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On absolute summability of orthogonal series

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1. In [7] the second of the authors investigated some questions of general absolute summability of orthogonal series. The aim of the present note is to continue these investigations.

In [6] the following theorem was proved:

Theorem A. *If*

$$\sum_{n=0}^{\infty}\Big\{\sum_{k=2^n+1}^{2^{n+1}} a_k^2\Big\}^{1/2} < \infty \tag{1}$$

holds true, then for any orthonormal system $\{\varphi_n(x)\}$ *on* (a, b) *the orthogonal series*

$$\sum_{k=0}^{\infty} a_k \varphi_k(x) \tag{2}$$

is absolutely Cesàro summable (or briefly $|C, 1|$*-summable) almost everywhere in* (a, b). *If* (1) *does not hold then there exists an orthonormal system* $\{\varphi_n(x)\}$ *such that series* (2) *is not* $|C, 1|$*-summable almost everywhere in* (a, b).

Moreover P. Billard [1] proved the following result.

Theorem B. *If the coefficient-sequence* $\{a_n\}$ *does not satisfy condition* (1) *then the Rademacher-series*

$$\sum_{k=0}^{\infty} a_k r_k(x) \tag{3}$$

is not $|C, 1|$*-summable almost everywhere in* $(0, 1)$.

Theorems A and B imply the following statement:

Let $\{a_n\}$ be a given coefficient-sequence. Then there are two cases. Either series (2) is $|C, 1|$-summable for any orthonormal system $\{\varphi_n(x)\}$ on (a, b) almost every-

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where in (a, b) or the Rademacher-series (3) is not $|C, 1|$-summable almost everywhere in $(0, 1)$.

Later F. MÓRICZ [3] established similar results in the case of absolute Riesz-summability. Very recently H. SCHWINN [5] proved an analogous theorem for Euler-means.

In the present paper we shall prove some theorems of this type for an arbitrary regular summability method T. Moreover we give a necessary and sufficient coefficient-condition in order that series (2) for any orthonormal system $\{\varphi_n(x)\}$ be absolutely T-summable (i.e. $|T|$-summable) almost everywhere in the domain of orthogonality.

2. Let $T=(t_{i,n})_{i,n=0}^{\infty}$ be a regular Toeplitz-matrix satisfying the usual conditions:

1. $\lim_{i\to\infty} t_{i,n} = 0 \quad (n = 0, 1, \ldots)$,

2. $\lim_{i\to\infty} \sum_{n=0}^{\infty} t_{i,n} = 1$,

3. $\sum_{n=0}^{\infty} |t_{i,n}| \leqq K \quad (<\infty) \quad (i = 0, 1, \ldots)$.

Let

$$t_i(a, \varphi; x) = \sum_{n=0}^{\infty} t_{i,n} s_n(x) \quad (i = 0, 1, \ldots), \quad t_{-1}(a, \varphi; x) \equiv 0, \tag{4}$$

where $s_n(x)$ denotes the nth partial sum of (2). Series (2) at a point x_0 is said to be $|T|$-summable if series (4) for each i at x_0 converges and

$$\sum_{i=0}^{\infty} |t_i(a, \varphi; x_0) - t_{i-1}(a, \varphi; x_0)| < \infty.$$

It is clear that the $|T|$-summability of series (2) at x_0 implies the existence of the limit of $t_i(a, \varphi; x_0)$ as $i\to\infty$, i.e. series (2) is also T-summable at x_0.

Let us define the terms $T_{i,k}$ as follows:

$$T_{i,k} = \sum_{n=k}^{\infty} t_{i,n} \quad (i, k = 0, 1, \ldots) \quad \text{and} \quad T_{-1,k} = 0 \quad (k = 0, 1, \ldots).$$

Henceforth let $\{\varphi_n(x)\}$ denote an arbitrary orthonormal system on the σ-finite measure space $(X, \quad, \mu)$. It is clear that if the matrix T is row-finite then

$$t_i(a, \varphi; x) = \sum_{n=0}^{\infty} t_{i,n}(a_0\varphi_0(x) + \ldots + a_n\varphi_n(x)) = \tag{5}$$

$$= \sum_{k=0}^{\infty} T_{i,k} a_k \varphi_k(x) \quad (i = 0, 1, \ldots)$$

holds true at any point x where each function $\varphi_k(x)$ has finite value, i.e. the equality in (5) holds true on X μ-almost everywhere.

