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On cell complexes generated by geodesics in the non-Euclidean elliptic plane

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Abstract. In this paper we consider some properties of cell complexes in the non-Euclidean elliptic space, which are generated by n geodesic lines. The cells are geodesically convex polygons and in non-degenerate case the number of cells equals $\binom{n}{2}+1$. Each cell-complex has at least one cell with maximal number of vertices. If we denote by ξ_n this maximal number and $\alpha_n = \min \xi_n$, where the minimum is taken over all possible complexes, then we show that $\alpha_3=3$, $\alpha_4=4$ and $\alpha_n=5$ for all $n \geqq 5$.

Introduction. Consider the space G of straight lines on the plane R^2. Let $O \in R^2$ be the origin and denote by $[O]$ the bundle of lines through O. For $g \in G \setminus [O]$ let (p, φ) be the polar coordinates of the foot of perpendicular from the origin on g. It is usual to consider the pair (p, φ) as coordinates of the line g, where $p \in]0, \infty[$, $\varphi \in [0, 2\pi]$. Thus $G \setminus [O]$ is mapped onto semi-cylinder without rim, having ordinary cylindric coordinates. Note that diametrically opposite points on the rim correspond to the same line from the bundle $[O]$. Hence for the space G we obtain the model C of a semi-cylinder with identified opposite points on the rim (see A. Baddeley in [2]). By means of central projection the manifold C can be mapped onto the elliptic plane E_2 with punctured pole N (see Fig. 1).

We shall denote the corresponding homeomorphism by $\Phi: G \to E_2 \setminus N$. Especially important is that under Φ the bundles of lines on R^2 correspond to the geodesics in E_2 (see [1]).

Note that the inverse mapping Φ^{-1} is well-defined if an origin and a reference direction are chosen.

Denote by Γ the space of geodesic lines γ in E_2. One can easily see that $\Phi^{-1}(\gamma)$ is either a bundle of parallels (if $N \in \gamma$), or a bundle of lines, passing through a point $\mathscr{P} \in R^2$. Thus we have the mapping $\Psi: \Gamma \setminus [N] \to R^2$. A collection of geodesics

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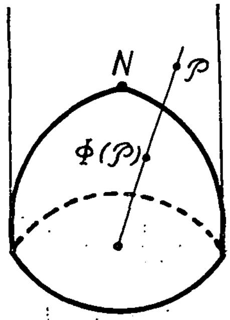

Figure 1. The points $\mathscr{P}$ and $\Phi(\mathscr{P})$ lie on a line through the centre of the circle.

$\{\gamma_i\}_{i=1}^n$ is called non-degenerate, if no three γ_i-s pass through a point. If we choose the pole $N\overline{\in}\bigcup_{i=1}^n \gamma_i$, then $\{\mathscr{P}_i=\Psi(\gamma_i)\}_{i=1}^n$ is a set of points in R^2. Note, that if $\{\gamma_i\}_{i=1}^n$ is non-degenerate, then the corresponding set $\{\mathscr{P}_i\}_{i=1}^n$ consists of n points in general position, i.e. no three lie on a line.

The application of mappings Φ^{-1} and Ψ facilitates the analysis of many properties of cell-complexes by reducing the problem to the investigation of the corresponding sets in G. The efficiency of such approach was suggested to the author by R. V. Ambartzumian.

The main problem. We consider a cell-complex in E_2, generated by non-degenerate collections of geodesic lines $\{\gamma_i\}_{i=1}^n$. Our problem is as follows. In non-degenerate case the number of cells equals $\binom{n}{2}+1$ and every cell-complex has at least one cell with maximal number of neighbours (the cells are polygons, they are regarded as neighbours, if they have a common side). If we denote this maximal number by ξ_n then the problem is to find $\min \xi_n$, where the minimum is with respect to all possible non-degenerate collections $\{\gamma_i\}_{i=1}^n$. Let us fix a non-degenerate $\{\gamma_i\}_{i=1}^n$ and consider the corresponding set $\{\mathscr{P}_i\}_{i=1}^n$. We call two lines $g_1, g_2 \in G$ $(g_1, g_2 \overline{\in} \bigcup_{i=1}^n [\mathscr{P}_i])$ equivalent if they produce the same separation of the set $\{\mathscr{P}_i\}_{i=1}^n$ into two subsets.

