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The approximate point spectrum of a pure quasinormal operator

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In this paper all Hilbert spaces are over the complex scalars. If $\mathscr{H}$ is a Hilbert space, let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. (In this paper the term operator shall mean an element of $\mathscr{L}(\mathscr{H})$.) If T is an operator, let $\sigma(T)$ denote the spectrum of T, let $\sigma_{ap}(T)$ denote the approximate point spectrum of T, let $\tilde{T}$ denote the image of T in the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}$ under the natural projection, where $\mathscr{C}$ denotes the ideal of all compact operators in $\mathscr{L}(\mathscr{H})$, and let $\sigma_e(T)$ denote the essential spectrum of T, i.e., $\sigma_e(T)=\sigma(\tilde{T})$. C. R. PUTNAM proved in [12] that the planar Lebesgue measure of the spectrum of a pure hyponormal operator is positive. There exist, however, pure hyponormal operators which have essential spectra of measure zero. (The unilateral shift is an example.) Let T be a pure hyponormal operator. It follows from Putnam's inequality [12] that $\pi\|\tilde{T}^*\tilde{T}-\tilde{T}\tilde{T}^*\|\leqq m_2(\sigma_e(T))$, where m_2 denotes planar Lebesgue measure. So if $m_2(\sigma_e(T))=0$, then the self-commutator T^*T-TT^* is compact. The converse is not true, even in the subnormal case. (See Example 2.4.) Yet the following question which was posed by the present author in [15] remains open: If T is a pure subnormal operator that has a finite rank self-commutator, then is $m_2(\sigma_e(T))=0$? The results of this paper were motivated by the above question. In Section 1 we show that the above question is equivalent to a similar question about the approximate point spectrum and to a question posed by J. Conway about the measure of the spectrum of the minimal normal extension of a pure subnormal operator. In Section 2 we compute the approximate point spectrum of a pure quasinormal operator and then present a formula for the planar Lebesgue measure of it. In Section 3 we present a class of pure subnormal operators for which the answer to the above question is affirmative.

We present here some terminology and notation. Let T be an operator. Recall that T is *hyponormal* if $T^*T-TT^*\geqq 0$, T is *subnormal* if T has a normal extension,

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and T is quasinormal if T commutes with T^*T. It is known that each quasinormal operator is subnormal and each subnormal operator is hyponormal. Each operator T is unitarily equivalent to $T_1 \oplus T_2$, where T_1 is normal and T_2 is pure, i.e., if $\mathcal{M}$ is a reducing subspace for T_2 and $T_2|\mathcal{M}$ is normal, then $\mathcal{M}=(0)$. The operator T_1 is the *normal part* of T and T_2 is the *pure part* of T. (Note that either of the operators T_1 or T_2 may be the zero operator on the zero Hilbert space.) Observe that if T is a hyponormal operator, then any eigenspace of T reduces T. Thus the point spectrum of a pure hyponormal operator is empty. We shall use this fact freely. Finally, let $\mathcal{K}(T)$ denote the kernel of T and $\mathcal{R}(T)$ the range of T.

1. Pure subnormal operators with finite rank self-commutators

We begin this section by observing that if T is a pure subnormal operator on a Hilbert space $\mathcal{H}$ and N is its minimal normal extension on a Hilbert space $\mathcal{K}$, where $\mathcal{H} \subseteq \mathcal{K}$, then N is unitarily equivalent to the operator

$$\begin{bmatrix} T & X \\ 0 & S^* \end{bmatrix} \tag{1}$$

on $\mathcal{H} \oplus \mathcal{H}_1$ for some Hilbert space $\mathcal{H}_1$. But since $\mathcal{K}$ is the closed linear span of $\{(N^*)^n x \colon x \in \mathcal{H},\ n$ a nonnegative integer$\}$ [6], we have $\dim(\mathcal{K}) = \dim(\mathcal{H})$. Olin has observed in [10] that N^* is the minimal normal extension of S if and only if T is pure (and it follows that N is the minimal normal extension of T if and only if S is pure). Thus, since T is pure, an argument similar to the one above shows that $\dim(\mathcal{K}) = \dim(\mathcal{H}_1)$. Hence N is unitarily equivalent to the operator in (1) on $\mathcal{H} \oplus \mathcal{H}$.

We now state the following questions.

Questions. Suppose that T is a pure subnormal operator that has a finite rank self-commutator and suppose that N is its minimal normal extension.

