$+O(1)\sum_{k=1}^{\infty} c_k^2\lambda^2(k)q^k \sum_{\nu=k}^{\infty}\binom{\nu-1}{k-1}(1-q)^{\nu-k} = O(1)\sum_{k=1}^{\infty} c_k^2 + O(1)\sum_{k=1}^{\infty} c_k^2\lambda^2(k) < \infty$$

which proves $\mu_i(x)=o_x((\lambda(m_{i+1}))^{-1})$ a.e.. Together with the assumption $|t_{m_i}(x)-f(x)|=$ $=o_x((\lambda(m_i))^{-1})$ a.e. the statement of the theorem is evident.

(c) This is a consequence of Lemma 3 and of the assertions (a) and (b).

4. Proof of the remarks

Remark 1 follows easily from Theorem A and the relation (analogous to (10))

$$|t_n(x)-f(x)| \leqq \tau_n^{(1)}(x) \leqq \tau_n^{(\gamma)}(x) \quad (\gamma > 1).$$

Proof of Remark 2. (a) We construct at first for a given (E, q) the following sequence $\{n_k^*\}$: With fixed ε, $0<q-\varepsilon<q$, we take at first n_0 such that $n_0^*(q-\varepsilon)>1$. Assuming that $n_0^*, \dots, n_{k-1}^*$ are determined we choose n_k^* as the smallest number which satisfies $(q-\varepsilon)n_k^*>n_{k-1}^*+\sqrt{n_{k-1}^*}$.

(b) Putting $\nu_k=[\sqrt{n_k^*}/4]$ we define our orthogonal system $\{\psi_n(x)\}$ with the aid of the Rademacher functions $r_n(x)=\operatorname{sign}(\sin(2^n\pi x))$, $0\leqq x\leqq 1$, $n=0, 1, \dots$. To this end we consider the sets

$$I_k^{(0)} = [0, 1-1/2^{n_k^*}), \quad I_k^{(1)} = [1-1/2^{n_k^*}, 1-1/2^{n_k^*+1}],$$
$$I_k^{(2)} = (1-1/2^{n_k^*+1}, 1-1/2^{n_k^*+2}), \quad I_k^{(3)} = [1-1/2^{n_k^*+2}, 1]$$

$(k=0, 1, \dots)$. Let us further denote, for an arbitrary interval $I=(a, b)$ (or $I=[a, b]$),

$$f(x;\ I) = f((x-a)/(b-a)) \quad (x\in I).$$

Then we define $\{\psi_n(x)\}$ in the following way:

$$\psi_n(x) = r_n(x), \quad 0 \leqq n \leqq n_0^*, \quad \text{resp.} \quad n_k^*+4\nu_k < n \leqq n_{k+1}^*, \quad k = 0, 1, \dots;$$

and if $n_k^*<n\leqq n_k^*+4\nu_k$ we distinguish four cases: if $n=n_k^*+4j+1$ $(0\leqq j<\nu_k)$:

$$\text{(14a)} \qquad \psi_n(x) = \begin{cases} (2^{n_k-2}/(2^{n_k}-1))^{1/2} r_n(x) & (x\in I_k^{(0)}),\\ (2^{n_k-1})^{1/2} r_{j+1}(x;\ I_k^{(1)}) & (x\in I_k^{(1)}),\\ (2^{n_k})^{1/2} r_{j+1}(x;\ I_k^{(2)}) & (x\in I_k^{(2)}),\\ (2^{n_k})^{1/2} r_{j+1}(x;\ I_k^{(3)}) & (x\in I_k^{(3)}); \end{cases}$$

if $n = n_k^*+4j+2$ $(0 \leqq j < \nu_k)$:

$$\text{(14b)} \qquad \psi_n(x) = \begin{cases} -\psi_{n-1}(x) & (x\in I_k^{(0)} \text{ resp. } x\in I_k^{(1)}),\\ \psi_{n-1}(x) & (x\in I_k^{(2)} \text{ resp. } x\in I_k^{(3)}); \end{cases}$$

if $n = n_k^*+4j+3$ $(0 \leqq j < \nu_k)$:

$$\text{(14c)} \qquad \psi_n(x) = \begin{cases} -\psi_{n-2}(x) & (x\in I_k^{(0)} \text{ resp. } x\in I_k^{(2)}),\\ \psi_{n-2}(x) & (x\in I_k^{(1)} \text{ resp. } x\in I_k^{(3)}); \end{cases}$$

if $n = n_k^* + 4j \quad (1 \leqq j \leqq \nu_k)$:

