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Singular perturbations of singular systems

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Let $\Omega\subset\mathbf{R}^p$ $(p\in\mathbf{N})$ be a bounded open domain with C^2-smooth boundary and consider for $\varepsilon\neq 0$ the system

$$(1)\qquad -\varepsilon\Delta z - z^n = v,\quad v\in L^2(\Omega),\quad z\in H^2(\Omega)\cap H_0^1(\Omega).$$

This system is well-posed if $\varepsilon<0$ and $n\in\{1, 3, \dots\}$, not well-posed otherwise. Fixing $z_d\in L^{2n}(\Omega)$ and a number $N>0$ arbitrarily, define

$$(2)\qquad J(v, z) = (1/2n)\,\|z-z_d\|^{2n}_{L^{2n}(\Omega)}+(N/2)\,\|v\|^2_{L^2(\Omega)}$$

and

$$(3)\qquad J_\varepsilon = \inf\,\{J(v, z)\mid (v, z)\ \text{satisfies}\ (1)\}.$$

One can see easily (see [1]) that for any $\varepsilon\neq 0$ there exists (at least) a pair $(u_\varepsilon, y_\varepsilon)$ such that

$$(4)\qquad (u_\varepsilon, y_\varepsilon)\ \text{satisfies}\ (1)\ \text{and}\ J(u_\varepsilon, y_\varepsilon) = J_\varepsilon.$$

The purpose of this paper is to investigate the behavior of the sequences (J_ε), (u_ε), (y_ε) when ε tends to 0.

In case $n=3$ such investigations were done for $\varepsilon<0$ by L. TARTAR (see [1]), A. HARAUX and F. MURAT [4], [6] and for $\varepsilon>0$ by A. BENSOUSSAN [3]. All these considerations remain valid for any $n\in\{1, 3, \dots\}$.

In the present paper, developing the method of A. Bensoussan, similar (and even stronger) results will be proved for the case $n\in\{2, 4, \dots\}$. We shall also improve the results of Bensoussan in case $n\in\{1, 3, \dots\}$.

Let us consider also the system

$$(5)\qquad -z^n = v,\quad v\in L^2(\Omega),\quad z\in L^{2n}(\Omega)$$

and put

$$(6)\qquad J_0 = \inf\{J(v, z)\mid (v, z)\ \text{satisfies}\ (5)\}.$$

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One can see easily that there exists a unique pair (u_0, y_0) such that

(7) $$(u_0, y_0) \text{ satisfies (5) and } J(u_0, y_0) = J_0.$$

Let us introduce the polynomial

(8) $$p_{n,N}(x) = (1-x)^{2n} - x^{2n} + 2x^n \frac{(x+M)^n - x^n}{M} \quad \text{where} \quad M = (nN)^{1/(2n-1)}$$

of degree $2n-2$ and set

(9) $$N_n = \sup\{N > 0 \mid \inf_{x \in \mathbf{R}} p_{n,N}(x) > 0\}.$$

We shall prove the following two theorems:

Theorem 1. *Suppose $N < N_n$. Then*

(10) $$|J_\varepsilon - J_0| \to 0,$$

(11) $$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \to 0,$$

(12) $$\|y_\varepsilon - y_0\|_{L^{2n}(\Omega)} \to 0.$$

(10) *and* (11) *are valid for* $N = N_n < \infty$, *too.*

Theorem 2. *Suppose $N < N_n$ and*

(13) $$z_d,\ z_d^n \in H^2(\Omega) \cap H_0^1(\Omega).$$

Then

(14) $$|J_\varepsilon - J_0| = O(\varepsilon),$$

(15) $$\|u_\varepsilon - u_0\|_{L^2(\Omega)} = O(\sqrt{\varepsilon}),$$

(16) $$\|y_\varepsilon - y_0\|_{L^{2n}(\Omega)} = O(\sqrt[2n]{\varepsilon}).$$

(14) *and* (15) *are valid for* $N = N_n < \infty$, *too.*

Naturally, it is important to have some information on the numbers N_n:

Proposition. *We have*

(17) $$0 < N_n < \infty \quad \text{if} \quad n \in \{3, 5, 7, \ldots\},$$

(18) $$N_n = +\infty \quad \text{if} \quad n = 1 \quad \text{and if} \quad n \in \{2, 4, 6, \ldots\}.$$

We turn to the proof of the theorems.

