Acta Sci. Math., 50 (1986), 169—182

On the asymptotic estimate of the maximum likelihood of parameters of the spectral density having zeros

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1. Introduction

1. Let x_t, $t=0, \pm 1, \cdots$ be a stationary Gaussian time-series with $E(x_t)=0$ and spectral density (SD) $f(\lambda)$. Suppose that the SD f is a function of an unknown vector parameter $\theta=(\theta_1, \ldots, \theta_p)' \in \Theta$, where Θ is a bounded closed set in the p-dimensional Euclidean space R^p. We wish to obtain an estimate of this parameter from data consisting of a part of a realisation of the series, which will be assumed to be n consecutive observations denoted by $x_1, \ldots, x_n$. Obviously, we can consider the maximum likelihood estimate $\tilde{\theta}_n$ of the parameter θ:

$$L_n(\tilde{\theta}_n; X) = \max_{\theta} L_n(\theta; X), \tag{1}$$

where $L_n(\theta; X)$ is the logarithm of the likelihood of the data $X=(x_1, \ldots, x_n)'$. The function $L_n(\theta; X)$ can be written in the form (see [9], [11])

$$L_n(\theta; X) = -(1/2)\{n \ln 2\pi + \ln \det B_{n,f_\theta} + X' B_{n,f_\theta}^{-1} X\}, \tag{2}$$

where $B_{n,f_\theta} = \|c_{k-j}(\theta)\|_{k,j=\overline{1,n}}$ is the Toeplitz matrix connected with the function $f(\lambda; \theta)$.

It follows from formulas (1) and (2) that in order to find the estimate $\tilde{\theta}_n$, it is necessary to obtain the explicit expressions for $\det B_{n,f_\theta}$ and B_{n,f_θ}^{-1}, and this is a very difficult problem. Even in the simplest case of the first order autoregression the explicit expression for $L_n(\theta; X)$ is complicated (see [3]).

Following WHITTLE [12] and WALKER [11], let us introduce the estimate $\hat{\theta}_n$ of parameter θ:

$$\tilde{L}_n(\hat{\theta}_n; X) = \max_{\theta} \tilde{L}_n(\theta; X), \tag{3}$$

Received January 11, 1984.

where $\tilde{L}_n(\theta; X)$ is "the main part" of the function $L_n(\theta; X)$ satisfying the condition

(4) $$n^{-1/2}[L_n(\theta; X)-\tilde{L}_n(\theta; X)] \to 0 \quad \text{as} \quad n \to \infty,$$

where the convergence is in probability. The estimate $\hat{\theta}_n$ will be called the asymptotical estimate of the maximum likelihood (AEML).

For the strictly positive SD case the asymptotical properties of the AEML $\hat{\theta}_n$ were investigated by WALKER [11] and DZHAPARIDZE [3]. The case in which the SD has "weak" zeros independent of the parameter θ was considered by the author [5]. In these papers it was shown that under wide conditions on the SD the function $\tilde{L}_n(\theta; X)$ can be chosen to have a much simpler form than $L_n(\theta; X)$. Moreover the estimates $\tilde{\theta}_n$ and $\hat{\theta}_n$ are asymptotically equivalent, i.e. the estimate $\hat{\theta}_n$ is also consistent, asymptotically normal and asymptotically efficient.

In the present paper we generalize the abovementioned results to the case in which the SD has both "weak" and "strong" zeros of polynomial type, i.e. when the function $f(\lambda; \theta)$ admits the representation

(5) $$f(\lambda; \theta) = |Q_m(e^{i\lambda})|^2 h(\lambda; \theta),$$

where $Q_m(e^{i\lambda})$ $(|Q_m(0)|=1)$ is a polynomial of degree m with roots on the unit circle, which are independent of the parameter θ, and the function $h(\lambda; \theta)$ has "weak" zeros also independent of the parameter θ.

Note that a similar case was considered by DZHAPARIDZE [3], [4], but under stronger restrictions on the function $f(\lambda; \theta)$. Namely he assumed that the function $h(\lambda; \theta)$ is strongly positive and the polynomial $Q_m(e^{i\lambda})$ has no multiple roots.

