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Involution algebras and the Anderson—Divinsky—Suliński property

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1. Introduction

In this paper we shall deal with special features of the general radical theory of involution algebras. We give necessary and sufficient conditions for a radical class $\mathbf{R}$ to have the A—D—S property:

$$I^* \lhd A^* \quad \text{implies} \quad \mathbf{R}(I^*) \lhd A^*$$

(where * indicates that algebras with involution * are considered). We prove that every radical class of involution algebras over a field K has the A—D—S property if and only if $\operatorname{char} K=2$. The A—D—S property implies trivially the hereditariness of the corresponding semisimple class. Nevertheless, there are radical classes which do not have the A—D—S property, but have hereditary semisimple classes. The semisimple class of a radical need not be hereditary, even if the radical is hereditary.

A K-algebra A is an involution algebra, if in A a unary operation * is defined so that $x^{**}=x$, $(x+y)^*=x^*+y^*$, $(xy)^*=y^*x^*$, $(kx)^*=kx^*$ for all $x, y \in A$ and $k \in K$. We shall work throughout with involution algebras over a commutative associative ring K with identity, and the universal class we shall use will be the variety $\mathfrak{V}$ of all K-algebras with involution. A^* will always stand for a K-algebra with involution *. In particular, id will denote the operation $x^{\mathrm{id}}=x$ and $^{-*}$ the operation $x^{-*}=-x^*$. An ideal I^* of an involution algebra A^* will always mean an ideal of the algebra A such that I^* is an involution algebra. This fact will be indicated by $I^* \lhd A^*$. By a homomorphism φ we mean an algebra homomorphism such that $(\varphi(x))^*=\varphi(x^*)$.

A *radical class* $\mathbf{R}$ (in the sense of Kurosh and Amitsur) of involution algebras is a subclass $\mathbf{R}$ of $\mathfrak{V}$ such that

i) $\mathbf{R}$ is homomorphically closed: if $A^* \in \mathbf{R}$ then $\varphi(A^*) \in \mathbf{R}$ for every homomorphism φ,

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ii) for every $A^*\in\mathfrak{V}$, the ideal $\mathbf{R}(A^*)=\sum(I^*\lhd A^*: I^*\in\mathbf{R})$ is in $\mathbf{R}$,

iii) $\mathbf{R}$ is closed under extensions: if $I^*\lhd A^*$, $I^*\in\mathbf{R}$ and $(A/I)^*\in\mathbf{R}$, then $A^*\in\mathbf{R}$.

Condition iii) can be replaced by $\mathbf{R}(A^*/\mathbf{R}(A^*))=0$ for all $A^*\in\mathfrak{V}$. The class

$$\mathcal{S}\mathbf{R}=\{A^*\in\mathfrak{V}:\ \mathbf{R}(A^*)=0\}$$

is called the *semisimple class* of the radical class $\mathbf{R}$. A semisimple class $\mathbf{S}$ is always *regular,* that is, if $0\neq I^*\lhd A^*\in\mathbf{S}$, then there exists an $M^*\lhd I^*$ such that $0\neq(I/M)^*\in\mathbf{S}$. If $\mathbf{M}$ is any regular class, in particular a semisimple class, then the class

$$\mathcal{U}\mathbf{M}=\{A^*\in\mathfrak{V}:\ A^*/I^*\in\mathbf{M}\Rightarrow I^*=A^*\}$$

is a radical class, which is referred to as the *upper radical class* of $\mathbf{M}$. For further details of the basic facts of radical theory we refer to [9]. Radicals of involution algebras have been studied in the recent papers [3], [5], [6] and [8].

Given a radical class $\mathbf{R}$, it may happen that

$$I^*\lhd A^*\in\mathfrak{V}\quad\text{implies}\quad\mathbf{R}(I^*)\lhd A^*.$$

In this case we say that $\mathbf{R}$ *satisfies A—D—S.* If $\mathbf{R}$ satisfies A—D—S, then it follows trivially that

$$I^*\lhd A^*\in\mathfrak{V}\quad\text{implies}\quad\mathbf{R}(I^*)\subseteq\mathbf{R}(A^*).$$