Lemma 1. *We have for all $N > 0$*

(19) $$J_\varepsilon \leqq J_0 + o(1);$$

f condition (13) *is satisfied, we have also*

(20) $$J_\varepsilon \leqq J_0 + O(\varepsilon).$$

Proof. One can see by explicit calculation that

(21) $$y_0 = z_d/(1+(nN)^{1/(2n-1)})$$

and

(22) $$u_0 = -y_0^n.$$

If condition (13) is satisfied then

$$J_\varepsilon \leqq J(-\varepsilon\Delta y_0 - y_0^n, y_0) = J(-y_0^n, y_0)+O(\varepsilon) = J_0+O(\varepsilon)$$

whence (20) follows. In the general case fix a sequence $(z_m)\subset\mathcal{D}(\Omega)$ such that $\|z_m-y_0\|_{L^{2n}(\Omega)}\to 0$. Then for any fixed m

$$\overline{\lim}\, J_\varepsilon \leqq \overline{\lim}\, J(-\varepsilon\Delta z_m - z_m^n, z_m) = J(-z_m^n, z_m)$$

and

$$\overline{\lim}\, J_\varepsilon \leqq \lim J(-z_m^n, z_m) = J(-y_0^n, y_0) = J_0;$$

(19) is shown and the lemma is proved.

Now we fix for each $\varepsilon\neq 0$ a function $\bar{y}_\varepsilon$ such that

(23) $$\bar{y}_\varepsilon,\ \bar{y}_\varepsilon^n \in H^2(\Omega)\cap H_0^1(\Omega),$$

(24) $$\|\Delta\bar{y}_\varepsilon\|_{L^2(\Omega)}+\|\Delta(\bar{y}_\varepsilon^n)\|_{L^{2n/(2n-1)}(\Omega)} \leqq |\varepsilon|^{-1/2},$$

(25) $$\|\bar{y}_\varepsilon - y_0\|_{L^{2n}(\Omega)} \leqq |\varepsilon| + \inf\{\|\bar{y}-y_0\|_{L^{2n}(\Omega)} \mid \bar{y},\ \bar{y}^n\in H^2(\Omega)\cap H_0^1(\Omega),\ \|\Delta\bar{y}\|_{L^2(\Omega)}+\|\Delta(\bar{y}^n)\|_{L^{2n/(2n-1)}(\Omega)} \leqq |\varepsilon|^{-1/2}\}.$$

Furthermore, we put

(26) $$\tilde{u}_\varepsilon = u_\varepsilon + \bar{y}_\varepsilon^n + \varepsilon\Delta\bar{y}_\varepsilon,$$

(27) $$\tilde{y}_\varepsilon = y_\varepsilon - \bar{y}_\varepsilon,$$

(28) $$\xi_\varepsilon = (\bar{y}_\varepsilon - z_d)^{2n-1} + nN\bar{y}_\varepsilon^{2n-1}.$$

Lemma 2. *We have*

(29) $$J_\varepsilon = J(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n, \bar{y}_\varepsilon) + (N/2)\int_\Omega \tilde{u}_\varepsilon^2\,dx + \int_\Omega \xi_\varepsilon\tilde{y}_\varepsilon\,dx + \varepsilon N\int_\Omega -\tilde{u}_\varepsilon\Delta\tilde{y}_\varepsilon + \tilde{y}_\varepsilon\Delta(\bar{y}_\varepsilon^n)\,dx + \int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2[(2n-1)(\bar{y}_\varepsilon - z_d+\lambda\mu\tilde{y}_\varepsilon)^{2n-2} + (n-1)nN\bar{y}_\varepsilon^n(\bar{y}_\varepsilon+\lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu\,dx.$$