A. Then is $m_2(\sigma_e(T))=0$?
B. Then is $m_2(\sigma(N))=0$?
C. Then is $m_2(\sigma_{ap}(T))=0$?

As mentioned earlier Question A was posed by the present author in [15] and Question B was posed by J. Conway in [6]. We shall show that the three questions are equivalent. In order to see that Questions A and B are equivalent, let T and N be as above and observe that N is unitarily equivalent to the operator in (1) on $\mathcal{H} \oplus \mathcal{H}$. The operator S is also a pure subnormal operator (and is called the *dual* of T). Since N is normal, a matrix calculation shows that $T^*T - TT^* = XX^*$

and $S^*S-SS^*=X^*X$. Since T has a finite rank self-commutator, S also has a finite rank self-commutator and X has finite rank. Hence $\sigma_e(N)=\sigma_e(T)\cup\sigma_e(S^*)$. Since N is normal, $\sigma(N)\setminus\sigma_e(N)$ is countable; thus $m_2(\sigma(N))=m_2(\sigma_e(N))$. It follows that $m_2(\sigma(N))=0$ if and only if $m_2(\sigma_e(T))=0$ and $m_2(\sigma_e(S))=0$. Now suppose that the answer to Question A is affirmative. Then, since both T and S are pure subnormal operators having finite rank self-commutators, $m_2(\sigma_e(T))=$ $=m_2(\sigma_e(S))=0$. Thus $m_2(\sigma(N))=0$. So the answer to Question B is affirmative also. On the other hand it is clear from the above discussions that if the answer to Question B is affirmative, then the answer to Question A is affirmative also. So Questions A and B are equivalent.

The following theorem and corollary will show that Question C is equivalent to Questions A and B. Recall that an operator T is *semi-Fredholm* if either $\mathcal{K}(T)$ or $\mathcal{K}(T^*)$ is finite dimensional and $\mathcal{R}(T)$ is closed and is *Fredholm* if both $\mathcal{K}(T)$ and $\mathcal{K}(T^*)$ are finite dimensional and $\mathcal{R}(T)$ is closed. Recall also that $\sigma_e(T)=$ $=\{\lambda\in\mathbf{C}\colon T-\lambda$ is not Fredholm$\}$ and $\sigma_{ap}(T)=\{\lambda\in\mathbf{C}\colon T-\lambda$ is not bounded below$\}$. If T is a pure hyponormal operator, then, since the point spectrum of T is empty, $\sigma_{ap}(T)=\{\lambda\in\mathbf{C}\colon \mathcal{R}(T-\lambda)$ is not closed$\}$ and thus $\sigma_{ap}(T)\subseteq\sigma_e(T)$. If T is semi-Fredholm, let $i(T)=\dim(\mathcal{K}(T))-\dim(\mathcal{K}(T^*))$ denote the index of T.

Theorem 1.1. *Suppose that T is a hyponormal operator and that T^*T-TT^* has rank n, where n is a nonnegative integer. Then if λ is a complex number and $T-\lambda$ is semi-Fredholm, then $0\geqq i(T-\lambda)\geqq -n$.*

The following corollary follows from Theorem 1.1 and from the above characterizations of the essential spectrum and the approximate point spectrum of a pure hyponormal operator.

Corollary 1.2. *If T is a pure hyponormal operator and T^*T-TT^* has finite rank, then $\sigma_{ap}(T)=\sigma_e(T)$.*

The following lemma is needed in the proof of Theorem 1.1. Its proof is an easy exercise.

Lemma 1.3. *If $\mathcal{H}$ is a Hilbert space, $\mathcal{M}$ an arbitrary subspace of $\mathcal{H}$, $\mathcal{N}$ a finite dimensional subspace of $\mathcal{H}$, and $\mathcal{H}=\mathcal{M}+\mathcal{N}$, then* $\dim(\mathcal{M}^{\perp})\leqq\dim(\mathcal{N})$.

Proof of Theorem 1.1. We first consider the case that T is a pure hyponormal operator on a Hilbert space $\mathcal{H}$. Let $P=\sqrt{T^*T-TT^*}$. Then for each complex number λ, $(T-\lambda)^*(T-\lambda)=(T-\lambda)(T-\lambda)^*+P^2$. By Theorem 2.2 of [8] we have $\mathcal{R}((T-\lambda)^*)\subseteq\mathcal{R}((T-\lambda))+\mathcal{R}(P)$. Since $\mathcal{R}(P)$ is finite dimensional, $\mathcal{R}((T-\lambda))^-+$ $+\mathcal{R}(P)$ is closed and contains $\mathcal{R}((T-\lambda)^*)^-$. But $\mathcal{R}((T-\lambda)^*)^-=\mathcal{H}$ since $\mathcal{K}(T-\lambda)=$ $=(0)$. Thus $\mathcal{H}=\mathcal{R}((T-\lambda))^-+\mathcal{R}(P)$. Lemma 1.3 implies that $\dim(\mathcal{K}((T-\lambda)^*))=$ $=\dim(\mathcal{R}((T-\lambda))^{\perp})\leqq\dim(\mathcal{R}(P))=n$. Thus if $T-\lambda$ is semi-Fredholm, then $0\geqq i(T-\lambda)\geqq -n$. The general case follows readily.