This latter condition is equivalent to demanding that

$$I^*\lhd A^*\in\mathcal{S}\mathbf{R}\quad\text{implies}\quad I^*\in\mathcal{S}\mathbf{R},$$

that is, that the semisimple class $\mathcal{S}\mathbf{R}$ is *hereditary.* Every radical class of algebras (without involution) satisfies A—D—S; this statement is the Anderson—Divinsky—Suliński Theorem [1], which is of fundamental importance in the general theory of radicals. For involution algebras, however, this is not always so, SALAVOVÁ ([5] Example 1, 9) gave a radical class $\mathbf{R}$ whose semisimple class is not hereditary, and consequently $\mathbf{R}$ does not satisfy A—D—S. In [8] WICHMANN suggested to deal with the problem whether a given radical class $\mathbf{R}$ of involution algebras satisfies A—D—S. This and related questions will be the topic of our investigations.

2. The A—D—S property

The next propositions show us that the variety $\mathfrak{V}$ of involution algebras is quite a good one.

Proposition 1. *If* $I^*\lhd A^*$, *then* $(I^2)^*\lhd A^*$.

Proof. It is clear that I^2 is an ideal of the algebra A. If $a, b \in I$, then

$$(ab)^* = b^* a^* \in I^2,$$

and so the assertion holds by the additivity of *.

Proposition 2. *For involution algebras the Andrunakievich Lemma holds; if* $M^* \lhd I^* \lhd A^*$ *and* L^* *denotes the ideal of* A^* *generated by* M^*, *then* $(L^*)^3 \subseteq M^*$.

Proof. We shall prove that the ideal J of the algebra A (without involution) generated by M is closed under involution. If $d \in J$, then

$$d = m + \sum_i a_i x_i + \sum_j y_j b_j + \sum_l u_l c_l v_l$$

where m, a_i, b_j, c_l are in M and x_i, y_j, u_l, v_l are in A. Since

$$d^* = m^* + \sum_i x_i^* a_i^* + \sum_j b_j^* y_j^* + \sum_l v_l^* c_l^* u_l^* \in M + AM + MA + AMA = J,$$

it follows that J^* is an involution algebra. Thus the Andrunakievich Lemma for algebras infers the assertion.

In view of Propositions 1 and 2, the proof of [2] Theorem 3, 2 and [2] Proposition 3, 5 yield immediately the following (see also [5] Предложение 3.4).

Corollary 1. *If* $\mathbf{R}$ *is a radical class of involution algebras which either contains all involution algebras with zero-multiplication or consists of idempotent involution algebras, then* $\mathbf{R}$ *satisfies A—D—S and hence the semisimple class* $\mathcal{S}\mathbf{R}$ *is hereditary.*

In order to prove necessary and sufficient conditions for a radical class $\mathbf{R}$ to satisfy A—D—S, we have to develop some techniques.

Proposition 3. *Let* A^* *be an involution algebra such that* $A^2=0$. *Then* A^{-*} *is also an involution algebra.*

Proof. Obvious.

Proposition 4. *Let* A^* *be an involution algebra such that* $A^2=0$. *If the unary operation* $^\square$ *of* $B=A\oplus A$ *is defined by* $(x, y)^\square=(y^*, x^*)$, *then* $B^\square$ *is an involution algebra.*

Proof. Straightforward.

Proposition 5. *Let* $\mathbf{R}$ *be a radical class of involution algebras,* $I^* \lhd A^*$ *and* $L^*=\mathbf{R}(I^*)$. *If* $(L^2/L^3)^* \in \mathbf{R}$, *then for any element* $a \in A$,

(i) $aL^2a^* \subseteq L$,

(ii) $(aL+La^*+L)^* \lhd I^*$,

(iii) *the mapping*

$$\varphi_a\colon ((L/L^2)\oplus(L/L^2))^\square \to ((aL+La^*+L)/L)^*$$

defined by $\varphi_a(x+L^2, y+L^2)=ax+ya^*+L$ *is a homomorphism onto*

$$((aL+La^*+L)/L)^*.$$

Proof. Let us define the mapping

$$\psi\colon (L^2/L^3)^* \to ((aL^2a^*+L)/L)^*$$

by $\psi(\sum x_i y_i+L^3)=a(\sum x_i y_i)a^*+L$. It is easy to check that ψ is a homomorphism onto $((aL^2a^*+L)/L)^*$, hence by $(L^2/L^3)^*\in\mathbf{R}$ we have

$$((aL^2a^*+L)/L)^*\in\mathbf{R}.$$

Since $((aL^2a^*+L)/L)^*\lhd I^*/L^*$, it follows

$$((aL^2a^*+L)/L)^* \subseteq \mathbf{R}(I^*/L^*) = \mathbf{R}(I^*/\mathbf{R}(I^*)) = 0.$$

Hence $aL^2a^*\subseteq L$ holds proving (i).