2. The following notations will be used: L_f^2 is the weight L^2 space with weight f; $H_n(f)$ is the space of polynomials of degree n, considered as a subspace of L_f^2; P_n^f is the projector from L_f^2 to $H_n(f)$; $G_n^f(\lambda, \mu)$ is the reproducing kernel of the space $H_n(f)$; $\|\cdot\|_f$ and $(\cdot, \cdot)_f$ are respectively the norm and inner product in L_f^2; $|A|_f$ and $\|A\|_f$ are respectively the uniform and Hilbert—Schmidt norms of the operator A in L_f^2.

Remark. In all notations the symbol f will be omitted if $f(\lambda)\equiv 1$.

We shall use the main result of [5], therefore for completeness of presentation we reproduce it here.

Theorem A [5]. *Let the* SD $f(\lambda; \theta)$ *of the stationary Gaussian time-series* x_t *admit the representation* (5), *where* $Q_m(e^{i\lambda})$ $(|Q_m(0)|=1)$ *is a polynomial of degree* m *with roots on the unit circle, which are independent of* θ, *and the function* $h(\lambda; \theta)$ *satisfies the following conditions:*

1. $\ln h(\lambda; \theta)=u(\lambda; \theta)+\tilde{v}(\lambda; \theta), \quad \theta\in\Theta,$

where $u(\cdot\,;\theta)$ and $v(\cdot\,;\theta)$ are bounded functions ($\tilde{v}$ is the harmonic conjugate of v) and $\|v\|_\infty<\pi/2$;

2. $\sum_{|k|>n} |a_k(\theta)|^2 = o\left(1/\sqrt{n}\right), \quad n\to\infty;$

3. $\sum_{|k|>n} |c_k(\theta)|^2 = o\left(1/(\sqrt{n}\ln n)\right), \quad n\to\infty;$

4. $\sum_{|k|>n} |b_k(\theta)|^2 = o\left(1/(\sqrt{n}\ln n)\right), \quad n\to\infty,$

for all $\theta\in\Theta$, where $a_k(\theta)$, $c_k(\theta)$ and $b_k(\theta)$ are the Fourier coefficients of the functions $\ln h(\cdot\,;\theta)$, $h(\cdot\,;\theta)$ and $1/h(\cdot\,;\theta)$, respectively. Then the limiting relation (4) *holds, where the function $L_n(\theta;X)$ is given by* (2) *and*

(6)
$$\tilde{L}_n(\theta;\,X) = -(n/2)\left\{\ln 2\pi+(1/2\pi)\int_{-\pi}^{\pi}\ln h(\lambda;\,\theta)\,d\lambda+(1/2\pi)\int_{-\pi}^{\pi}(\tilde{I}_n(\lambda)/h(\lambda;\,\theta))\,d\lambda\right\},$$

where

(7)
$$\tilde{I}_n(t) = (1/n)\iint_{-\pi}^{\pi} G_n^{|Q_m|^2}(\lambda,t)\,G_n^{|Q_m|^2}(t,\mu)\,|Q_m(t)|^2 Z^f(d\lambda)\overline{Z^f(d\mu)}$$

is the generalized periodogram of x_t. ($Z^f(d\lambda)$ is the orthogonal stochastic measure participating in the spectral representation of x_t: $x_t=\int_{-\pi}^{\pi}\exp(i\lambda t)Z^f(d\lambda)$.)

2. Auxiliary results

Let the functions $f(\lambda;\theta)$ and $h(\lambda;\theta)$ be connected by the relation (5). We have the obvious inclusion

$$Q_m H_n(f) \subseteqq H_{n+m}(h).$$

Let us denote by N_m the orthogonal complement of $Q_m H_n(f)$ in $H_{n+m}(h)$:

(8)
$$H_{n+m}(h) = Q_m H_n(f)\oplus N_m.$$

Denoting by $G_{n+m}(\lambda,\mu)$, $G_n^{|Q_m|^2}(\lambda,\mu)$ and $R_m(\lambda,\mu)$ the reproducing kernels of the spaces H_{n+m}, $H_n(|Q_m|^2)$ and N_m, respectively, from (8) we have

(9)
$$G_{n+m}(\lambda;\,\mu) = Q_m(\lambda)\overline{Q_m(\mu)}\,G_n^{|Q_m|^2}(\lambda,\mu)+R_m(\lambda,\mu).$$

From (8) we also obtain that

(10)
$$P_{n+m} = \Phi_n+T_m,$$

where P_{n+m}, Φ_n and T_m are the projectors from L^2 to subspaces H_{n+m}, $Q_m H_n(|Q_m|^2)$ and N_m, respectively.

