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On relations of coefficient conditions

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In the theory of orthogonal series several different kinds of coefficient condition are being used and among them the three most frequently investigated conditions are

(i) $$\sum_{n=1}^{\infty} c_n^2 \varrho_n < \infty,$$

(ii) $$\sum_{k=1}^{\infty} \lambda_k \Big(\sum_{n=\nu_k+1}^{\nu_{k+1}} c_n^2\Big)^{\varepsilon/2} < \infty$$

and

(iii) $$\sum_{k=1}^{\infty} \varkappa_k \Big(\sum_{n=k}^{\infty} c_n^2\Big)^{\varepsilon/2} < \infty.$$

Here $\varepsilon > 0$, $\{\varrho_n\}$, $\{\lambda_n\}$ and $\{\varkappa_n\}$ are certain monotone sequences of real numbers and $\{c_n\}$ is a real coefficient sequence. For different results incorporating (i), (ii) or (iii) we refer to [2], a paper devoted to the systematic study of the connections between (i), (ii) or (iii). In it L. Leindler gave sufficient conditions for the equivalence of (ii) and (iii) (for any sequence $\{c_n\}$) and investigated the relation between (i) and (ii). Our aim in this paper is to give necessary and sufficient conditions for the equivalences above in a somewhat more general setting.

Let us consider the conditions

(1) $$\sum_{n=1}^{\infty} \varrho_n |c_n|^q < \infty$$

(2) $$\sum_{k=1}^{\infty} \lambda_k \Big(\sum_{n=\nu_k+1}^{\nu_{k+1}} |c_n|^q\Big)^{p/q} < \infty$$

and

(3) $$\sum_{k=1}^{\infty} \varkappa_k \Big(\sum_{n=k}^{\infty} |c_n|^q\Big)^{p/q} < \infty,$$

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where p and q are positive numbers, $\{\varrho_n\}$ and $\{\varkappa_k\}$ positive sequences, $\{\lambda_k\}$ is a positive monotone sequence and $\{\nu_k\}$ is a subsequence of the natural numbers.

We have the following theorems:

Theorem 1. a) *If $p=q$ then conditions* (2) *and* (3) *are equivalent for every* $\{c_n\}$ *if and only if*

$$(1/A)\sum_{k=1}^{\nu_{m+1}} \varkappa_k \leqq \lambda_m \leqq A \sum_{k=1}^{\nu_m+1} \varkappa_k \quad (m = 1, 2, 3, \ldots) \tag{4}$$

is satisfied with some constant A.

b) *If $p \neq q$ then* (2) *and* (3) *are equivalent if and only if* (4) *and*

$$\lambda_{m+A} \geqq 2\lambda_m \quad (m = 1, 2, \ldots) \tag{5}$$

are satisfied with some natural number A.

Theorem 2. *Conditions* (1) *and* (2) *cannot be equivalent unless $p=q$. If $p=q$ then they are equivalent if and only if there is an A with*

$$(1/A)\lambda_m \leqq \varrho_n \leqq A\lambda_m \quad (m = 1, 2, 3, \ldots;\ \nu_m < n \leqq \nu_{m+1}).$$

Theorem 3. a) *If $p \neq q$ then* (1) *and* (3) *are equivalent if and only if the three sequences*

$$\{\varrho_n\}, \quad \{1/\varrho_n\}, \quad \Big\{\sum_{k=1}^{n} \varkappa_k\Big\}$$

are bounded.

b) *If $p=q$ then* (1) *and* (3) *are equivalent if and only if*

$$(1/A)\varrho_m \leqq \sum_{k=1}^{m} \varkappa_k \leqq A\varrho_m \quad (m = 1, 2, 3, \ldots)$$

is satisfied with a constant A.

Our theorems have several consequences, the most remarkable one is that, since e.g. (4) and (5) are independent of p and q $(p \neq q)$, the equivalence of (2) and (3) for a pair p, q $(p \neq q)$ implies their equivalence for any other pair p', q'. Another direct corollary of Theorem 1 is [2, Theorem 2.1].