Using (i), one can verify easily that

$$(aL+La^*+L)^*\lhd I^*.$$

The last assertion is a straightforward calculation with an application of Proposition 4.

Proposition 6. *Let* $\mathbf{R}$ *be a radical class of involution algebras satisfying the following condition:*

(*) *if* $A^*\in\mathbf{R}$ *and* $A^2=0$, *then* $A^\circ\in\mathbf{R}$ *for any involution* $^\circ$ *on* A.

Then for any $L^*\in\mathbf{R}$, *also* $(L^2/L^3)^*\in\mathbf{R}$.

Proof. Let us consider an arbitrary element $a\in L$ and the mapping

$$f_a\colon L/L^2 \to L^2/L^3$$

defined by $f_a(x+L^2)=ax+L^3$ for all $x\in L$. The mapping f_a is obviously a homomorphism of the algebra $(L/L^2)^{\mathrm{id}}$ into $(L^2/L^3)^{\mathrm{id}}$ and $f_a(L/L^2)^{\mathrm{id}}\lhd(L^2/L^3)^{\mathrm{id}}$. Since $L^*\in\mathbf{R}$, condition (*) implies $(L/L^2)^{\mathrm{id}}\in\mathbf{R}$. Thus also $f_a(L/L^2)^{\mathrm{id}}\in\mathbf{R}$. Hence by

$$(L^2/L^3)^{\mathrm{id}} = \sum_{a\in L} f_a(L/L^2)^{\mathrm{id}} \subseteq \mathbf{R}((L^2/L^3)^{\mathrm{id}}) \subseteq (L^2/L^3)^{\mathrm{id}}$$

we have $(L^2/L^3)^{\mathrm{id}}\in\mathbf{R}$ and so condition (*) yields $(L^2/L^3)^*\in\mathbf{R}$.

Let A^* be an involution algebra over the ring K such that $A^2=0$. The ring K can be regarded as an involution algebra K^{id}. Let us consider the Cartesian product $E=K\times K\times A\times A$. On E we define operations by the following rules:

$$(a, b, x, y)+(c, d, u, v) = (a+c, b+d, x+u, y+v),$$

$$(a, b, x, y)(c, d, u, v) = (ac-bd, ad+bc, au-bv+cx-dy, av+bu+cy+dx),$$

$$k(a, b, x, y) = (ka, kb, kx, ky),$$

$$(a, b, x, y)^\circ = (a, -b, x^*, -y^*)$$

for all $a, b, c, d, k\in K$ and $x, y, u, v\in A$.

Proposition 7. *E° is an involution algebra with identity* $(1, 0, 0, 0)$.

$$I^\circ = \{(0, 0, x, y): x, y \in A\}$$

is an ideal of E° and $I^\circ \cong A^* \oplus A^{-*}$,

$$L^\circ = \{(0, 0, x, 0): x \in A\}$$

is an ideal of I°, $L^\circ \cong A^*$ but L° is not an ideal of E°.

Proof. The proof that E° is an involution algebra, is an exhausting verification and therefore we omit it. The further assertions are straightforward. For the last assertion we notice that by

$$(0, 1, 0, 0)(0, 0, x, 0) = (0, 0, 0, x)$$

we have $E^\circ \cdot L^\circ \nsubseteq L^\circ$.

In the following theorem we shall prove necessary and sufficient conditions for a radical class **R** to satisfy A—D—S. This will exhibit the decisive role of the behaviour of involution algebras with zero-multiplication.