The following assumptions will be made throughout the paper.

A1. The true value θ_0 of the parameter θ belongs to a bounded closed set Θ contained in an open set S in the p-dimensional Euclidean space R^p.

A2. If θ_1 and θ_2 are any two points of Θ, $f(\lambda;\theta_1)$ and $f(\lambda;\theta_2)$ are not equal almost everywhere (λ).

A3. For SD$f(\lambda;\theta)$ all the conditions of Theorem A are satisfied.

Lemma 1. *Let the partial derivatives* $\partial \ln h(\lambda;\theta)/\partial\theta_k$, $k=\overline{1,p}$, *be continuous functions of* (λ,θ) *for* $\lambda\in[-\pi,\pi]$, $\theta\in S$. *Then for any* $\theta_1\in\Theta$ *such that* $\theta_1\neq\theta_0$ *(θ_0 is the true value of θ)*

(11)

$$\lim_{n\to\infty}(1/n)\iint_{-\pi}^{\pi}|G_n^{|Q_m|^2}(\lambda,t)Q_m(\lambda)\overline{Q_m(t)}|^2 r(t;\theta_0,\theta_1)\big(h(\lambda;\theta_0)/h(t;\theta_0)\big)\,d\lambda\,dt=$$
$$=\int_{-\pi}^{\pi} r(t;\theta_0,\theta_1)\,dt,$$

where

$$r(t;\theta_0,\theta_1)=1-h(t;\theta_0)/h(t;\theta_1).$$

Proof. Under the given conditions we have (see [5, proof of Lemma 6])

$$\lim_{n\to\infty}(1/n)\iint_{-\pi}^{\pi}|G_{n+m}(\lambda,t)|^2 r(t;\theta_0,\theta_1)\big(h(\lambda;\theta_0)/h(t;\theta_1)\big)\,d\lambda\,dt= \tag{12}$$
$$=\int_{-\pi}^{\pi} r(t;\theta_0,\theta_1)\,dt.$$

To prove Lemma 1 it therefore suffices to show that

$$\lim_{n\to\infty}(1/n)\iint_{-\pi}^{\pi}[|G_{n+m}(\lambda,t)|^2-|G_n^{|Q_m|^2}(\lambda,t)Q_m(\lambda)\overline{Q_m(t)}|^2]\times \tag{13}$$
$$\times r(t;\theta_0,\theta_1)\big(h(\lambda;\theta_0)/h(t;\theta_1)\big)\,d\lambda\,dt=0.$$

It is easy to see that

$$\iint_{-\pi}^{\pi}[|G_{n+m}(\lambda,t)|^2-|G_n^{|Q_m|^2}(\lambda,t)Q_m(\lambda)\overline{Q_m(t)}|^2]\times \tag{14}$$
$$\times r(t;\theta_0,\theta_1)\big(h(\lambda;\theta_0)/h(t;\theta_1)\big)\,d\lambda\,dt=\operatorname{tr}\big(P_{n+m}(r_{01}/h_0)P_{n+m}h_0-\Phi_n(r_{01}/h_0)\Phi_n h_0\big),$$

where r_{01}/h_0 and h_0 are the operators of multiplication by the functions $r(t;\theta_0,\theta_1)/h(t;\theta_0)$ and $h(t;\theta_0)$, respectively.