2. The approximate point spectrum of a pure quasinormal operator

If $\mathscr{H}$ is a Hilbert space, let $\hat{\mathscr{H}}=\sum_{k=1}^{\infty}\oplus\mathscr{H}_k$, where for each positive integer k, $\mathscr{H}_k=\mathscr{H}$. If $T\in\mathscr{L}(\mathscr{H})$, define an operator $\hat{T}$ in $\mathscr{L}(\hat{\mathscr{H}})$ by $\hat{T}=\sum_{k=1}^{\infty}\oplus T_k$, where for each positive integer k, $T_k=T$. Let $V_{\mathscr{H}}$ denote the unilateral shift on $\hat{\mathscr{H}}$, i.e., $V_{\mathscr{H}}(x_1, x_2, \ldots)=(0, x_1, x_2, \ldots)$ for each $(x_1, x_2, \ldots)$ in $\hat{\mathscr{H}}$. ARLEN BROWN proved in [3] that each pure quasinormal operator is unitarily equivalent to $V_{\mathscr{H}}\hat{P}$, for some Hilbert space $\mathscr{H}$ and for some positive definite operator P in $\mathscr{L}(\mathscr{H})$, i.e., P is positive and $\mathscr{K}(P)=(0)$. The present author showed in [14] that

$$\sigma(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\leqq\|P\|\},$$

$$\sigma_e(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}\cup\{\lambda\in\mathbf{C}\colon |\lambda|\leqq\|\tilde{P}\|\}$$

if $\mathscr{H}$ is infinite dimensional, and

$$\sigma_e(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}$$

if $\mathscr{H}$ is finite dimensional. Here we compute the approximate point spectrum of $V_{\mathscr{H}}\hat{P}$.

Theorem 2.1. *If P is a positive definite operator on a nonzero Hilbert space $\mathscr{H}$, then $\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}$.*

Proof. Let $\Gamma=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}$ and let E be the spectral measure of P. Suppose that $\lambda\in\mathbf{C}\setminus\Gamma$. If $|\lambda|>\|P\|=\|V_{\mathscr{H}}\hat{P}\|$, then $\lambda\notin\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. So assume that $|\lambda|<\|P\|$. There exists a positive number ε such that $(|\lambda|-\varepsilon, |\lambda|+\varepsilon)\cap\sigma(P)=\emptyset$. Let $\mathscr{M}=\mathscr{R}(E([|\lambda|+\varepsilon, \|P\|]))$, let $R=P|\mathscr{M}$, and let $Q=P|\mathscr{M}^{\perp}$. Then $V_{\mathscr{H}}\hat{P}$ is unitarily equivalent to $V_{\mathscr{M}}\hat{R}\oplus V_{\mathscr{M}^{\perp}}\hat{Q}$. Since $\|V_{\mathscr{M}^{\perp}}\hat{Q}\|=\|Q\|\leqq|\lambda|-\varepsilon$, $\lambda\notin\sigma(V_{\mathscr{M}^{\perp}}\hat{Q})$. Also since $\sigma(R)\subseteq[|\lambda|+\varepsilon, \|P\|]$, we have $\|\hat{R}x\|\geqq(|\lambda|+\varepsilon)\|x\|$ for each x in $\hat{\mathscr{M}}$. Thus $\|(V_{\mathscr{M}}\hat{R}-\lambda)x\|\geqq\|V_{\mathscr{M}}\hat{R}x\|-\|\lambda x\|=\|\hat{R}x\|-\|\lambda x\|\geqq\varepsilon\|x\|$ for each x in $\hat{\mathscr{M}}$. It follows that $\lambda\notin\sigma_{\mathrm{ap}}(V_{\mathscr{M}}\hat{R})$, and, therefore, $\lambda\notin\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. We have shown that $\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})\subseteq\Gamma$.

Now suppose that $\mu\in\Gamma$. For each positive integer n, let $\mathscr{M}_n=\mathscr{R}(E([|\mu|-1/n, |\mu|+1/n]))$, let $R_n=P|_{\mathscr{M}_n}$, and let $Q_n=P|_{\mathscr{M}_n^{\perp}}$. Note that $V_{\mathscr{H}}\hat{P}$ is unitarily equivalent to $V_{\mathscr{M}_n}\hat{R}_n\oplus V_{\mathscr{M}_n^{\perp}}\hat{Q}_n$ and that $\sigma(V_{\mathscr{M}_n}\hat{R}_n)=\{\lambda\in\mathbf{C}\colon |\lambda|\leqq\|R_n\|\}$. Since for any operator T, $\partial\sigma(T)\subseteq\sigma_{\mathrm{ap}}(T)$, we have $\{\lambda\in\mathbf{C}\colon |\lambda|=\|R_n\|\}\subseteq\sigma_{\mathrm{ap}}(V_{\mathscr{M}}\hat{R}_n)\subseteq\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. But $|\mu|\leqq\|R_n\|\leqq|\mu|+1/n$, $n=1, 2, \ldots$; thus $\|R_n\|\to|\mu|$. Hence, since $\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$ is closed, $\{\lambda\in\mathbf{C}\colon |\lambda|=|\mu|\}\subseteq\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. This argument shows that $\Gamma\subseteq\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$, and the proof is complete.

We next discuss the relation between quasinormal operators and some other important classes of operators. In the following let $\mathscr{H}$ be a separable, infinite dimensional Hilbert space, let P be a positive definite operator on $\mathscr{H}$, and let (BQT) denote the class of biquasitriangular operators in $\mathscr{L}(\mathscr{H})$. The present author showed in [15] that a hyponormal operator T belongs to (BQT) if and only if $\sigma_{ap}(T)=\sigma(T)$. This fact shall be used freely in the following discussions. The following corollary is easy to verify.