(14d)
$$\psi_n(x) = \begin{cases} \psi_{n-3}(x) & (x \in I_k^{(0)} \text{ resp. } x \in I_k^{(3)}), \\ -\psi_{n-3}(x) & (x \in I_k^{(1)} \text{ resp. } x \in I_k^{(2)}). \end{cases}$$

We note that for $n_k^* < n, m \leqq n_k^* + 4\nu_k$, $l = 0, \ldots, 3$

$$\int_{I_k^{(l)}} \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1/4 & n_k^* + 4j < m,\ n \leqq n_k^* + 4(j+1) \\ 0 & \text{otherwise} \end{cases}$$

and for $0 < n \leqq n_0^*$ resp. $n_k^* + 4\nu_k < n \leqq n_{k+1}^*$, $0 \leqq m < \infty$,

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

It is then easy to see that $\{\psi_n(x)\}$ is an orthonormal system on $[0, 1]$.

Finally we define the coefficients c_n with an arbitrarily chosen $\alpha > 1/4$:

$$c_n = \begin{cases} 0 & 0 \leqq n \leqq n_0^* \text{ resp. } n_k^* + 4\nu_k < n \leqq n_{k+1}^*, \quad k = 0, 1, \ldots, \\ (n_k^*)^{-\alpha} & n_k^* < n \leqq n_k^* + 4\nu_k, \quad k = 0, 1, \ldots. \end{cases}$$

Because of the relation $n_{k+1}^*/n_k^* \geqq 1/(q-\varepsilon) > 1$,

$$\sum_{n=0}^{\infty} c_n^2 = \sum_{k=0}^{\infty} (n_k^*)^{-2\alpha} \sum_{n=n_k^*+1}^{n_k^*+4\nu_k} 1 = O(1) \sum_{k=0}^{\infty} (n_k^*)^{-2\alpha+1/2} < \infty,$$

(c) To prove the statement of Remark 2 we consider the partial sums $s_m(x)$ of the series $\sum_{n=0}^{\infty} c_n \psi_n(x)$. If $m > n_{k_0}^* + 4\nu_{k_0}$ we have on $I_{k_0}^{(0)}$ (cf. (14a)—(14d))

(15a)
$$s_m(x) = s_{n_{k_0}^* + 4\nu_{k_0}}(x) = f(x),$$

$$n_k^* + 4\nu_k \leqq m \leqq n_{k+1}^* \quad \text{resp.} \quad m = n_k^* + 2j \quad (1 \leqq j \leqq 2\nu_k), \quad x \in I_{k_0}^{(0)},$$

resp.

(15b)
$$s_m(x) = s_{n_{k_0}^* + 4\nu_{k_0}}(x) + (n_k^*)^{-\alpha}(2^{n_k^*-2}/(2^{n_k^*}-1))^{1/2} r_m(x),$$

$$m = n_k^* + 4j + 1 \quad (0 \leqq j < \nu_k;\ k > k_0), \quad x \in I_{k_0}^{(0)},$$

resp.

(15c)
$$s_m(x) = s_{n_{k_0}^* + 4\nu_{k_0}}(x) - (n_k^*)^{-\alpha}(2^{n_k^*-2}/(2^{n_k^*}-1))^{1/2} r_{m-2}(x),$$

$$m = n_k^* + 4j + 3 \quad (0 \leqq j < \nu_k;\ k > k_0), \quad x \in I_{k_0}^{(0)}.$$

Consequently $\{s_m(x)\}$ converges on $[0, 1]$ and $f(x) = s_{n_k^* + 4\nu_k}(x)$ if $x \in I_k^{(0)}$ $(k = 0, 1, \ldots)$. Let us now consider $t_n(x)$ on $I_{k_0}^{(0)}$; we assume $n_k^* < n \leqq n_{k+1}^*$ $(k > k_0)$ and get with

(15a)—(15c)

$$t_n(x)-f(x) = \sum_{m=0}^{n} e_{nm}(s_m(x)-f(x)) =$$

$$= \sum_{m=0}^{n^*_{k_0}+1} e_{nm}(s_m(x)-f(x)) + \sum_{m=n^*_{k_0}+1+1}^{n^*_{k-1}+4\nu_{k-1}} e_{nm}(s_m(x)-s_{n^*_{k_0}+4\nu_{k_0}}(x)) +$$

$$+ \sum_{j:0\leqq j<(1/2)(n-n^*_k)} e_{n,n^*_k+2j+1}(s_m(x)-s_{n^*_{k_0}+4\nu_{k_0}}(x)) = t_n^{(1)}(x)+t_n^{(2)}(x)+t_n^{(3)}(x).$$