Therefore using formula (10) we obtain

$$\operatorname{tr}\left(P_{n+m}(r_{01}/h_0)P_{n+m}h_0-\Phi_n(r_{01}/h_0)\Phi_n h_0\right)= \tag{15}$$
$$=\operatorname{tr}\left(\Phi_n(r_{01}/h_0)T_m h_0+T_m(r_{01}/h_0)\Phi_n h_0+T_m(r_{01}/h_0)T_m h_0\right).$$

Further, using the inequalities (see e.g. [2])

$$\operatorname{tr}(A_n B_n)\leqq \|A_n\|_h\|B_n\|_h, \tag{16}$$

$$\|A_n\|_h\leqq \sqrt{n}\,|A_n|_h \tag{17}$$

and the relation $|A_n|_{1/h}=|(1/h)A_n h|_h$, from (15) we find

$$(1/n)\left|\operatorname{tr}\left(P_{n+m}(r_{01}/h_0)P_{n+m}h_0-\Phi_n(r_{01}/h_0)\Phi_n h_0\right)\right|\leqq \tag{18}$$
$$\leqq\left(\sqrt{m/n}\sup_{\theta,\lambda}|r(\lambda;\ \theta)|\left[|\Phi_n|_h|T_m|_{1/h}+|\Phi_n|_{1/h}|T_m|_h+\sqrt{m/n}\,|T_m|_{1/h}|T_m|_h\right]\right).$$

The right-hand side of (18) tends to zero as $n\to\infty$, since by Lemma 1 in [5] and by formula (10)

$$\sup_n|\Phi_n|_h<\infty\quad\text{and}\quad\sup_n|\Phi_n|_{1/h}<\infty.$$

Lemma 2. *Under the conditions of Lemma* 1

$$\lim_{n\to\infty}(1/n)\iint_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}G_n^{|Q_m|^2}(\lambda,t)G_n^{|Q_m|^2}(t,\mu)Q_m(\lambda)\overline{Q_m(\mu)}\times\right. \tag{19}$$
$$\left.\times|Q_m(t)|^2\left(r(t;\ \theta_0,\theta_1)/h(t;\ \theta_0)\right)dt\right|^2h(\lambda;\ \theta_0)h(\mu;\ \theta_0)\,d\lambda\,d\mu=\int_{-\pi}^{\pi}r^2(t;\ \theta_0,\theta_1)\,dt.$$

Proof. It is known that under the given conditions (see [5, proof of Lemma 6])

$$\lim_{n\to\infty}(1/n)\iint_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}G_{n+m}(\lambda,t)G_{n+m}(t;\ \mu)\left(r(t;\ \theta_0,\theta_1)/h(t;\ \theta_0)\right)dt\right|^2\times$$
$$\times h(\lambda;\ \theta_0)h(\mu;\ \theta_0)\,d\lambda\,d\mu=\int_{-\pi}^{\pi}r^2(t;\ \theta_0,\theta_1)\,dt.$$

Hence to prove Lemma 2 it is enough to show that

$$\lim_{n\to\infty}(1/n)\iint_{-\pi}^{\pi}\left[\left|\int_{-\pi}^{\pi}G_{n+m}(\lambda,t)G_{n+m}(t,\mu)\left(r(t;\ \theta_0,\theta_1)/h(t;\ \theta_0)\right)dt\right|^2-\right. \tag{20}$$
$$-\left|\int_{-\pi}^{\pi}G_n^{|Q_m|^2}(\lambda,t)G_n^{|Q_m|^2}(t,\mu)Q_m(\lambda)\overline{Q_m(\mu)}|Q_m(t)|^2\times\right.$$
$$\left.\left.\times\left(r(t;\ \theta_0,\theta_1)/h(t;\ \theta_0)\right)dt\right|^2\right]h(\lambda;\ \theta_0)h(\mu;\ \theta_0)\,d\lambda\,d\mu=0.$$

It is easy to see that the relation (20) is equivalent to the following

(21) $$\lim_{n\to\infty}[\|P_{n+m}(r_{01}/h_0)P_{n+m}h_0\|_h^2-\|\Phi_n(r_{01}/h_0)\Phi_n h_0\|_h^2]=0.$$

Using the well-known inequality (see [1])

$$\|A_n\|_h^2-\|B_n\|_h^2\leqq\|A_n-B_n\|_h^2+2\|A_n-B_n\|_h\|B_n\|_h$$

and the fact that

$$\|\Phi_n(r_{01}/h_0)\Phi_n h_0\|_h\leqq\sqrt{n}\sup_{\lambda,\theta}|r(\lambda;\theta)|\,|\Phi_n|_h|\Phi_n|_{1/h}=o(\sqrt{n})$$

when $n\to\infty$ (which follows from inequality (17) and Lemma 1 in [5]), it is easy to see that to prove (21) it suffices to show that