The sufficiency of our conditions can be verified by more or less direct considerations using some well-known inequalities (such as Jensen's inequality) from the theory of sequences. Our necessity proofs, however, have a very general character — a certain boundedness principle is applied in them. Since the proofs run on similar lines we shall give a detailed proof only for Theorem 1. However, we emphasize that the method can be applied to Theorems 2 and 3 and to other equivalence problems of this kind.

Proof of Theorem 1. We separately prove the necessity and sufficiency of our conditions.

I. Necessity. Let X_1 and X_2 be the set of the sequences $\{c_n\}$ for which (2) and (3) are satisfied, respectively. Then with the usual operations X_1 and X_2 are linear spaces. We introduce on them length functions $\|\ \|_i$: $X_i \to R_+$ $(i=1, 2)$ as follows: for $c=\{c_n\}$ let

$$|c|_1 = \sum_{k=1}^{\infty} \lambda_k \Big(\sum_{n=\nu_k+1}^{\nu_{k+1}} |c_n|^q \Big)^{p/q},$$

$$\|c\|_1 = \begin{cases} (|c|_1)^{1/p} & \text{if } \ p, q \geqq 1, \\ |c|_1 & \text{if } \ 0 < p \leqq q, \quad 0 < p < 1, \\ (|c|_1)^{q/p} & \text{if } \ 0 < q \leqq p, \quad 0 < q < 1, \end{cases}$$

and

$$|c|_2 = \sum_{k=1}^{\infty} \varkappa_k \Big(\sum_{n=k}^{\infty} |c_n|^q \Big)^{p/q},$$

$$\|c\|_2 = \begin{cases} (|c|_2)^{1/p} & \text{if } \ p, q \geqq 1, \\ |c|_2 & \text{if } \ 0 < p \leqq q, \quad 0 < p < 1, \\ (|c|_2)^{q/p} & \text{if } \ 0 < q < p, \quad 0 < q < 1. \end{cases}$$

These length functions induce two metrics:

$$d_1(c_1, c_2) = \|c_1 - c_2\|_1, \quad d_2(c_1, c_2) = \|c_1 - c_2\|_2$$

and it is easy to see that (X_1, d_1) and (X_2, d_2) are complete metric spaces, the metrics d_1 and d_2 are invariant, i.e. $d_i(c_1, c_2) = d_i(c_1 - c_2, 0)$ $(i=1, 2)$, furthermore, the mappings $(\lambda, c) \to \lambda c$ of $R \times X_i \to X_i$ $(i=1, 2)$ are continuous in λ for each c and in c for each λ.

Summarizing and putting into the terminology of function spaces we can say that (X_1, d_1) and (X_2, d_2) are F-spaces (see [1, pp. 50—51]).

Now suppose that (2) implies (3), i.e. $X_1 \subseteqq X_2$. For $c=\{c_n\} \in X_1$ and a natural number m let

$$T_m c = d \quad \text{where} \quad d = \{c_1, c_2, \ldots, c_m, 0, 0, \ldots\}.$$

By the assumption the sequence $\{T_m\}$ of the bounded linear operators T_m: $X_1 \to X_2$ is pointwise bounded, i.e. for every $c \in X_1$ there is a bound K_c such that

$$\|T_m c\|_2 \leqq K_c \quad (m = 1, 2, \ldots).$$

Since the length functions $\|\cdot\|_1$ and $\|\cdot\|_2$ are homogeneous of the same degree, the theorem of Banach and Steinhaus valid for operators between F-spaces (see [1, p. 52]) yields that there exists a uniform bound A such that

$$\|T_m c\|_2 \leqq A \|c\|_1 \quad (c \in X_1, \ m = 1, 2, \ldots)$$

is satisfied, and letting here m tend to infinity we arrive at

$$\|c\|_2 \leqq A\|c\|_1 \quad (c\in X_1).$$

Similarly, it can be proved that if (3) implies (2) then with some constant

$$\|c\|_1 \leqq A\|c\|_2 \quad (c\in X_2).$$

Now we introduce the perhaps somewhat awkward, nonetheless suggestive notations

$$((2)) = \sum_{k=1}^{\infty} \lambda_k \Big(\sum_{n=\nu_k+1}^{\nu_{k+1}} |c_n|^q\Big)^{p/q},$$

$$((3)) = \sum_{k=1}^{\infty} \varkappa_k \Big(\sum_{n=k}^{\infty} |c_n|^q\Big)^{p/q}$$

for the sums involved in (2) and (3).