Theorem 1. *For a radical class* **R** *of involution algebras the following conditions are equivalent:*

1) **R** *satisfies A—D—S,*
2) *if $A^* \in \mathbf{R}$ and $A^2 = 0$, then $A^\circ \in \mathbf{R}$ for any involution $^\circ$ built on A,*
3) *if $A^* \in \mathbf{R}$ and $A^2 = 0$, then $A^{-*} \in \mathbf{R}$,*
4) *$A^{\mathrm{id}} \in \mathbf{R}$ if and only if $A^{-\mathrm{id}} \in \mathbf{R}$ whenever $A^2 = 0$.*

Proof. 1)⇒3) Suppose that condition 3) is not satisfied, that is, there exists an involution algebra A^* such that $A^2 = 0$, $A^* \in \mathbf{R}$ and $A^{-*} \notin \mathbf{R}$. Obviously the ideals of A^* are exactly those of A^{-*}. Hence without loss of generality we may assume that $A^* \in \mathbf{R}$ and $A^{-*} \in \mathcal{S}\mathbf{R}$. Applying Proposition 7 we get $\mathbf{R}(I^\circ) = L^\circ$ and that L° is not an ideal of E°, though $I^\circ \lhd E^\circ$. Hence **R** does not satisfy A—D—S.

Next we show the equivalence of conditions 2), 3) and 4). The implications 2)⇒3)⇒4) are trivial. To prove the implication 4)⇒2), let us assume that $A^* \in \mathbf{R}$, $A^2 = 0$. First we shall show that $A^{\mathrm{id}} \in \mathbf{R}$. The set

$$D = \{x + x^*: x \in A\}$$

is clearly an ideal of A^*, moreover, $D^* = D^{\mathrm{id}}$ holds. The mapping

$$g: A^* \to D^*$$

defined by $g(x) = x + x^*$ is obviously a homomorphism of A^* onto D^*, and by $A^* \in \mathbf{R}$ it follows $D^{\mathrm{id}} = D^* \in \mathbf{R}$. In the factor algebra $(A/D)^*$ we have

$$x + D = -x^* + D$$

and

$$(x+D)^* = (-x^*+D)^* = -x+D.$$

Hence we have

$$(A/D)^{-\mathrm{id}} = (A/D)^* = A^*/D^* \in \mathbf{R}.$$

Applying condition 4) it follows $A^{\mathrm{id}}/D^{\mathrm{id}} = (A/D)^{\mathrm{id}} \in \mathbf{R}$. Since $\mathbf{R}$ is closed under extensions, $D^{\mathrm{id}} \in \mathbf{R}$ and $A^{\mathrm{id}}/D^{\mathrm{id}} \in \mathbf{R}$ implies $A^{\mathrm{id}} \in \mathbf{R}$.

Applying condition 4) again, we get $A^{-\mathrm{id}} \in \mathbf{R}$.

Let $^\circ$ denote an arbitrary involution on A and let us consider the set

$$C = \{x+x^\circ \colon x \in A\}.$$

As above, we get that $C^{-\circ} = C^{-\mathrm{id}}$ is a homomorphic image of $A^{-\mathrm{id}} \in \mathbf{R}$ and hence also $C^{-\mathrm{id}} \in \mathbf{R}$ holds. Since we have also

$$A^\circ/C^{-\mathrm{id}} = A^\circ/C^\circ = (A/C)^{-\mathrm{id}} = A^{-\mathrm{id}}/C^{-\mathrm{id}} \in \mathbf{R},$$

and since $\mathbf{R}$ is closed under extensions, we get $A^\circ \in \mathbf{R}$, proving the validity of condition 2).

Finally we shall show the implication 2)$\Rightarrow$1). Let $I^* \lhd A^*$ and $L^* = \mathbf{R}(I^*)$. By 2) and Proposition 4 we have $(L^2/L^3)^* \in \mathbf{R}$ and hence Proposition 5 (iii) yields that $((aL+La^*+L)/L)^*$ is a homomorphic image of the involution algebra $((L/L^2)\oplus(L/L^2))^\square$ which is in $\mathbf{R}$ in view of condition 2) and of $(L/L^2)^* \in \mathbf{R}$. Hence we have $((aL+La^*+L)/L)^* \in \mathbf{R}$. Applying Proposition 5 (ii) it follows that

$$((aL+La^*+L)/L)^* \lhd (I/L)^* = I^*/L^* = I^*/\mathbf{R}(I^*) \in \mathcal{S}\mathbf{R}.$$

Thus we get

$$((aL+La^*+L)/L)^* \subseteq \mathbf{R}(I^*/L^*) = 0,$$

that is, $aL+La^* \subseteq L^*$ holds. Hence

$$ax = ax+0a^* \in L \quad \text{and} \quad xa^* = a0+xa^* \in L^*$$

is valid for all $x \in L$. Since the choice of $a \in A$ was arbitrary, we have got $AL \cup \cup LA^* \subseteq L^*$, implying $\mathbf{R}(I^*) = L^* \lhd A^*$.