Corollary 2.2. *The operator* $V_{\mathscr{K}}\hat{P}\in$(BQT) *if and only if* $\sigma(P)=[0, \|P\|]$.

Let $(\mathrm{Ni})^-$ denote the norm-closure of the class of nilpotent operators in $\mathscr{L}(\mathscr{H})$. (See [11] for a discussion of the classes (BQT) and $(\mathrm{Ni})^-$.) Apostol, Foiaş, and Voiculescu gave the following characterization of $(\mathrm{Ni})^-$ in [1]: $(\mathrm{Ni})^-=\{T\in(\mathrm{BQT})$: both $\sigma(T)$ and $\sigma_e(T)$ are connected and $0\in\sigma_e(T)\}$. Hence we have the following corollary.

Corollary 2.3. *The operator* $V_{\mathscr{K}}\hat{P}\in$(BQT) *if and only if* $V_{\mathscr{K}}\hat{P}\in(\mathrm{Ni})^-$.

Let (EN) denote the class of essentially normal operators, i.e., $T\in$(EN) if and only if T^*T-TT^* is compact, and let $(N+K)=\{N+K\in\mathscr{L}(\mathscr{H})$: N is normal and K is compact$\}$. It is known that $(N+K)=(\mathrm{BQT})\cap(\mathrm{EN})$ [4], [11]. Suppose that $V_{\mathscr{K}}\hat{P}\in$(EN). Then P is compact and Corollary 2.2 implies that $V_{\mathscr{K}}\hat{P}\notin$(BQT). Hence there are no pure quasinormal operators in the class $(N+K)$.

In [11] C. Pearcy observed that each operator in $\mathscr{L}(\mathscr{H})$ has a nontrivial invariant subspace if and only if each operator in $(\mathrm{Ni})^-$ does. He then wrote $(\mathrm{Ni})^-$ as the disjoint union of four subsets and he conjectured that if there exists an operator in $\mathscr{L}(\mathscr{H})$ that does not have a nontrivial invariant subspace, then that operator belongs to the "mysterious" fourth subset which consists of those operators in $(\mathrm{Ni})^-$ that are neither essentially normal nor quasinilpotent. The above shows that pure quasinormal operators in (BQT) are examples of operators in this fourth subset of $(\mathrm{Ni})^-$. But, of course, pure quasinormal operators do have nontrivial invariant (and hyperinvariant) subspaces.

Even though there are no pure quasinormal operators in $(N+K)$, there are general nonnormal quasinormal operators in this class. For example, let N be a normal operator such that $\sigma(N)=\sigma_e(N)=\overline{\mathbf{D}}$, where $\mathbf{D}$ denotes the open unit disk, and let V denote the unilateral shift (of multiplicity one). Then $N\oplus V$ is clearly quasinormal and essentially normal, and, since $\sigma_{ap}(N\oplus V)=\sigma(N\oplus V)=\overline{\mathbf{D}}$, $N\oplus V\in$(BQT). Thus $N\oplus V\in(N+K)$.

The following example shows that there are also pure subnormal operators in $(N+K)$.

Example 2.4. Let S denote the Bergman shift, i.e., the Bergman operator for $\mathbf{D}$, and let N be its minimal normal extension. (See [6] for a discussion of Bergman operators.) The operator N is unitarily equivalent to the operator

$$\begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}$$

on $\mathscr{H}\oplus\mathscr{H}$, where $\mathscr{H}$ is the Hilbert space on which S acts. We shall show that the dual T of S belongs to $(N+K)$. The operator T is a pure subnormal operator. It is known that $\sigma(N)=\sigma_e(N)=\sigma(S)=\overline{\mathbf{D}}$ and $\sigma_e(S)=\partial\mathbf{D}$ [6], and it is easy to verify that $\sigma(T)\subseteq\overline{\mathbf{D}}$. Since S is a weighted unilateral shift, S has a compact self-commutator. Thus X is compact and $T\in(\mathrm{EN})$ because $S^*S-SS^*=XX^*$ and $T^*T-TT^*=X^*X$. Observe that N is a compact perturbation of $S\oplus T^*$. Suppose that $\lambda\in\mathbf{D}$. Then $N-\lambda$ is not semi-Fredholm; thus $T^*-\lambda$ is not semi-Fredholm since $S-\lambda$ is Fredholm. Hence $\bar\lambda\in\sigma_{ap}(T)$ since the point spectrum of T is empty. It follows that $\sigma_{ap}(T)=\sigma(T)=\overline{\mathbf{D}}$; thus $T\in(\mathrm{BQT})$. Consequentially, $T\in(N+K)$.

Clancey and Morrell gave an example of a pure hyponormal operator T that is not subnormal and that has a rank one self-commutator such that $\sigma_e(T)=\sigma(T)=\overline{\mathbf{D}}$ [5]. By Corollary 1.2 $\sigma_{ap}(T)=\overline{\mathbf{D}}$. Thus $T\in(\mathrm{BQT})\cap(\mathrm{EN})=(N+K)$. On the other hand if T is a pure quasinormal operator that has a finite rank self-commutator, then $T\notin(N+K)$. These facts motivate the following question.