According to what we have proved above the equivalence of (2) and (3) implies the inequalities $((3))\leqq A((2))$ and $((2))\leqq A((3))$. Applying the first one of these to the sequence

$$c_n = \begin{cases} 1 & \text{if} \quad n = \nu_{m+1} \\ 0 & \text{otherwise} \end{cases}$$

and the second one to

$$c_n = \begin{cases} 1 & \text{if} \quad n = \nu_m + 1 \\ 0 & \text{otherwise} \end{cases}$$

where m is a fixed natural number we get that

$$\sum_{k=1}^{\nu_{m+1}} \varkappa_k \leqq A\lambda_m \quad \text{and} \quad \lambda_m \leqq A \sum_{k=1}^{\nu_m+1} \varkappa_k$$

are satisfied and the necessity of (4) has been verified.

To prove that (5) also holds we remark first that in the case $\lambda_m \searrow$ we have by (4) $\lambda_1 \geqq \lambda_k \geqq (1/A^2)\lambda_1$ and so from our point of view the sequence $\{\lambda_k\}$ is the same as the sequence $\lambda_k^* \equiv 1$ (condition (2) does not change if we replace $\{\lambda_k\}$ by $\{\lambda_k^*\}$). Thus, we may assume $\{\lambda_k\}$ to be nondecreasing (and the proof below shows that $\lambda_m \searrow$ cannot occur at all).

Let m and $s/2$ be two integers. Applying the inequality $((2))\leqq A((3))$ to the sequence

$$(6) \qquad c_n = \begin{cases} 1 & \text{if} \quad n = \nu_{m+2}, \quad \nu_{m+3}, \dots, \nu_{m+s} \\ 0 & \text{otherwise} \end{cases}$$

we obtain for $p/q < 1$ that

$$s\lambda_m \leqq \sum_{k=m}^{m+s-1} \lambda_k = ((2)) \leqq A((3)) \leqq A \sum_{k=1}^{\nu_{m+s}} \varkappa_k s^{p/q} \leqq A^2 s^{p/q} \lambda_{m+s}$$

and so

$$\lambda_{m+s} \geqq s^{1-p/q} \cdot A^{-2} \cdot \lambda_m \geqq 2 \cdot \lambda_m$$

if $s > (2A)^{2(1-p/q)}$, which proves (5).

Similarly, if $p/q > 1$ then we obtain from $((3)) \leqq A((2))$ applied to the sequence (6) that

$$\lambda_m (s/2)^{p/q} \leqq A\Big(\sum_{k=1}^{\nu_m + s/2} \varkappa_k\Big)(s/2) \leqq A((3)) \leqq A^2((2)) = \sum_{k=m}^{m+s-1} \lambda_k \leqq s \cdot \lambda_{m+s}$$

and the proof is over.

II. Sufficiency. Let us assume now (4) and (5) and we shall separately prove that (2) implies (3) and (3) implies (2) for any sequence $\{c_n\}$. Let

$$d_m = \sum_{n=\nu_m+1}^{\nu_{m+1}} |c_n|^q.$$

1) (2) implies (3). Since

$$((3)) \leqq \sum_{m=1}^{\infty} \Big(\sum_{k=\nu_m+1}^{\nu_{m+1}} \varkappa_k\Big)\Big(\sum_{j=m}^{\infty} d_j\Big)^{p/q} = I$$

it is enough to show that (2) implies $I < \infty$.