Proof. We recall that if $f\colon \mathbf{R}\to\mathbf{R}$ is a C^2-smooth function then

(30) $$f(a+b) = f(a)+f'(a)b+\int_0^1\int_0^1 \lambda b^2 f''(a+\lambda\mu b)\,d\lambda\,d\mu$$

for any $a, b \in \mathbf{R}$. Now using (1), (2), (4), (26), (27), (28), (30), we have the following three relations:

$$J_\varepsilon = J(u_\varepsilon, y_\varepsilon) = J(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n + \tilde{u}_\varepsilon, \bar{y}_\varepsilon + \tilde{y}_\varepsilon) = J(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n, \bar{y}_\varepsilon) + \frac{N}{2}\int_\Omega \tilde{u}_\varepsilon^2\,dx +$$

$$+N\int_\Omega(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n)\tilde{u}_\varepsilon\,dx + \int_\Omega(\bar{y}_\varepsilon - z_d)^{2n-1}\tilde{y}_\varepsilon\,dx + \tag{31}$$

$$+\int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2(2n-1)(\bar{y}_\varepsilon - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2}\,d\lambda\,d\mu\,dx,$$

$$\int_\Omega(\bar{y}_\varepsilon - z_d)^{2n-1}\tilde{y}_\varepsilon\,dx = \int_\Omega \xi_\varepsilon\tilde{y}_\varepsilon\,dx - N\int_\Omega \bar{y}_\varepsilon^n(n\bar{y}_\varepsilon^{n-1}\tilde{y}_\varepsilon)\,dx =$$

(32)

$$= \int_\Omega \xi_\varepsilon\tilde{y}_\varepsilon\,dx - N\int_\Omega \bar{y}_\varepsilon^n(y_\varepsilon^n - \bar{y}_\varepsilon^n)\,dx + \int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2 n(n-1)N\bar{y}_\varepsilon^n(\bar{y}_\varepsilon + \lambda\mu\tilde{y}_\varepsilon)^{n-2}\,d\lambda\,d\mu\,dx,$$

$$-N\int_\Omega \bar{y}_\varepsilon^n(y_\varepsilon^n - \bar{y}_\varepsilon^n)\,dx + N\int_\Omega(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n)\tilde{u}_\varepsilon\,dx =$$

$$= N\int_\Omega \bar{y}_\varepsilon^n(\varepsilon\Delta\tilde{y}_\varepsilon + \tilde{u}_\varepsilon)\,dx + N\int_\Omega(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n)\tilde{u}_\varepsilon\,dx = \tag{33}$$

$$= \varepsilon N\int_\Omega \tilde{y}_\varepsilon\Delta(\bar{y}_\varepsilon^n)\,dx - \varepsilon N\int_\Omega \tilde{u}_\varepsilon\Delta\bar{y}_\varepsilon\,dx;$$

(31), (32) and (33) imply (29).

Lemma 3. *We have the following estimates for the terms of the formula* (29) *when* ε *tends to* 0:

$$J(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n, \bar{y}_\varepsilon) = J_0 + o(1), \tag{34}$$

$$(N/2)\int_\Omega \tilde{u}_\varepsilon^2\,dx \geqq 0, \tag{35}$$

$$\int_\Omega \xi_\varepsilon\tilde{y}_\varepsilon\,dx = o(1), \tag{36}$$

$$\varepsilon N\int_\Omega -\tilde{u}_\varepsilon\Delta\bar{y}_\varepsilon + \tilde{y}_\varepsilon\Delta(\bar{y}_\varepsilon^d)\,dx = o(1), \tag{37}$$

$$\int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2[(2n-1)(\bar{y}_\varepsilon - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)N\bar{y}_\varepsilon^n(\bar{y}_\varepsilon + \lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu\,dx =$$