Let us notice that the assertion of Corollary 1 follows immediately also from Theorem 1.

Corollary 2. *For a radical class* $\mathbf{R}$ *the following conditions are equivalent:*

1) $\mathbf{R}$ *satisfies A—D—S,*

5) *if* $A^* \in \mathcal{S}\mathbf{R}$ *and* $A^2=0$, *then* $A^{-*} \in \mathcal{S}\mathbf{R}$,

6) $A^{\mathrm{id}} \in \mathcal{S}\mathbf{R}$ *if and only if* $A^{-\mathrm{id}} \in \mathcal{S}\mathbf{R}$ *whenever* $A^2=0$.

Proof. We show that 3) of Theorem 1 implies 5). Suppose that $A^* \in \mathcal{S}\mathbf{R}$. If $A^{-*} \notin \mathcal{S}\mathbf{R}$, then $0 \neq L^{-*} = \mathbf{R}(A^{-*}) \in \mathbf{R}$. By condition 3) we have $L^* \in \mathbf{R}$. Hence $L^* \subseteq \mathbf{R}(A^*) = 0$ contradicts $L^{-*} \neq 0$.

5)$\Rightarrow$4) Trivial.

6)$\Rightarrow$4) of Theorem 1. This can be proved similarly to the implication 3)$\Rightarrow$5) and therefore it is left to the reader.

Corollary 3. *Let* **R** *be a radical class in* $\mathfrak{B}$. *If all involution algebras of* **R** *with zero-multiplication are of characteristic* 2, *then* **R** *satisfies A—D—S.*

Proof. Condition 3) of Theorem 1 is satisfied, as $x^* = -x^*$ whenever $x \in A^* \in \mathbf{R}$ and $A^2 = 0$.

For varieties of not-necessarily associative algebras (without involution) over a field satisfying some weaker conditions than the assertions of Propositions 1 and 2, Anderson and Gardner [2] have proved that any radical class **R** either contains all algebras with zero-multiplication or consists of idempotent algebras, and therefore it satisfies A—D—S (cf. [2] Theorem 3. 9). The corresponding assertion for involution algebras is not true, as follows from the following theorem.

Theorem 2. *Let* $\mathfrak{B}$ *be the variety of all involution algebras over a field K. Every radical class in* $\mathfrak{B}$ *satisfies A—D—S if and only if* $\operatorname{char} K = 2$.

Proof. If $\operatorname{char} K = 2$, then Corollary 3 yields that every radical class satisfies A—D—S. In the case $\operatorname{char} K \neq 2$ on the underlying set K one can build two involution algebras K^{id} and K^{-id} such that $K^2 = 0$ and K^{id} is not isomorphic to K^{-id}. As K is a field, K^{id} is a simple involution algebra. Now the upper racical $\mathbf{R} = \mathcal{U}(K^{id})$ does not satisfy A—D—S because $K^{-id} \in \mathbf{R}$ and $K^{id} \in \mathcal{S}\mathbf{R}$.

3. The hereditariness of semisimple classes

If a radical class **R** satisfies A—D—S, then the corresponding semisimple class must be hereditary. The converse of this assertion is not true, and varieties of involution algebras provide natural examples to show that the hereditariness of a semisimple class $\mathcal{S}\mathbf{R}$ does not imply that **R** satisfies A—D—S.

Theorem 3. *A radical class of involution algebras with hereditary semisimple class, need not satisfy A—D—S.*

Proof. We shall construct a radical class **R** which has the desired properties. Let Z denote the algebra of integers (over the ring of integers) with zero-multiplication. The upper radical $\mathbf{R} = \mathcal{U}(Z^{id})$ does not satisfy A—D—S, as $Z^{-id} \in \mathbf{R}$ and $Z^{id} \in \mathcal{S}\mathbf{R}$.