It follows immediately as a consequence of Lemma 1 (with s sufficiently large), regarding $n^*_{k-1}+4\nu_{k-1}<(q-\varepsilon)\cdot n^*_k$

(16) $$t_n^{(1)}(x) = o_x(n^{-\alpha});\ t_n^{(2)}(x) = o_x(n^{-\alpha}) \quad (x\in I^{(0)}_{k_0}).$$

Putting now $\mu(n)=\min\{(n-n^*_k)/4, \nu_k\}$ and $\delta(n)=1$ if $n=n^*_k+4j+1$ $(0\leqq j<\nu_k)$ resp. $\delta(n)=0$ for all other n, $n^*_k<n\leqq n^*_{k+1}$, we find with (15b), (15c)

$$t_n^{(3)}(x) = (n^*_k)^{-\alpha}(2^{n^*_k-2}/(2^{n^*_k}-1))^{1/2}\Big\{\sum_{j=0}^{\mu(n)-1} r_{n^*_k+4j+1}(x)(e_{n,n^*_k+4j+1}-e_{n,n^*_k+4j+3})+$$

$$+\delta(n)r_n(x)e_{nn}\Big\} \quad (x\in I^{(0)}_{k_0})$$

which leads with respect to the formerly used estimation $e_{nk}=O(n^{-1/2})$ and to the relations $e_{nk}<e_{n,k+1}$ $(k\leqq q(n+1)-1)$ resp. $e_{nk}>e_{n,k+1}$ $(k>q(n+1)-1)$ to

$$t_n^{(3)}(x) = O(n^{-\alpha})\Big\{\sum_{j=0}^{\nu_k} |e_{n,n^*_k+4j+1}-e_{n,n^*_k+4j+3}|+1/\sqrt{n}\Big\} =$$

$$= O(n^{-\alpha-1/2}) = o(n^{-\alpha}) \quad (x\in I^{(0)}_{k_0}).$$

Together with (16) we finally get

(17) $$t_n(x) = o_x(1/n^{\alpha}) \quad (x\in[0, 1]).$$

(d) On the other hand let us consider for $\tilde{n}_k$ with $[q\cdot\tilde{n}_k]=n^*_k$ the means

$$\tau^{(\gamma)}_{\tilde{n}_k}(x) = \Big\{\sum_{m=0}^{\tilde{n}_k} e_{\tilde{n}_k,m}|s_m(x)-f(x)|^{\gamma}\Big\}^{1/\gamma} \geqq \Big\{\sum_{m=n^*_k}^{n^*_k+4\nu_k} e_{\tilde{n}_k,m}|s_m(x)-f(x)|^{\gamma}\Big\}^{1/\gamma}.$$

Taking into account equations (15a)—(15c) we find on an arbitrarily chosen $I^{(0)}_{k_0}$ for $k>k_0$

$$\tau_{\tilde{n}_k}(x) \geqq \tilde{C}\Big\{(n^*_k)^{-\gamma\alpha}\sum_{j=0}^{2\nu_k-2} e_{\tilde{n}_k,n^*_k+2j+1}\Big\}^{1/\gamma} \quad \text{a.e.} \quad (x\in I^{(0)}_{k_0}).$$

Regarding the estimation $e_{nk}\geqq\tilde{C}(q)n^{-1/2}$ if $qn\leqq k\leqq qn+\sqrt{n}$ (cf. A. RÉNYI [6], p. 31), the last relation yields because of $\nu_k\geqq\sqrt{n^*_k}/4$: $\tau^{(k)}_{\tilde{n}_k}(x)\geqq\tilde{C}^*(\tilde{n}_k)^{-\alpha}$ a.e.

($x \in I_{k_0}^{(0)}$), i.e.

(18) $$\tau_n^{(\gamma)}(x) \neq o_x(1/n^{\alpha}) \quad \text{a.e.} \quad (x \in [0, 1]).$$

Now we may define $\lambda(n) = n^{\alpha}$. The statement of Remark 2 is proved by (17) and (18).

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