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Integrability of Rees—Stanojević sums

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1. A sequence $\{a_n\}$ of positive numbers is called quasi-monotone if $n^{-\beta}a_n\downarrow 0$ for some β, or equivalently, if $a_{n+1} \leqq a_n(1+\alpha/n)$.

RAM [3] defined that a sequence $\{a_k\}$ of numbers satisfies condition (S*) if $a_k\to 0$ as $k\to\infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is quasi-monotone,

(1) $$\sum_{k=0}^{\infty} A_k < \infty$$

and

(2) $$|\Delta a_k| \leqq A_k \quad \text{for all} \quad k.$$

Condition (S*) is weaker than the condition (S) of Sidon introduced in [4].

REES and STANOJEVIĆ [2] (see also GARRETT and STANOJEVIĆ [1]) introduced the modified cosine sums

(1.1) $$g_n(x) = (1/2)\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx$$

and obtained a necessary and sufficient condition for the integrability of the limit of (1.1).

RAM [3] proved the following theorem in which he showed that condition (S*) is sufficient for the integrability of the limit of (1.1).

Theorem A. *Let the sequence $\{a_k\}$ satisfy condition* (S*). *Then*

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[(1/2)\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx\right]$$

exists for $x\in(0, \pi]$, and $g(x)\in L(0, \pi)$.

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We say that a sequence $\{a_n\}$ of numbers satisfies condition (S**) if $\{a_n\}$ is a null sequence and

(3) $$n\Delta a_n = O(1) \quad (n \to \infty).$$

We claim that our condition (S**) includes a more general class of sequences $\{a_n\}$ than that of Ram's condition (S*).

Example. The sequence

$$a_n = \frac{(-1)^{n+1}}{n \log(n+1)} \quad (n = 1, 2, \ldots)$$

does not satisfy the conditions (S*) of Ram as $|\Delta a_n| \geqq (n \log(n+1)^{-1})$ and so $\sum |\Delta a_n| = \infty$. This in fact contradicts conditions (1) and (2) of (S*). On the other hand this sequence satisfies the condition (3) of (S**).

The object of this paper is to show that condition (S**) is sufficient for the integrability of the limit of (1.1).

Theorem. *Let the sequence* $\{a_n\}$ *satisfy condition* (S**). *Then*

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{n} [(1/2)\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx]$$

exists for $x \in (0, \pi]$, *and* $g(x) \in L(0, \pi)$.

2. We require the following lemma for the proof of our theorem.

Lemma. *Let* $\{a_n\}$ *be a null sequence and* $n\Delta a_n = O(1)$, $n \to \infty$. *Then*

$$\sum (n+1)\Delta^2 a_n < \infty.$$

Proof. Applying Abel's transformation, we find

$$\sum_{m=0}^{n} \Delta a_m = \sum_{m=0}^{n} 1 \cdot \Delta a_m = \sum_{m=0}^{n-1} (m+1)\Delta^2 a_m + (n+1)\Delta a_n,$$

and since $(n+1)\Delta a_n \to 0$ but $\sum_{m=0}^{n} \Delta a_m = a_0 - a_n \to a_0$ as $a_n \to 0$, $n \to \infty$, then

$$\sum_{m=0}^{n-1} (m+1)\Delta^2 a_m \to a_0,$$

i.e. the series $\sum_{m=0}^{\infty} (m+1)\Delta^2 a_m$ converges.

3. Proof of the Theorem. We have

$$g_n(x) = \sum_{k=1}^{n} [(1/2)\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx] =$$

$$= \sum_{k=1}^{n} (1/2)\Delta a_k + \sum_{k=1}^{n} \Delta a_k \cos kx - a_{n+1} D_n(x) + (1/2) a_{n+1}.$$

Using Abel's transformation, we obtain

$$(4.1)\quad \begin{aligned} g_n(x) &= \sum_{k=1}^{n} (1/2)\Delta a_k + \sum_{k=1}^{n-1} \Delta a_k(D_k(x)+1/2) + a_n(D_n(x)+1/2) - \\ &- a_{n+1}D_n(x) - a_1 + (1/2)a_{n+1} = \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x). \end{aligned}$$

Applying again Abel's transformation, we have

$$(4.2)\quad g_n(x) = \sum_{k=1}^{n-2} (k+1)\Delta^2 a_k F_k(x) + n\Delta a_{n-1} F_{n-1}(x) + a_n D_n(x) - a_{n+1} D_n(x),$$

where $D_n(x)$ and $F_n(x)$ denotes the Dirichlet and Fejér kernels respectively.

If $x \not\equiv 0 \pmod{2\pi}$, then since $a_n \to 0$, the last two terms of the right hand side of (4.2) tends to zero as $n \to \infty$. Moreover, at $x \not\equiv 0 \pmod{2\pi}$ $F_n(x)$ always remains finite as $n \to \infty$ and since $n\Delta a_n \to 0$ therefore $n\Delta a_{n-1} F_{n-1}(x) \to 0$ as $n \to \infty$.

Since $F_k(x) = o(1/(k+1)x^2)$ if $x \not\equiv 0$ and $\sum (k+1)\Delta^2 a_k$ is convergent then the series $\sum_{k=1}^{\infty} (k+1)\Delta^2 a_k F_k(x)$ converges. Hence for $x \not\equiv 0 \pmod{2\pi}$

$$g(x) = \lim_{n\to\infty} g_n(x) = \sum_{k=1}^{\infty} (k+1)\Delta^2 a_k F_k(x).$$

The integrability of $g(x)$ follows from the lemma; indeed, we have

$$\int_0^\pi g(x)\,dx = \sum_{k=1}^{\infty} (k+1)\Delta^2 a_k \int_0^\pi F_k(x)\,dx = (\pi/2) \sum_{k=1}^{\infty} (k+1)\Delta^2 a_k < \infty;$$

since $\int_0^\pi F_n(x)\,dx = \pi/2$.

Corollary. *Let $\{a_n\}$ be a null sequence and $n\Delta a_n = o(1)$, $n \to \infty$. Then*

$$(1/x) \sum_{k=1}^{\infty} \Delta a_k \sin(k+1/2)x = h(x)/x$$

converges for $x \in (0, \pi]$, and $h(x)/x \in L(0, \pi)$.

Proof. From (4.1) we have

$$g(x) = \sum_{k=1}^{\infty} \Delta a_k D_k(x) = \left(\sum_{k=1}^{\infty} \Delta a_k \sin(k+1/2)x\right)/2\sin(x/2) = h(x)/2\sin(x/2).$$

According to the theorem, $g(x)$ exists for $x \not\equiv 0$, and $g(x) \in L[0, \pi]$ if $n\Delta a_n = o(1)$, which implies our result.

References

[1] J. W. GARRETT and C. V. STANOJEVIĆ, On L^1 convergence of certain cosine sums, *Proc. Amer. Math. Soc.*, **54** (1976), 101—105.

[2] C. S. REES and C. V. STANOJEVIĆ, Necessary and sufficient condition for integrability of cosine sums, *J. Math. Anal. Appl.*, **43** (1973), 579—586.

[3] B. RAM, Integrability of Rees—Stanojević sums, *Acta Sci. Math.*, **42** (1980), 153—155.

[4] S. SIDON, Hinreichende Bedingungen für den Fourier-Charakter einer trigonometrischen Reihe, *J. London Math. Soc.*, **14** (1939), 158—160.

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