If the matrix T is not row-finite, but the sequence $\{a_n\}\in l^2$, then it is easy to show that the equality in (5) also holds true on X μ-almost everywhere. Indeed, if the series on the left-hand side of (5) converges on X μ-almost everywhere to a function $F_i(x)$ and the series on the right-hand side of (5) converges in the metric $L^2(X, \mathscr{A}, \mu)$ to a function $G_i(x)\in L^2(X, \mathscr{A}, \mu)$, i.e.

$$\lim_{N\to\infty} \int_X \Big(\sum_{k=0}^{N} T_{i,k}a_k\varphi_k(x)-G_i(x)\Big)^2 d\mu = 0,$$

then the equality $F_i(x)=G_i(x)$ holds on X μ-almost everywhere $(i=0, 1, \dots)$.

We prove the following theorems:

Theorem 1. *If T is a row-finite matrix then condition*

$$\sum_{i=0}^{\infty}\Big\{\sum_{k=0}^{\infty}(T_{i,k}-T_{i-1,k})^2a_k^2\Big\}^{1/2}<\infty \tag{6}$$

implies that series (2) *is* $|T|$*-summable on* X μ*-almost everywhere. If T is not a row-finite matrix then* (6) *and* $\{a_n\}\in l^2$ *together imply the* $|T|$*-summability of series* (2) *on* X μ*-almost everywhere.*

Theorem 2. *If*

$$\sum_{i=0}^{\infty}|T_{i,k}-T_{i,k-1}|\,|a_k|<\infty \quad (k=0, 1, \dots) \tag{7}$$

and (6) *does not hold, then the Rademacher-series* (3) *is not* $|T|$*-summable almost everywhere in* $(0, 1)$.

Theorem 3. *If the coefficient-sequence* $\{a_k\}$ *does not satisfy condition* (6) *then there exists an orthonormal system* $\{\varphi_n(x)\}$ *such that series* (2) *is not* $|T|$*-summable almost everywhere in* $(0, 1)$.

Remarks. I. It is clear that if the matrix T satisfies the following conditions

$$\sum_{i=0}^{\infty}|T_{i,k}-T_{i-1,k}|<\infty \quad (k=0, 1, \dots) \tag{8}$$

then (7) holds true for any coefficient-sequence $\{a_n\}$. An easy calculation shows that the methods of summation $(C, \alpha>0)$ and Riesz satisfy (8).

II. Theorems 1 and 2 imply the cited theorems of P. Billard and F. Móricz; moreover they include the results concerning the $|C, \alpha\geq 1/2|$-summability of the first author [2], and the theorems of H. Schwinn [5] published very recently in connection with Euler summability. All of these assertions can be shown by elementary calculations.

III. By Theorem 3 condition (6) is always necessary in order that series (2) for any orthonormal system $\{\varphi_n(x)\}$ should be $|T|$-summable almost everywhere in the domain of orthogonality.

3. Proofs. Proof of Theorem 1. Let $E\in\mathscr{A}$ with $\mu(E)<\infty$. Then, by (5), we have

$$\sum_{i=0}^{\infty}\int_E |t_i(a,\varphi;x)-t_{i-1}(a,\varphi;x)|\,d\mu \leqq$$

$$\leqq \{\mu(E)\}^{1/2}\sum_{i=0}^{\infty}\Big\{\int_E (t_i(a,\varphi;x)-t_{i-1}(a,\varphi;x))^2\,d\mu\Big\}^{1/2}\leqq$$

$$\leqq \{\mu(E)\}^{1/2}\sum_{i=0}^{\infty}\Big\{\int_X (t_i(a,\varphi;x)-t_{i-1}(a,\varphi;x))^2\,d\mu\Big\}^{1/2}\leqq$$

$$\leqq \{\mu(E)\}^{1/2}\sum_{i=0}^{\infty}\Big\{\sum_{k=0}^{\infty}(T_{i,k}-T_{i-1,k})^2a_k^2\Big\}^{1/2},$$

which implies that the series

$$\sum_{i=0}^{\infty}|t_i(a,\varphi;x)-t_{i-1}(a,\varphi;x)| \tag{9}$$

converges on E μ-almost everywhere. By the assumption the measure space $(X,\mathscr{A},\mu)$ is σ-finite, so it also follows that series (9) converges on X μ-almost everywhere, that is, series (2) is $|T|$-summable on X μ-almost everywhere, as desired.

Proof of Theorem 2. We distinguish two cases. If $\{a_n\}\notin l^2$ then by a well-known theorem of A. Zygmund [8] the Rademacher-series (3) is not T-summable almost everywhere in $(0,1)$, and consequently it is not $|T|$-summable almost everywhere in $(0,1)$. In this case our theorem is already proved.

Next let us assume that $\{a_n\}\in l^2$. In this case we need a slightly modified version of a well-known theorem of Orlicz [4]. We formulate it as a lemma.

Lemma. *For any Lebesgue-measurable set $E(\subseteqq(0,1))$ there exist a positive number $K=K(E)$ and a natural number $k_0=k_0(E)$ such that if $\{a_n\}\in l^2$ and $k_1\geqq k_0$ then*

$$K(E)(\operatorname{mes}E)\Big\{\sum_{k=k_1}^{\infty}a_k^2\Big\}^{1/2}\leqq\int_E\Big|\sum_{k=k_1}^{\infty}a_kr_k(t)\Big|\,dt$$

holds true.