Question 2.5. Does $(N+K)$ or, equivalently, does (BQT) contain any pure subnormal operators that have finite rank self-commutators?

Now let $\mathscr{H}$ be a nonzero Hilbert space of arbitrary dimension, let P be a positive definite operator in $\mathscr{L}(\mathscr{H})$, and let m_1 denote Lebesgue measure on the real line. It is easy to verify that

$$m_2(\sigma(V_{\mathscr{H}}\hat{P}))=\int_{[0,a]}2\pi r\,dm_1(r)\quad\text{and}\quad m_2(\sigma_e(V_{\mathscr{H}}\hat{P}))=\int_{[0,b]}2\pi r\,dm_1(r),$$

where $a=\|P\|$ and $b=\|\tilde{P}\|$ if $\mathscr{H}$ is infinite dimensional and $b=0$ otherwise. (To get the second equation we used the fact that $\{c\in\sigma(P)\colon c>\|\tilde{P}\|\}$ is countable.) We shall now develop a similar formula for $m_2(\sigma_{ap}(V_{\mathscr{H}}\hat{P}))$. We shall need the following notation and lemma.

Let $\mathscr{B}$ denote the family of Borel subsets of $[0,+\infty)$. If $E\in\mathscr{B}$, let $\Lambda(E)=\{\lambda\in\mathbf{C}\colon|\lambda|\in E\}$, and let $\mathscr{D}=\{\Lambda(E)\colon E\in\mathscr{B}\}$. It is clear that $\mathscr{D}$ is a σ-algebra consisting of Borel subsets of $\mathbf{C}$ and that $\Lambda\colon\mathscr{B}\to\mathscr{D}$ is a one-to-one mapping of $\mathscr{B}$ onto $\mathscr{D}$ that preserves all of the Boolean operations.

Lemma 2.6. *Suppose that* $E\in\mathscr{B}$. *Then* $m_2(\Lambda(E))=\int_E 2\pi r\,dm_1(r)$.

Proof. For in $\mathscr{B}$, let $\mu(\Lambda(E))=\int_E 2\pi r\,dm_1(r)$. It is clear that μ is a measure on $\mathscr{D}$ and that if $E=(a,b]$, then $\mu(\Lambda(E))=m_2(\Lambda(E))$. An application of the theorem of Caratheodory shows that $\mu(\Lambda(E))=m_2(\Lambda(E))$ for all E in $\mathscr{B}$.

Theorem 2.1 and Lemma 2.6 imply the following theorems.

Theorem 2.7. $m_2(\sigma_{ap}(V_{\mathscr{H}}\hat{P}))=\int_{\sigma(P)}2\pi r\,dm_1(r)$.

Theorem 2.8. $m_2(\sigma_{ap}(V_{\mathscr{H}}\hat{P}))=0$ *if and only if* $m_1(\sigma(P))=0$.

For comparison, we state the following theorem.

Theorem 2.9. $m_2(\sigma_e(V_{\mathscr{H}}\hat{P}))=0$ *if and only if P is compact.*

The spectrum of a pure quasinormal operator is connected and its essential spectrum has at most countably many connected components. In the following example, we present a pure quasinormal operator whose approximate point spectrum has uncountably many connected components each of which is a circle. We then use Theorem 2.7 to compute the measure of its approximate point spectrum.

Example 2.9. Let C denote the Cantor set and let $g\colon [0, 1]\to[0, 1]$ be the Cantor ternary function (cf. [13]). Recall that g is defined as follows: Let $r=\sum_{n=1}^{\infty} a_n/3^n$ be the ternary expansion of a number in $[0, 1]$. Let $N=+\infty$ if $a_n\neq 1$ for each positive integer n, and otherwise let N be the smallest positive integer such that $a_N=1$. Let $b_n=a_n/2$ for $n<N$ and let $b_N=1$. Then $g(r)=\sum_{n=1}^{N} b_n/2^n$. Recall also that g is a continuous, monotonic increasing function of $[0, 1]$ onto itself that is constant on the intervals in the complement in $[0, 1]$ of C. Let $t>0$. Define a function $f\colon [0, 1]\to[0, 1+t]$ by $f(r)=g(r)+tr$. The function f is a monotone homeomorphism of $[0, 1]$ onto $[0, 1+t]$. Let $F=f(C)$, let P be a positive definite operator on a Hilbert space $\mathscr{H}$ such that $\sigma(P)=F$, and let $T=V_{\mathscr{H}}\hat{P}$. Since F is uncountable and totally disconnected, it follows from Theorem 2.1 that $\sigma_{ap}(T)$ has uncountably many connected components.