Further we shall call each class of equivalent lines an atom in G. Each atom corresponds (via Φ^{-1}) to a cell in E_2 and there is exactly one unbounded atom, which corresponds to the cell in E_2 containing the pole N. We shall use the following algorithm to determine the number of neighbours of a cell α from a cell-complex on E_2.

Algorithm. Denote by g_{ij} the straight line through the points $\mathscr{P}_i$ and $\mathscr{P}_j$. The number of neighbours of the atom α is equal to the number of lines from the collection g_{ij}, which belong to the boundary of the atom $\Phi^{-1}(\alpha)$ (the latter lines will be termed "limiting" lines of the atom).

For example (see Fig. 2) the atom containing the line g is a pentagon.

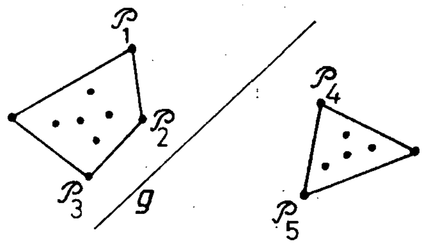

Figure 2. The limiting lines of the atom generated by g are $g_{12}, g_{23}, g_{15}, g_{34}, g_{45}$.

We state (without proof) the following simple lemma.

Lemma. *The number of sides of the minimal convex hull of the set* $\{\mathcal{P}_i\}_{i=1}^n$ *is equal to the number of neighbours of the unbounded atom.*

Let us make some remarks on the properties of cell-complexes. Here and below the term "cell-complex" (c.c.) will mean "partition of the non-Euclidean elliptic plane E_2 by a non-degenerate family of geodesics".

Remark I. Let $n \geqq 5$, and suppose that among the atoms of c.c. there is at least one n-gon. Then the c.c. consists of exactly one n-gon, n triangles and $n(n-3)/2$ quadrangles.

Proof. Let us denote the n-gon by α. We choose the pole N in α and construct the mapping $\Psi\colon \Gamma \setminus [N] \to R^2$. By the Lemma $\{\mathcal{P}_i\}_{i=1}^n$ forms a convex n-gon.

Applying the above algorithm one can easily show (Fig. 3), that the remaining atoms are either triangles or quadrangles. Namely, the atoms, which contain a line

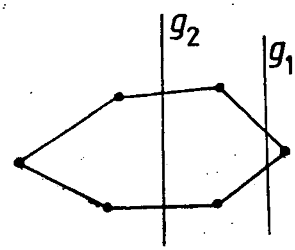

Figure 3. The line g_1 belongs to a triangular atom and g_2 belongs to a quadrangular atom.

separating one vertex from the others, are triangles and their number is exactly n. All other atoms are quadrangles and their number is $\binom{n}{2}-n=n(n-3)/2$.

Remark II. If $n>3$, then $\min \xi_n>3$.

Proof. Suppose, to the contrary, that there exists a c.c. consisting of triangles only. Then, by the Lemma, for any choice of N, the minimal convex hull of the set $\{\mathscr{P}_i=\Psi(\gamma_i)\}_{i=1}^n$ is a triangle.

It is not difficult to see that the atom defined by the line shown in Fig. 4 is a k-gon, with $k\geqq 4$. This contradiction proves Remark II.

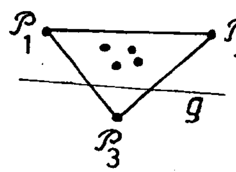

Figure 4. $\mathscr{P}_3$ is separated from all other points.

Remark III. If $n>4$, then $\min \xi_n>4$.

Proof. Suppose, to the contrary, that there exists a c.c. consisting of triangles and quadrangles only. By (II) we can find a quadrangular atom. If we choose the pole N in this atom then, by the Lemma, let the points $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$ and $\mathscr{P}_4$ form the minimal convex hull of $\{\mathscr{P}_i\}_{i=1}^n$ (Fig. 5).