Returning to the proof of Theorem 2, if now we assume the contrary of the statement of Theorem 2, that is, that series (3) is $|T|$-summable on a set $E(\subseteqq(0,1))$

of positive measure, then there exist a positive number M and a set $F(\subseteqq E)$ of positive Lebesgue measure such that

$$\sum_{i=0}^{\infty} |t_i(a, \varphi; x)-t_{i-1}(a, \varphi; x)| \leqq M \tag{10}$$

holds for any $x \in F$.

Then, by Lemma, there exists a natural number $k_0=k_0(F)$ such that

$$\int_F \Big| \sum_{k=k_0}^{\infty} (T_{i,k}-T_{i-1,k}) a_k r_k(x) \Big| dx \geqq \tag{11}$$

$$\geqq K(F)(\text{mes } F) \Big\{ \sum_{k=k_0}^{\infty} (T_{i,k}-T_{i-1,k})^2 a_k^2 \Big\}^{1/2}.$$

Moreover, by (7), we have

$$\sum_{i=0}^{\infty} \Big| \sum_{k=0}^{k_0-1} (T_{i,k}-T_{i-1,k}) a_k r_k(x) \Big| \leqq \sum_{k=0}^{k_0-1} M(k), \tag{12}$$

where

$$M(k) := \sum_{i=0}^{\infty} |T_{i,k}-T_{i-1,k}| \cdot |a_k| \quad (k=0, 1, \ldots).$$

Now, using (5), (7), (10), (11) and (12) yield

$$K(F)(\text{mes } F) \sum_{i=0}^{\infty} \Big\{ \sum_{k=k_0}^{\infty} (T_{i,k}-T_{i-1,k})^2 a_k^2 \Big\}^{1/2} \leqq \tag{13}$$

$$\leqq (\text{mes } F)\Big(M+\sum_{k=0}^{k_0-1} M(k)\Big) < \infty,$$

furthermore, using (7) once more, we get

$$\sum_{i=0}^{\infty} \Big\{ \sum_{k=0}^{k_0-1} (T_{i,k}-T_{i-1,k})^2 a_k^2 \Big\}^{1/2} \leqq \sum_{k=0}^{k_0-1} M(k) < \infty. \tag{14}$$

Estimations (13) and (14) imply that (6) holds true, which is a contradiction, and this proves Theorem 2.

Proof of Theorem 3. We distinguish two cases again.

If (7) holds true for each k then, by Theorem 2, the Rademacher-series (3) is not $|T|$-summable almost everywhere in $(0, 1)$.

If (7) does not hold for a certain natural number k_0, that is,

$$\sum_{i=0}^{\infty} |T_{i,k_0}-T_{i-1,k_0}| \cdot |a_{k_0}| = \infty,$$

then we define a special orthonormal system $\{\psi_n(x)\}$ as follows. Let

$$\psi_{k_0}(x) = \begin{cases} \sqrt{2}, & x \in (0, 1/2), \\ 0, & x \in (1/2, 1), \end{cases}$$

furthermore let us choose the functions $\psi_k(x)$ $\{k=0, 1, \ldots; k\neq k_0\}$ such that they are zero on $(0, 1/2)$ and form an orthonormal system on the interval $(1/2, 1)$. Then the system $\{\psi_n(x)\}_0^\infty$ is orthonormal on $(0, 1)$. For this system we obviously have

$$\int_0^1 \Big(\sum_{i=0}^{\infty} |t_i(a, \psi; x)-t_{i-1}(a, \psi; x)|\Big)\, dx \geqq$$

$$\geqq \int_0^{1/2} \Big(\sum_{i=0}^{\infty} |t_i(a, \psi; x)-t_{i-1}(a, \psi; x)|\Big)\, dx =$$

$$= (\sqrt{2}/2)\, |a_{k_0}| \sum_{i=0}^{\infty} |T_{i,k_0}-T_{i-1,k_0}| = \infty,$$

whence

$$\|a; T\| := \sup_{\{\varphi_k\}} \int_0^1 \Big(\sum_{i=0}^{\infty} |t_i(a, \varphi; x)-t_{i-1}(a, \varphi; x)|\Big)\, dx = \infty \tag{15}$$

follows, where the supremum is taken for all orthonormal systems $\{\varphi_n(x)\}$ on (a, b). On account of a theorem of the second author [7] statement (15) implies the existance of an orthonormal system $\{\varphi_n(x)\}$ for which series (2) is not $|T|$-summable almost everywhere in $(0, 1)$.

This completes the proof.

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