We now compute $m_2(\sigma_{ap}(T))$ by evaluating $\int_F 2\pi r\, dm_1(r)$. (It is easy to see that $m_2(\sigma_e(T))=m_2(\sigma(T))=\pi\|P\|^2=\pi(1+t)^2$.) Let S_n^k, $k=1, 2, \ldots, 2^{n-1}$, be the disjoint subintervals of $[0, 1]\setminus C$ that have measure equal to $1/3^n$, and let $T_n^k= =f(S_n^k)$, $k=1, 2, \ldots, 2^{n-1}$, $n=1, 2, \ldots$. Let $S=[0, 1]\setminus C$ and $T=[0, 1+t]\setminus F$. Then $\bigcup_{n,k} S_n^k=S$ and $\bigcup_{n,k} T_n^k=T$. We will first evaluate $\int_T 2\pi r\, dm_1(r)$. Fix n and k. Now S_n^k and T_n^k are open intervals and $g=(2k-1)/2^n$ on S_n^k. Thus f is differentiable on S_n^k and, therefore, by the change of variable theorem,

$$\int_{T_n^k} 2\pi r\, dm_1(r) = \int_{S_n^k} 2\pi f(r) f'(r)\, dm_1(r) = 2\pi t(2k-1)/6^n + t^2 \int_{S_n^k} 2\pi r\, dm_1(r).$$

Thus

$$\int_T 2\pi r\, dm_1(r) = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \Big(2\pi t(2k-1)/6^n + t^2 \int_{S_n^k} 2\pi r\, dm_1(r)\Big) =$$

$$= 2\pi t\Big(\sum_{n=1}^{\infty} (1/6^n) \sum_{k=1}^{2^{n-1}} (2k-1)\Big) + t^2 \int_S 2\pi r\, dm_1(r).$$

Using the facts that

$$\sum_{k=1}^{2^{n-1}} (2k-1) = (2^{n-1})^2 \quad \text{and} \quad \int_S 2\pi r \, dm_1(r) = \int_{[0,1]} 2\pi r \, dm_1(r) = \pi$$

$$\text{(since } m_1(C) = 0),$$

we get $\int_T 2\pi r \, dm_1(r) = \pi t + \pi t^2$. Hence

$$\int_F 2\pi r \, dm_1(r) = \int_{[0,1+t]} 2\pi r \, dm_1(r) - \int_T 2\pi r \, dm_1(r) = \pi(1+t);$$

thus $m_2(\sigma_{ap}(T)) = \pi(1+t)$.

3. Quasinormals plus commuting normals

In this section we present a class of pure subnormal operators that contains the class of pure quasinormal operators and show that for this class of operators the answer to the equivalent questions posed in Section 1 is affirmative.

Let S be a subnormal operator on a Hilbert space $\mathscr{H}$. Halmos has shown that S has a normal extension of the form

$$\begin{bmatrix} S & X \\ 0 & R^* \end{bmatrix} \tag{1}$$

on $\mathscr{H} \oplus \mathscr{H}$ (cf. [2], [9]). In fact, as mentioned earlier, if S is pure, then the minimal normal extension of S is unitarily equivalent to an operator of the form (1) on $\mathscr{H} \oplus \mathscr{H}$. If S is pure and is unitarily equivalent to its dual R, then we say that S is a *self-dual* subnormal operator. (See [7] for a discussion of the dual of a pure subnormal operator.) If S is self-dual, then the minimal normal extension of S is unitarily equivalent to the operator

$$\begin{bmatrix} S & Z \\ 0 & S^* \end{bmatrix} \tag{2}$$

on $\mathscr{H} \oplus \mathscr{H}$. As we mentioned earlier, Olin has observed in [10] that the operator in (1) is the minimal normal extension of S if and only if R is pure. Also note that any matrix of form (1) is normal if and only if $S^*S - SS^* = XX^*$, $R^*R - RR^* = X^*X$, and $S^*X = XR$.

Theorem 3.1. *Suppose that S is a subnormal operator on a Hilbert space $\mathscr{H}$ that has a normal extension of the form* (1) *and suppose that N is a normal operator in $\mathscr{L}(\mathscr{H})$ that commutes with S, R, and X. Then $T = S + N$ is also subnormal. Moreover, T is pure if and only if S is pure.*

Proof. An application of Fuglede's theorem shows that both N and N^* commute with S, R, X, S^*, R^*, and X^*. Let $Q=R+N^*$. Then $T^*T-TT^*=S^*S-SS^*=XX^*$, $Q^*Q-QQ^*=R^*R-RR^*=X^*X$, and $T^*X=XQ$. Therefore, the operator

$$\begin{bmatrix} T & X \\ 0 & Q^* \end{bmatrix}$$

on $\mathcal{H}\oplus\mathcal{H}$ is a normal extension of T, i.e., T is subnormal.

Now suppose that $S=S_1\oplus S_2$ on $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$, where S_1 is a normal operator on the Hilbert space $\mathcal{H}_1$ and S_2 is a pure operator on the Hilbert space $\mathcal{H}_2$. Then $N=[N_{ij}]$, relative to the decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$. Since $NS=SN$ and $N^*S=SN^*$, a matrix calculation shows that $N_{21}S_1=S_2N_{21}$ and $N_{12}^*S_1=S_2N_{12}^*$. Theorem 1.2 of [15] shows that $N_{12}=N_{21}=0$. Thus $N=N_{11}\oplus N_{22}$ and $T=(S_1+N_{11})\oplus(S_2+N_{22})$, where S_1+N_{11} is normal since S_1 and N_{11} are commuting normal operators. This argument shows that if T is pure, then S is also pure. Since $S=T-N$, a similar argument shows that if S is pure, then T is also pure.