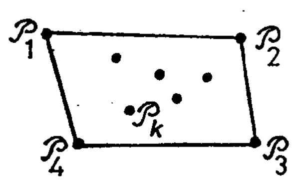

Figure 5.

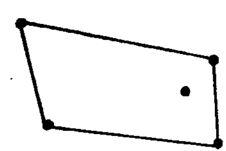

Figure 6.

Consider the collection $\{\mathscr{P}_i\}_{i=1}^n\diagdown\{\mathscr{P}_k\}$ (we delete the point $\mathscr{P}_k$), where $\mathscr{P}_k$ belongs to the interior of $\mathscr{P}_1\mathscr{P}_2\mathscr{P}_3\mathscr{P}_4$. This corresponds to the deletion of the geodesic γ_k on E_2. It can be proved that the deletion of this geodesic cannot result in the formation a new polygon with more than 4 sides. Deleting successively the points different from $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$, $\mathscr{P}_4$, we obtain a five-point set (see Fig. 6). Here we can easily show the pentagonal atom. This contradiction proves Remark III.

Remark IV. Denote by q_k the number of k-gons of the c.c. The method described above answers the following question: What are the possible sequences $(q_1, q_2, \ldots, q_n)$, generated by c.c.? We have found that for $n=3, 4, 5$ all possible cases are as follows:

$$n = 3, \quad q_3 = 4,$$
$$n = 4, \quad q_3 = 4, \quad q_4 = 5,$$
$$n = 5, \quad q_3 = 5, \quad q_4 = 5, \quad q_5 = 1.$$

For $n=6$ we have the following possibilities.

$$q_3 = 6, \quad q_4 = 9, \quad q_5 = 0, \quad q_6 = 1,$$
$$q_3 = 10, \quad q_4 = 0, \quad q_5 = 6, \quad q_6 = 0,$$
$$q_3 = 6, \quad q_4 = 8, \quad q_5 = 2, \quad q_6 = 0.$$

In particular, we obtain that $\min \xi_3=3$, $\min \xi_4=4$, $\min \xi_5=5$. What is the $\min \xi_n$, when $n>5$? The answer is given by the following

Theorem.
$$\min \xi_n = \begin{cases} n, & n \leqq 5 \\ 5, & n > 5. \end{cases}$$

Proof. It is sufficient to construct such a set of points on R^2, which have only "triangular", "quadrangular" or "pentagonal" atoms.

Consider a unit square $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4$ on R^2 (see Fig. 7).

We shall place the points $\{\mathcal{P}_i\}_{i=1}^n$ on congruent arcs σ_i, which emanate from vertices $\mathcal{P}_i$ $(i=1, 2, 3, 4)$ and lie within the square. Two points $Q_1 \in \sigma_i$ and $Q_2 \in \sigma_j$ $(i \neq j)$ are called corresponding, if Q_1 goes in Q_2 under euclidean motion, which brings σ_i in to σ_j.

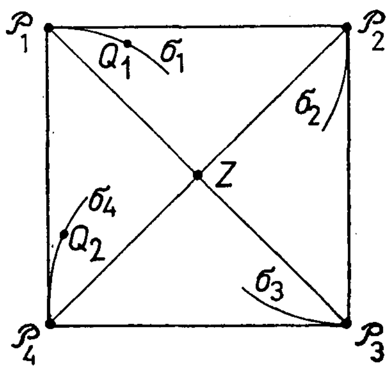

Figure 7.

Let $y=f(x)$ be an equation of σ_1. We shall find $f(x)$, using the following condition:

The tangent at every point of σ_1 crosses σ_2 in the point, which corresponds to the point of tangency on σ_1.

From this we derive the differential equation

$$\text{(A)} \qquad \frac{1-x-y}{y-x} = \frac{dy}{dx} \quad \text{with initial condition} \quad y(0) = 1.$$

The solution of this Cauchy problem exists, it is unique and it is a logarithmic helix. From (A) we deduce that the curve $y=f(x)$ is convex in the neighbourhood of $\mathscr{P}_1$ and it has horizontal tangent at the point $\mathscr{P}_1$ (side of the square).