(38)

$$= \int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2[(2n-1)(y_0 - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)Ny_0^n(y_0 + \lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu\,dx + o(1).$$

Proof. It follows from (23), (24), (25) that

(39) $$\|\tilde{y}_\varepsilon - y_0\|_{L^{2n}(\Omega)} = o(1),$$

(40) $$\|\tilde{y}_\varepsilon\|_{L^{2n}(\Omega)} = O(1),$$

(41) $$\varepsilon\|\Delta\tilde{y}_\varepsilon\|_{L^2(\Omega)} = o(1),$$

(42) $$\varepsilon\|\Delta(\tilde{y}_\varepsilon^n)\|_{L^{2n/(2n-1)}(\Omega)} = o(1);$$

(2), (5), (6), (23), (39), (40) and (41) imply (34).

(35) is obvious.

Using the obvious estimate $J(u_\varepsilon, y_\varepsilon) = J_\varepsilon \leqq J(0, 0)$ and (2), we obtain

(43) $$\|u_\varepsilon\|_{L^2(\Omega)} = O(1),$$

(44) $$\|y_\varepsilon\|_{L^{2n}(\Omega)} = O(1);$$

(26), (27), (40), (41), (43) and (44) imply

(45) $$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)} = O(1),$$

(46) $$\|\tilde{y}_\varepsilon\|_{L^{2n}(\Omega)} = O(1).$$

Furthermore we note that

(47) $$\|\xi_\varepsilon\|_{L^{2n/(2n-1)}(\Omega)} = o(1)$$

by (21), (28) and (39).

Now (36) follows from (47) and (46), (37) follows from (41), (42), (45), (46), finally (38) is a consequence of (39) and (46). The lemma is proved.

Lemma 4. *Putting*

(48) $$C_{n,N} = (2n)^{-1} \inf_{x\in\mathbf{R}} p_{n,N}(x),$$

we have

(49) $$\int_\Omega \int_0^1 \int_0^1 \lambda\tilde{y}_\varepsilon^2[(2n-1)(y_0 - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)Ny_0^n(y_0 + \lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu\,dx \geqq C_{n,N} \int_\Omega \tilde{y}_\varepsilon^{2n}\,dx.$$

Proof. We show the stronger inequality

(50)
$$\int_0^1 \int_0^1 \lambda[(2n-1)(y_0 - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)\,Ny_0^n(y_0 + \lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu \geqq C_{n,N}\,\tilde{y}_\varepsilon^{2n-2}.$$

This is obvious if $\tilde{y}_\varepsilon(x) = 0$. Otherwise, putting

(51) $$M = (nN)^{1/(2n-1)},$$

(52) $$y = M(y_0/\tilde{y}_\varepsilon)$$

and taking into account that

(53) $$z_d = (1+M)y_0$$

by (21), after integration we see that (50) is equivalent to

(54) $$p_{n,N}(y) \geqq 2nC_{n,N}.$$

The lemma is proved.

Lemma 5. *We have for all* $0<N<N_n$

(55) $$\inf_{x\in\mathbf{R}} p_{n,N}(x) > 0.$$

We prove Lemma 5 simultaneously with the Proposition (i.e. with (17) and (18)). First we note that

(56) $$p_{n,N}(x) = \left((1-x)^{2n} - x^{2n} + 2nx^{2n-1}\right) + (2/M)x^n\left((x+M)^n - x^n - nMx^{n-1}\right).$$

Consider first the case when $n=1$ or $n\in\{2, 4, \dots\}$. It suffices to show that

$$p_{n,N}(x) > 0 \quad \text{for all} \quad 0 < N < \infty \quad \text{and} \quad x\in\mathbf{R}.$$

This is obvious if $n=1$ because then $p_{n,N}(x)\equiv 1$. If $n\in\{2, 4, \dots\}$ then it follows from the formula (56), taking into account that $x^n \geqq 0$ and that the functions $t\mapsto t^{2n}$, $t\mapsto t^n$ are strictly convex.