Corollary 3.2. *Suppose that S is a subnormal operator on a Hilbert space $\mathcal{H}$ that has a normal extension of the form* (2) *and suppose that N is a normal operator in $\mathcal{L}(\mathcal{H})$ that commutes with S and Z. Then $T=S+N$ is also subnormal.*

We remark that if S is a subnormal operator with a normal extension of form (2) and if U is a unitary operator that commutes with S (for example, if $U=\alpha 1_{\mathcal{H}}$, where α is a complex number such that $|\alpha|=1$), then the operator in (2) is unitarily equivalent to the operator

$$\begin{bmatrix} S & ZU \\ 0 & S^* \end{bmatrix}$$

on $\mathcal{H}\oplus\mathcal{H}$. Thus even a normal extension of the form (2) is not unique.

The following theorem is well-known. (Recall that if S is a quasinormal operator, then $S^*S-SS^*\geqq 0$.)

Theorem 3.3. *If S is a quasinormal operator, then the matrix in* (2) *is a normal extension of S if $Z=\sqrt{S^*S-SS^*}$. In particular, if S is a pure quasinormal operator, then S is self-dual.*

Proof. It is clear that $S^*S-SS^*=Z^2$. To show that $S^*Z=ZS$, observe that $Z^2S=(S^*S-SS^*)S=0$. Hence $ZS=0$ since Z is self-adjoint, and thus $S^*Z=(ZS)^*=0$. Therefore, the operator in (2) is normal.

Corollary 3.4. *If S is a quasinormal operator on a Hilbert space $\mathcal{H}$ and N is a normal operator in $\mathcal{L}(\mathcal{H})$ that commutes with S, then $T=S+N$ is subnormal.*

Proof. By Fuglede's theorem N also commutes with S^* and, therefore, with $Z=\sqrt{S^*S-SS^*}$. Hence T is subnormal by Corollary 3.2.

Let $\mathscr{K}$ be a Hilbert space, and let $\mathscr{S}=\{S+N\in\mathscr{L}(\mathscr{K})$: S is a pure quasinormal, N is normal, and $NS=SN\}$. The set $\mathscr{S}$ is a subset of the set of all pure subnormal operators in $\mathscr{L}(\mathscr{K})$ and contains the pure quasinormals. We show that the operators in $\mathscr{S}$ have a fairly simple structure.

Theorem 3.5. *If $T\in\mathscr{S}$, then there exist a Hilbert space $\mathscr{H}$ and commuting operators P and N in $\mathscr{L}(\mathscr{H})$, where P is positive definite and N is normal, such that T is unitarily equivalent to the operator $V_{\mathscr{H}}\hat{P}+\hat{N}$ on $\hat{\mathscr{H}}$.*

Proof. We have $T=S+N_1$, where S is a pure quasinormal operator, N_1 is normal, and $N_1S=SN_1$. There exist a Hilbert space $\mathscr{H}$ and a positive definite operator P in $\mathscr{L}(\mathscr{H})$ such that S is unitarily equivalent to $V_{\mathscr{H}}\hat{P}$. So T is unitarily equivalent to $V_{\mathscr{H}}\hat{P}+N_0$, where N_0 is normal and commutes with $V_{\mathscr{H}}\hat{P}$. Then $N_0=[N_{ij}]$ on $\hat{\mathscr{H}}$. Since N_0 commutes with $V_{\mathscr{H}}\hat{P}$, a matrix calculation shows that $N_{1,j+1}P=0$ and $N_{i+1,j+1}P=PN_{ij}$, $i,j=1,2,\dots$. An induction argument shows that $N_{ij}=0$ for $i<j$. Since Fuglede's theorem implies that N_0^* commutes with $V_{\mathscr{H}}\hat{P}$, we have by a similar argument that $N_{ij}=0$ for $i>j$. Thus $[N_{ij}]$ is diagonal, N_{ii} is normal, and $N_{i+1,i+1}P=PN_{ii}$, $i=1,2,\dots$. Using the Putnam—Fuglede's theorem, we can see that $PN_{i+1,i+1}=N_{ii}P$ also. Hence P^2, and thus P, commutes with N_{ii}. It follows that $(N_{i+1,i+1}-N_{ii})P=0$; thus $N_{i+1,i+1}=N_{ii}$, $i=1,2,\dots$. Therefore, $N_0=\hat{N}_{11}$, and the proof is complete.

It follows from Theorem 3.5 that if S is a pure quasinormal and N is a nonzero normal operator that commutes with S, then $S+N$ is not quasinormal. Thus $\mathscr{S}$ contains operators that are not quasinormal.

We can say more about the structure of those operators in $\mathscr{S}$ that have compact self-commutators. Let V denote the unilateral shift of multiplicity one.