We take each σ_i to be a "piece" of logarithmic helix. Denote by λ the common length of the arcs σ_i $(i=1, 2, 3, 4)$. Let $\delta_1, \delta_2, \ldots, \delta_k, \ldots$ be a sequence of positive numbers such that $\sum \delta_i < \lambda$. Now we proceed to construct the desired set. First we construct an auxiliary sequence of points $\{Q_i\}$ on the curve σ_1. Let Q_1 be the endpoint of σ_1 and if the points $Q_1, Q_2, \ldots, Q_j$ have been constructed, then Q_{j+1} is constructed as follows. We draw the line $\mathscr{P}_1 Q_j$. Let Q_j' be the intersection point of this line with σ_2 (see Fig. 8). Starting from Q_j' we move along σ_2 in the direction of $\mathscr{P}_2$ at distance δ_j. In this way we obtain the point Q_j''. Now we draw a line through Q_j'', which is tangent to σ_1 and let Q_{j+1} be the point of tangency. It is clear that in this way an infinite sequence of points $\{Q_i\}_{i=1}^{\infty}$ can be constructed. Now we describe how we construct the collection $\{\mathscr{P}_i\}_{i=1}^{n}$. On σ_1 we construct $[n/4]$ points Q_i, where $[n/4]=$"entier" of $n/4$. Further, we construct the corresponding points on the arcs σ_i $(i=2, 3, 4)$. Together with the vertices of the square we have now $4[n/4]+4$ points. The set $\{\mathscr{P}_i\}_{i=1}^{n}$ is obtained by deletion of $4-n$ (mod 4) extremal points on the arcs σ_i (which are distinct from the vertices of the square). The so obtained set is denoted by P.

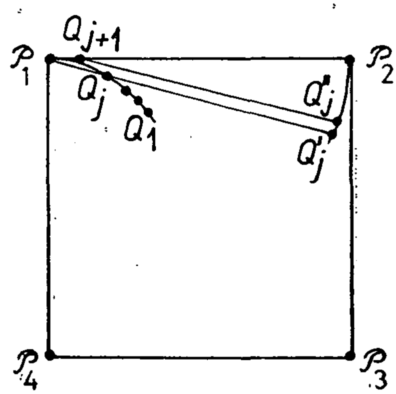

Figure 8. The length of arc $Q_j'Q_j''$ is δ_j

Now we shall verify that all atoms generated by P (equivalently the cells of the corresponding complex on E_2) are triangles, quadrangles or pentagons. For the description of an arbitrary atom α it is sufficent to determine those two subsets of P, which are separated by the lines of the atom. Therefore we shall use the following notations: $\alpha = F|P\setminus F$, where F, $P\setminus F$ are the two subsets in question.

Denote by $\mathcal{M}_i$ the set of points $\{\mathcal{P}_j\}$ lying on σ_i, then $P=\bigcup_{i=1}^{4}\mathcal{M}_i$. By choosing a sufficiently small λ, it is possible to satisfy the following conditions:

(a) Every σ_i is contained in the triangle $\mathcal{P}_i Z \mathcal{P}_j$, where $j=i+1 \pmod 4$, $i\in\{1, 2, 3, 4\}$.

(b) The segments $\mathcal{P}_j Q_1$ $(j=2, 3, 4)$ are intersected by no σ_i $(i=2, 3, 4)$. The same is true for segments, joining $\mathcal{P}_1$ with the endpoints of σ_i $(i=2, 3, 4)$.

(c) The lines g intersect the same σ_i in at most two points.