Consider now the case $n\in\{3, 5, \dots\}$. One can see easily that

$$\lim_{N\to 0} \inf_{x\in\mathbf{R}} p_{n,N}(x) > 0$$

whence $N_n>0$. Now fix $0<\alpha<1$ such that

$$(1-\alpha)^n + \alpha^n - n\alpha^{n-1} > 0.$$

An easy computation shows that

$$M^{1-2n} p_{n,N}(-\alpha M) = -2\alpha^n\left((1-\alpha)^n + \alpha^n - n\alpha^{n-1}\right) + o(1) \quad (N\to\infty).$$

Therefore $N_n<\infty$ and (17) is proved.

To finish the proof of the lemma we show that for any fixed $x\in\mathbf{R}$ there exists a number $0<M_1\leqq\infty$ such that

$$p_{n,N}(x) > 0 \ \text{ if } \ M < M_1 \quad \text{and} \quad p_{n,N}(x) < 0 \ \text{ if } \ M > M_1.$$

Taking into account that $p_{n,N}(x)$ is a polynomial of degree $\geqq 1$ in M and that $\lim_{N\to 0} p_{n,N}(x) > 0$, this would follow from the concavity of the function $f(M) := p_{n,N}(x)$ $(M>0)$. And f is concave because, applying the Taylor formula,

$$f''(M) = 2M^{-3}\left(x^n - (x+M)^n + n(x+M)^{n-1}M - \binom{n}{2}(x+M)^{n-2}M^2\right) =$$
$$= 2M^{-3}\binom{n}{3}\xi^{n-3}(-M)^3 = -2\binom{n}{3}\xi^{n-3} \leqq 0.$$

The lemma and the proposition are proved.

Proof of Theorem 1. It follows from Lemmas 1—5 that

$$o(1) \geqq J_\varepsilon - J_0 \geqq (N/2)\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} + C_{n,N}\|\tilde{y}_\varepsilon\|^{2n}_{L^{2n}(\Omega)} - o(1). \tag{57}$$

If $N \leqq N_n$ then $C_{n,N} \geqq 0$ and therefore (57) implies (10) and

$$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)} = o(1). \tag{58}$$

(11) follows from (58), (26), (39), (22) and (41). If $N < N_n$ then $C_{n,N} > 0$ and (57) implies also

$$\|\tilde{y}_\varepsilon\|_{L^{2n}(\Omega)} = o(1). \tag{59}$$

(12) follows from (59), (27), (39) and the theorem is proved.

Proof of Theorem 2. In view of (13) and (21) we can put $\bar{y}_\varepsilon := y_0$; then (23), (24), (25) remain valid for $|\varepsilon|$ sufficiently small. Furthermore, in the estimates (39), (41), (42), (47) and therefore also in (34), (36), (37), (38) the term $o(1)$ can be replaced by $O(\varepsilon)$. (Moreover, in (39), (47), (36), (38) we can also write 0.) Therefore, repeating the proof of Theorem 1, we can change the terms $o(1)$ to $O(\varepsilon)$ in (57), (58), (59), too. Hence the theorem follows.

Remarks. (i) In case $n=3$ the condition $N < N_3$ is weaker than the original condition of Bensoussan:

$$(0{,}3N)^{1/3}/(1+(3N)^{1/5}) < 4/3. \tag{60}$$

Indeed, up to decimals $N < N_3$ signifies $N < 5207$ while (60) signifies $N < 2841$.

(ii) All the results of this paper remain valid with the same proof if we replace in (1) the condition $v \in L^2(\Omega)$ by the more general condition $v \in K$ where K is a closed convex subset of $L^2(\Omega)$ such that

$$(z_d/(1+(nN)^{1/(2n-1)}))^n \in \operatorname{int} K \tag{61}$$

(this is a problem with constraints).

(iii) A more general investigation of the influence of the different constraints is given by Haraux and Murat [4], [6]. A systematic study of the control of nonlinear singular systems can be found in the book of J.-L. Lions [1].

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