We claim that $A^* = A^{id}$ for all $A^* \in \mathcal{S}\mathbf{R}$. Suppose that $x^* \neq x$ for some $x \in A$, and consider the ideal B^* of A^* generated by the element $x^* - x$. Since $A^* \in \mathcal{S}\mathbf{R}$, there exists a homomorphism φ of B^* onto Z^{id}. Hence

$$\varphi(x^*-x) = \varphi(x^*)-\varphi(x) = \varphi(x)^{\mathrm{id}}-\varphi(x) = 0$$

holds, that is, $x^*-x \in \ker\varphi$. This implies $0 = B^*/\ker\varphi \cong Z^{\mathrm{id}}$, a contradiction. Thus $A^* = A^{\mathrm{id}}$

If $I^* \lhd A^* \in \mathcal{S}\mathbf{R}$, then by $A^* = A^{\mathrm{id}}$ we have $I^* = I^{\mathrm{id}}$. Hence the standard proof of the hereditariness of semisimple classes of algebras (without involution) works, yielding $I^* \in \mathcal{S}\mathbf{R}$ (cf. [1] or [9]).

We shall see that a semisimple class $\mathcal{S}\mathbf{R}$ need not be hereditary, even if $\mathbf{R}$ is a hereditary radical class. Prior to this we prove some assertions.

Proposition 8. *Let A^* be an involution algebra such that $2a \neq 0$ whenever $0 \neq a \in A$. If A^* has an ideal I^* such that $I^* = I^{\mathrm{id}}$ and $(A/I)^* = (A/I)^{\mathrm{id}}$, then also $A^* = A^{\mathrm{id}}$ holds.*

Proof. In $(A/I)^{\mathrm{id}}$ we have

$$x^*+I = (x+I)^* = (x+I)^{\mathrm{id}} = x+I,$$

yielding $x^*-x \in I$ for all $x \in A$. Hence

$$x^*-x = (x^*-x)^{\mathrm{id}} = (x^*-x)^* = x-x^*$$

holds, so we get $2(x^*-x) = 0$ for all $x \in A$. By the assumption we conclude that $x^* = x$, that is, $A^* = A^{\mathrm{id}}$.

The Tangeman—Kreiling [7] lower radical construction carries over to involution algebras without difficulty. Given a class $\mathbf{C}$ of involution algebras, define inductively

$\mathbf{C}_1 = \{A^*\colon A^*$ is a homomorphic image of an involution algebra $B^* \in \mathbf{C}\}$,

$\mathbf{C}_\lambda = \{A^*\colon$ there exists an $I^* \lhd A^*$ such that $I^* \in \mathbf{C}_{\lambda-1}$ and $(A/I)^* \in \mathbf{C}_{\lambda-1}\}$

if $\lambda-1$ exists, and

$\mathbf{C}_\lambda = \{A^*\colon A^*$ is the union of an ascending chain of ideals each belonging to one of the classes $\mathbf{C}_\mu$, $\mu < \lambda\}$

if λ is a limit ordinal. Then the smallest radical class containing $\mathbf{C}$, called the *lower radical* of $\mathbf{C}$, is given as

$$\mathcal{L}\mathbf{C} = \cup(C_\lambda \text{ for all ordinals}).$$

If, in addition, $\mathbf{C}$ is a hereditary class of involution algebras, then so is the lower radical $\mathcal{L}\mathbf{C}$.

Proposition 9. *Let $\mathbf{C}$ be the class of involution algebras such that in each $A^* \in \mathbf{C}$, $2a = 0$ implies $a = 0$. If $A^* = A^{\mathrm{id}}$ for each $A^* \in \mathbf{C}$, then $A^* = A^{\mathrm{id}}$ holds also for every $A^* \in \mathcal{L}\mathbf{C}$.*

Proof. By Proposition 8 each class $\mathbf{C}_\lambda$ consists of algebras with involution id. This yields the assertion.

Corollary 4. *In the variety* $\mathfrak{V}$ *of involution algebras over a field* K *with* char $K\neq 2$, *the class*

$$\mathbf{I} = \{A^*\in\mathfrak{V}:\ A^* = A^{\mathrm{id}}\}$$

is a radical class.

Proof. By Proposition 9 we have $\mathbf{I}=\mathscr{L}\mathbf{I}$.

Theorem 4. *Let* $\mathfrak{V}$ *be the variety of all involution algebras over a ring* K *and assume that a homomorphic image* K/M *of* K *is an integral domain such that the quotient field* F *of* K/M *is not of characteristic* 2. *In* $\mathfrak{V}$ *there exists a hereditary radical* $\mathbf{R}$ *whose semisimple class* $\mathscr{S}\mathbf{R}$ *is not hereditary.*

Let us notice that the condition imposed on K is relatively mild as it includes, for instance, the case when K is a subdirect sum of subdirectly irreducible rings such that at least one of them is a field of characteristic $\neq 2$.