Theorem 3.6. *Suppose that $T\in\mathscr{S}$ and T^*T-TT^* is compact. Then there exist an index set A, a set of positive numbers $\{c_\alpha\}_{\alpha\in A}$, and a set of complex numbers $\{\lambda_\alpha\}_{\alpha\in A}$ such that T is unitarily equivalent to $\sum_{\alpha\in A}\oplus(\lambda_\alpha+c_\alpha V)$. If the rank of T^*T-TT^* is n, then $A=\{1,2,\dots,n\}$; otherwise $A=\{1,2,\dots\}$.*

Proof. By Theorem 3.5 T is unitarily equivalent to $V_{\mathscr{H}}\hat{P}+\hat{N}$, where P is positive definite in $\mathscr{L}(\mathscr{H})$, N is normal in $\mathscr{L}(\mathscr{H})$, and $PN=NP$. Since T^*T-TT^* is compact, P is compact. Suppose that c is an eigenvalue of P. Then $\mathscr{K}(P-c)$ is finite dimensional and reduces N. Hence $\mathscr{K}(P-c)$ has an orthonormal basis consisting of eigenvectors of N. It follows that $\mathscr{H}$ has an orthonormal basis $\{e_\alpha\}_{\alpha\in A}$ consisting of vectors that are eigenvectors of both P and N. For $\alpha\in A$, let c_α and

λ_α be the eigenvalues of P and N, respectively, associated with e_α, and let $\mathcal{M}_\alpha$ be the one-dimensional subspace of $\mathcal{H}$ spanned by e_α. Then $\mathcal{H}=\sum_{\alpha\in A}\oplus\mathcal{M}_\alpha$ and $\hat{\mathcal{H}}$ is Hilbert space isomorphic to $\sum_{\alpha\in A}\oplus\hat{\mathcal{M}}_\alpha$. Hence $V_{\mathcal{H}}\hat{P}$ is unitarily equivalent to $\sum_{\alpha\in A}\oplus c_\alpha V_{\mathcal{M}_\alpha}$ and $\hat{N}$ is unitarily equivalent to $\sum_{\alpha\in A}\oplus\lambda_\alpha 1_{\hat{\mathcal{M}}_\alpha}$; thus T is unitarily equivalent to $\sum_{\alpha\in A}\oplus(\lambda_\alpha+c_\alpha V_{\mathcal{M}_\alpha})$. The proof is complete since for each α in A, $V_{\mathcal{M}_\alpha}$ is unitarily equivalent to V.

Corollary 3.7. *If* $T\in\mathcal{S}$ *and* T^*T-TT^* *has finite rank, then* $m_2(\sigma_{ap}(T))=m_2(\sigma_e(T))=0$.

Proof. By Theorem 3.6 T is unitarily equivalent to $\sum_{k=1}^{n}\oplus(\lambda_k+c_kV)$. The proof is complete since $m_2(\sigma_{ap}(V))=m_2(\sigma_e(V))=0$.

Corollary 3.7 shows that the answer to the three equivalent questions posed in Section 1 is affirmative for the class of operators $\mathcal{S}$. In regard to Question 2.5, note that Corollary 3.7 also implies that if $T\in\mathcal{S}$ and T has a finite rank self-commutator, then $T\notin(N+K)$, since $\sigma_{ap}(T)\neq\sigma(T)$. Recall that if T is a pure quasinormal operator, then T^*T-TT^* is compact if and only if $m_2(\sigma_e(T))=0$ (see Theorem 2.9), and if T^*T-TT^* is compact, then $m_2(\sigma_{ap}(T))=0$ (see Theorem 2.8). The next example shows that this is not the case for the class of operators $\mathcal{S}$.

Example 3.8. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, let $\{\lambda_k\}_{k=1}^\infty$ be an enumeration of all the "rational" complex numbers in $\mathbf{D}$, let $\{c_k\}_{k=1}^\infty$ be a sequence of positive numbers such that $\sum_{k=1}^\infty c_k^2<1/2$, let $G_k=\{\lambda\in\mathbf{D}: |\lambda-\lambda_k|<c_k\}$, $k=1,2,\dots$, and let $G=\bigcup_{k=1}^\infty G_k$. Note that G is an open subset of $\mathbf{C}$ and that $m_2(G)\leqq\sum_{k=1}^\infty \pi c_k^2<\pi/2$. Since $\bar{G}=\bar{\mathbf{D}}$, $m_2(\partial G)>\pi/2$. Define a positive definite operator P and a normal operator N in $\mathcal{L}(\mathcal{H})$ by $Pe_k=c_ke_k$ and $Ne_k=\lambda_ke_k$. Let $T=V_{\mathcal{H}}\hat{P}+\hat{N}$. Observe that $T\in\mathcal{S}$ and that T^*T-TT^* is compact (since P is compact). Observe also that T is unitarily equivalent to $\sum_{k=1}^\infty\oplus(\lambda_k+c_kV)$ and that $\partial G_k=\sigma_{ap}(\lambda_k+c_kV)\subseteqq\sigma_{ap}(T)$. Thus $\left(\bigcup_{k=1}^\infty\partial G_k\right)^-\subseteqq\sigma_{ap}(T)$. It follows that $m_2(\sigma_{ap}(T))>0$ since $\partial G\subseteqq\left(\bigcup_{k=1}^\infty\partial G_k\right)^-$. We also have $m_2(\sigma_e(T))>0$ since $\sigma_{ap}(T)\subseteqq\sigma_e(T)$.

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