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On uniqueness and the Lifting Theorem

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It is the purpose of this note to present a simple proof of Theorem 1.1 in [1]. This theorem gives a necessary and sufficient condition for the existence of a unique intertwining dilation in the Lifting Theorem.

We follow the notation in [4]. If C is a contraction, then D_C is the positive square root of $I-C^*C$. The closure of the range of D_C is $\mathfrak{D}_C$.

A factorization C_1C_2 is regular [4] if

$$\mathfrak{D}_{C_1}\oplus D_{C_2} = \{D_{C_1}C_2h\oplus D_{C_2}h:\ h\in\mathfrak{H}\}^-.$$

Throughout T on $\mathfrak{H}$, T' on $\mathfrak{H}'$ and $A\colon \mathfrak{H}\to\mathfrak{H}'$ are contractions such that $T'A=AT$. The minimal isometric dilations of T on $\mathfrak{K}$ and T' on $\mathfrak{K}'$ are denoted by U and U', respectively. It is always assumed that U is in its matrix form with respect to the decomposition $\mathfrak{K}=\mathfrak{H}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T\oplus\dots$, i.e.,

$$(1)\qquad U=\begin{bmatrix} T & 0 & 0 & \cdots \\ D_T & 0 & 0 & \\ 0 & I & 0 & \cdots \\ 0 & 0 & I & \\ 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & \ddots \end{bmatrix},$$

and analogously for U'. An operator B mapping $\mathfrak{K}$ into $\mathfrak{K}'$ is a contractive intertwining dilation (CID) of A if B is a contraction, and

$$(2)\qquad U'B=BU \quad\text{and}\quad AP_{\mathfrak{H}}=P_{\mathfrak{H}'}B.$$

(The orthogonal projection onto $\mathfrak{H}$ is denoted by $P_{\mathfrak{H}}$.) The famous Lifting Theorem of Sz.-Nagy and Foiaş [3], [4] states that there exists a CID for A. The following shows when there is only one CID for A.

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Theorem 1 ([1]). *The contraction A has a unique CID if and only if AT or $T'A$ is a regular factorization.*

Proof. Let B be a CID for A. Matrix multiplication with (1) shows that B must be in the form:

$$B = \begin{bmatrix} A & 0 & 0 & 0 & 0 & \cdots \\ Z_1 & Y_1 & 0 & 0 & 0 & \ddots \\ Z_2 & Y_2 & Y_1 & 0 & 0 & \\ Z_3 & Y_3 & Y_2 & Y_1 & 0 & \cdots \\ Z_4 & Y_4 & Y_3 & Y_2 & Y_1 & \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{bmatrix} \tag{3}$$

where $Z_i\colon \mathfrak{H}\to\mathfrak{D}_{T'}$ and $Y_i\colon \mathfrak{D}_T\to\mathfrak{D}_{T'}$ are contractions for all $i\geqq 1$. Note the first row in B follows from the second equation in (2). Since $\|Bh\|^2\leqq\|h\|^2$ for all h in $\mathfrak{H}$, Equation (3) implies $\|Z_ih\|\leqq\|D_A h\|$. Thus, $Z_i=X_iD_A$ for $i\geqq 1$ where X_i is a contraction from $\mathfrak{D}_A$ into $\mathfrak{D}_{T'}$. Finally, using $U'Bh=BUh$ for h in $\mathfrak{H}$ with (3) gives:

$$[X_1, Y_1]\begin{bmatrix} D_A Th \\ D_T h\end{bmatrix} = D_{T'}Ah, \quad [X_{n+1}, Y_{n+1}]\begin{bmatrix} D_A Th \\ D_T h\end{bmatrix} = X_n D_A h \quad (n\geqq 1). \tag{4}$$

Assume AT is a regular factorization. This implies that the X_i's and Y_i's in (3) are uniquely determined by (4). Hence, B is unique. Now assume $T'A$ is a regular factorization. By Proposition VII.3.2 in [4] (or Lemma 3.1 in [2]) the factorization $A^*T'^*$ is regular. Therefore, A^* admits a unique CID. Lemma 2.1 in [1] shows that A has a unique CID if and only if A^* has a unique CID. Hence, A has a unique CID.

The other half of the proof follows from the one-step dilations for A in [2]. For completeness it is given. Assume $T'A$ and AT are not regular factorizations. By [2], there exist two different contractions

$$A_1 = \begin{bmatrix} A & 0 \\ Z_1 & Y_1\end{bmatrix} \quad\text{and}\quad A_1' = \begin{bmatrix} A & 0 \\ Z_1' & Y_1'\end{bmatrix} \tag{5}$$

where $T_1'A_1=A_1T_1$ and $T_1'A_1'=A_1'T_1$. Here

$$T_1 = \begin{bmatrix} T & 0 \\ D_T & 0\end{bmatrix} \quad\text{and}\quad T_1' = \begin{bmatrix} T' & 0 \\ D_{T'} & 0\end{bmatrix}. \tag{6}$$

Applying the Lifting Theorem to (5) and (6) shows that A does not have a unique CID. The proof is now complete.

References

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