If $p/q \leqq 1$ then from the concavity of $x^{p/q}$ we get

$$I \leqq \sum_{m=1}^{\infty} \Big(\sum_{k=\nu_m+1}^{\nu_{m+1}} \varkappa_k\Big)\Big(\sum_{j=m}^{\infty} d_j^{p/q}\Big) \leqq \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{\nu_{j+1}} \varkappa_k\Big) d_j^{p/q} \leqq A \sum_{j=1}^{\infty} \lambda_j d_j^{p/q} = A((2)) < \infty.$$

If, however, $p/q > 1$ then we have by (5) the inequalities

$$\sum_{j=1}^{m} \lambda_j^{q/p} \leqq c \cdot \lambda_m^{q/p}; \quad 1 \leqq \sum_{j=m}^{\infty} \lambda_m^{q/p} / \lambda_j^{q/p} \leqq c \quad (m = 1, 2, \ldots)$$

and so Jensen's inequality gives

$$I \leqq A \sum_{m=1}^{\infty} \lambda_m \Big(\sum_{j=m}^{\infty} d_j\Big)^{p/q} = A \sum_{m=1}^{\infty} \Big(\sum_{j=m}^{\infty} (\lambda_j^{q/p} d_j)(\lambda_m/\lambda_j)^{q/p}\Big)^{p/q} \leqq$$

$$\leqq K \sum_{m=1}^{\infty} \Big(\sum_{j=m}^{\infty} \lambda_j d_j^{p/q} (\lambda_m/\lambda_j)^{q/p}\Big) =$$

$$= K \sum_{j=1}^{\infty} d_j^{p/q} \Big(\sum_{m=1}^{j} \lambda_m^{q/p} \lambda_j^{1-q/p}\Big) \leqq K \sum_{j=1}^{\infty} \lambda_j d_j^{p/q} = K((2)) < \infty$$

and this is what we wanted to prove.

2) (3) implies (2). If $p/q \geqq 1$ then we get from the convexity of $x^{p/q}$ and from (4) that

$$((3)) \geqq \sum_{m=1}^{\infty} \Big(\sum_{k=\nu_m+1}^{\nu_{m+1}} \varkappa_k \Big) \Big(\sum_{j=m+1}^{\infty} d_j \Big)^{p/q} \geqq \sum_{m=1}^{\infty} \Big(\sum_{k=\nu_m+1}^{\nu_{m+1}} \varkappa_k \Big) \Big(\sum_{j=m+1}^{\infty} d_j^{p/q} \Big) =$$

$$= \sum_{j=1}^{\infty} d_j^{p/q} \Big(\sum_{m=1}^{j-1} \sum_{k=\nu_m+1}^{\nu_{m+1}} \varkappa_k \Big) \geqq$$

$$\geqq c \sum_{j=1}^{\infty} \lambda_j d_j^{p/q} - \sum_{j=1}^{\infty} \varkappa_{\nu_j+1} d_j^{p/q} \geqq c((2)) - ((3)) \quad (c > 0)$$

and so (3)⇒(2) follows.

If $p/q < 1$, then (5) is also satisfied and so there is an s with

$$\sum_{k=\nu_m+1}^{\nu_{m+s}} \varkappa_k \geqq 2 \sum_{k=1}^{\nu_m} \varkappa_k \quad (m = 1, 2, \ldots).$$

Thus, for

$$\gamma_m = \sum_{k=\nu_{ms}+1}^{\nu_{(m+1)s}} \varkappa_k$$

we have

$$\gamma_{m+1} \geqq 2\gamma_m \quad (m = 1, 2, \ldots). \tag{7}$$

Assuming (3), (2) will surely hold if

$$\sum_{m=2}^{\infty} \Big(\sum_{j=1}^{m-1} \gamma_j \Big) \Big(\sum_{l=ms+1}^{(m+1)s} d_l^{p/q} \Big)$$

is finite (take also into account (4)) and by (7) this amounts to the finiteness of

$$\sum_{m=2}^{\infty} \gamma_{m-1} \Big(\sum_{l=ms+1}^{(m+1)s} d_l^{p/q} \Big)$$

which clearly follows from (3).

We have completed our proof.

References

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