Proof. Let us build an involution algebra E° on the set $E=F\times F\times F\times F$ similarly as before Proposition 7 with involution defined by

$$(a, b, x, y)^\circ = (a, -b, x, -y).$$

The assertions of Proposition 7 remain valid. Using the notations of Proposition 7 we have that

$$L^\circ = \{(0, 0, x, 0):\ x\in F\}$$

and so

$$L^{-\circ} = \{(0, 0, 0, y):\ y\in F\}.$$

Further $L^\circ \lhd I^\circ \lhd E^\circ$ holds, but L° is not an ideal of E°. We claim that I° is a maximal ideal of E°. Let J° be an ideal of E° such that $I^\circ \subseteqq J^\circ$, and let $(a, b, x, y)\in J^\circ$ be any element of $J^\circ \diagdown I^\circ$. We have

$$(a, b, 0, 0) = (a, b, x, y)-(0, 0, x, y)\in J^\circ.$$

If $a\neq 0$, then

$$(a, -b, 0, 0) = (a, b, 0, 0)^\circ \in J^\circ$$

yields $(2a, 0, 0, 0)\in J^\circ$. Hence

$$(1, 0, 0, 0) = (2a, 0, 0, 0)((2a)^{-1}, 0, 0, 0)\in J^\circ$$

holds, implying $J^\circ=E^\circ$. If $a=0$, then $b\neq 0$ and so

$$(1, 0, 0, 0) = (0, b, 0, 0)(0, -b^{-1}, 0, 0)\in J^\circ$$

is valid, implying $J^\circ=E^\circ$. Thus $(E/I)^\circ$ is a simple idempotent involution algebra.

Let $\mathbf{F}$ denote the class consisting of all ideals of the involution algebra L°. The lower radical $\mathbf{R}=\mathscr{L}\mathbf{F}$ of $\mathbf{F}$ is hereditary and by Proposition 9 $A^*=A^{\mathrm{id}}$ holds for every $A^*\in\mathbf{R}$. Moreover, as $(E/I)^\circ$ is simple and idempotent, it follows $(E/I)^\circ\in\mathcal{S}\mathbf{R}$ and so $\mathbf{R}(E^\circ)\subseteqq I^\circ$. As $\mathbf{R}(L^{-\circ})\in\mathbf{R}$, by Proposition 9 we have $x=-x$ for all $x\in\mathbf{R}(L^{-\circ})$. Since char $F\neq 2$, it follows $x=0$, and so $L^{-\circ}\in\mathcal{S}\mathbf{R}$. Hence by the hereditariness of $\mathbf{R}$ we obtain

$$\mathbf{R}(E^\circ) = I^\circ\cap\mathbf{R}(E^\circ) \subseteqq \mathbf{R}(I^\circ) = L^\circ,$$

and the inclusion must be proper as L° is not an ideal in E°. Let $(0, 0, x, 0)$ be an arbitrary element of $\mathbf{R}(E^\circ)$. If $x\neq 0$, then for each $f\in F$ we have

$$(0, 0, f, 0) = (fx^{-1}, 0, 0, 0)(0, 0, x, 0)\in\mathbf{R}(E^\circ)$$

implying $\mathbf{R}(E^\circ)=L^\circ$, which is impossible. Hence $\mathbf{R}(E^\circ)=0$ and so $\mathbf{R}(I^\circ)\nsubseteq\mathbf{R}(E^\circ)$ holds. This means that $\mathcal{S}\mathbf{R}$ is not hereditary.

Recall that a class $\mathbf{C}$ of algebras (with or without involution) is said to be a *coradical class,* if $\mathbf{C}$ is hereditary and closed under subdirect sums and extensions. In [2] Anderson and Gardner posed the question whether the concepts "semisimple class" and "coradical class" coincide in a variety of rings satisfying some weaker conditions than the assertions of Proposition 1 and 2. [5] Example 1, 9 of Salavová or our Theorem 4 gives a negative answer to this question.

Corollary 5. *A semisimple class of involution algebras need not be a coradical class.*

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