Let us introduce a classification of the atoms. In our classification we denote by $\mathcal{A}_j$ the classes of the atoms. All the sets F, M in the description of the atoms will be non-empty. Below, the sign $\subset$ denotes only proper inclusion. We put

$\mathcal{A}_1 = \{\emptyset|P\}$,

$\mathcal{A}_2 = \{F|P\setminus F$, where $F\subset \mathcal{M}_i$ for some $i\}$,

$\mathcal{A}_3 = \{\mathcal{M}_i|P\setminus \mathcal{M}_i$ for some $i\}$,

$\mathcal{A}_4 = \{\mathcal{M}_i\cup \mathcal{M}_j|P\setminus(\mathcal{M}_i\cup \mathcal{M}_j)$, where $i\neq j\}$,

$\mathcal{A}_5 = \{\mathcal{M}_i\cup F|P\setminus(\mathcal{M}_i\cup F)$, where $F\subset \mathcal{M}_j$, $i\neq j\}$,

$\mathcal{A}_6 = \{F\cup M|P\setminus(F\cup M)$, where $F\subset \mathcal{M}_i$, $M\subset \mathcal{M}_j$ $(i\neq j)$ and every g intersects σ_i and σ_j in one point only$\}$,

$\mathcal{A}_7 = \{F\cup M|P\setminus(F\cup M)$, where $F\subset \mathcal{M}_i$, $M\subset \mathcal{M}_j$ $(i\neq j)$ and every g intersects σ_i (or σ_j) in exactly two points$\}$,

$\mathcal{A}_8 = \{\mathcal{M}_i\cup F\cup M|P\setminus(\mathcal{M}_i\cup F\cup M)$, where $F\subset \mathcal{M}_k$, $M\subset \mathcal{M}_j$ and $i\neq j$, $j\neq k$, $i\neq k.\}$

By our choice of λ this classification is complete. Direct verification shows that the atom of $\mathcal{A}_1$ is a quadrangle (by the Lemma),

the atoms of $\mathcal{A}_2$ are either quadrangles, or pentagons (by the construction of $\{Q_i\}$ and (a), (b).),

the atoms of $\mathcal{A}_3$ are either triangles or quadrangles (by the construction of $\{Q_i\}$ and (a), (b)),

the atoms $\mathcal{A}_4$ are either quadrangles or pentagons (by the construction of $\{Q_i\}$ and (a), (b)),

the atoms of $\mathcal{A}_5$ are either quadrangles or pentagons (by the construction of $\{Q_i\}$ and (a), (b)), the atoms of $\mathscr{A}_6$ are quadrangles (by the construction of σ_i and (a), (b)),

the atoms of $\mathscr{A}_7$ are either quadrangles or pentagons (by the construction of $\{Q_i\}$, σ_i and (a), (b), (c)),

the atoms of $\mathscr{A}_8$ are quadrangles (by the construction of σ_i and (a), (b)).

Let us consider one of the types of the atoms, say $\mathscr{A}_2$, in more detail (see Fig. 9). Let g be a line defining an atom from $\mathscr{A}_2$, say $F|P\setminus F$, where $F\subset\mathscr{M}_1$. Then there exist two points Q_k and Q_{k+1} belonging to $\mathscr{M}_1$ such that $Q_k\in P\setminus F$, $Q_{k+1}\in F$. Further let Q'_k and Q'_{k+1} be the points from $\mathscr{M}_4$ corresponding to Q_k and Q_{k+1}. Then the limiting lines are $Q_{k+1}\mathscr{P}_2$, $Q'_{k+1}Q_{k+1}$, Q'_kQ_k, $Q_k\mathscr{P}_2$, $Q'_kQ'_{k+1}$ (by the construction of $\{Q_i\}$ and the choice λ). Hence this atom is a pentagon. If $Q_k=Q_1$ and $n\not\equiv 0 \pmod 4$, then any atom from $\mathscr{A}_2$ is a quadrangle.

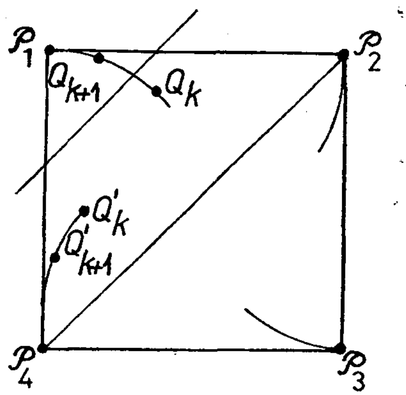

Figure 9

One can similarly treat the other seven cases. This will conclude the proof of the Theorem.

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