(22) $$\lim_{n\to\infty}(1/n)\|P_{n+m}(r_{01}/h_0)P_{n+m}h_0-\Phi_n(r_{01}/h_0)\Phi_n h_0\|_h^2=0.$$

From formula (10) we have

$$P_{n+m}(r_{01}/h_0)P_{n+m}h_0-\Phi_n(r_{01}/h_0)\Phi_n h_0=T_m(r_{01}/h_0)P_{n+m}h_0+\Phi_n(r_{01}/h_0)T_m h_0.$$

Using this fact, the inequalities (16) and $\|A_nB_n\|_h\leqq\|A_n\|_h|B_n|_h$, we find

$$\begin{aligned}&(1/n)\|P_{n+m}(r_{01}/h_0)P_{n+m}h_0-\Phi_n(r_{01}/h_0)\Phi_n h_0\|_h^2\leqq\\ &\leqq(2/n)\{\|T_m\|_h^2|(r_{01}/h_0)P_{n+m}h_0|_h^2+|\Phi_n|_h^2\|(r_{01}/h_0)T_m h_0\|_h^2\}\leqq\\ &\leqq(2/n)\sup_{\lambda,\theta}|r(\lambda;\theta)|\{\|T_m\|_h^2\sup_n|P_{n+m}|_{1/h}^2+\|T_m\|_{1/h}^2\sup_n|\Phi_n|_h^2\}\to 0\end{aligned}$$

as $n\to\infty$, since by Lemma 1 in [5] and formula (10)

$$\sup_n|\Phi_n|_h<\infty\quad\text{and}\quad\sup_n|P_{n+m}|_{1/h}<\infty.$$

Lemma 3. *Let* A_n *be an* $n\times n$ *matrix such that* $|A_n|\to 0$ *as* $n\to\infty$ *and* $\sup_n\|A_n\|<\infty$. *Then*

$$\lim_{n\to\infty}[\ln(E_n+A_n)-\operatorname{tr}(A_n)+(1/2)\|A_n\|^2]=0.$$

The proof easily follows from inequality (V) in [2].

3. The asymptotic properties of AEML $\hat{\theta}_n$

It follows from Theorem A, that the AEML $\hat{\theta}_n$ can be found from the relation

$$U_n(\hat{\theta}_n;X)=\min_\theta U_n(\theta;X),$$

where

(23) $$U_n(\theta;X)=(1/4\pi)\int_{-\pi}^{\pi}[\ln h(\lambda;\theta)+(\tilde{I}_n(\lambda)/h(\lambda;\theta))]\,d\lambda.$$

1. Consistency of AEML $\hat{\theta}_n$.

Theorem 1. *Let the partial derivatives* $\partial \ln h(\lambda; \theta)/\partial\theta_k$, $k=\overline{1,p}$, *be continuous functions of* (λ, θ) *for* $\lambda\in[-\pi, \pi]$, $\theta\in S$. *Then the* AEML $\hat{\theta}_n$ *is consistent, i.e.* $\hat{\theta}_n\to\theta_0$, *as* $n\to\infty$, *in probability.*

We first establish three lemmas.

Lemma 4. *Let* θ_0 *be the true value of* θ *and* θ_1 *be any other point of* Θ. *Then there is a positive constant* $K(\theta_0, \theta_1)$ *such that*

$$\lim_{n\to\infty} \mathrm{P}\{U_n(\theta_0)-U_n(\theta_1) < -K(\theta_0, \theta_1)\} = 1. \tag{24}$$

Proof. From formulas (7) and (23) we have

$$\begin{aligned}
W_n \overset{\text{def}}{=} U_n(\theta_0)-U_n(\theta_1) &= (1/4\pi) \int_{-\pi}^{\pi} (\ln h(t; \theta_0)/h(t; \theta_1))\, dt + \\
&+ (1/4\pi) \int_{-\pi}^{\pi} \tilde{I}_n(t)[1/h(t; \theta_0)-1/h(t; \theta_1)]\, dt = \\
= (1/4\pi) \int_{-\pi}^{\pi} (\ln h(t; \theta_0)/h(t; \theta_1)\, dt &+ (1/4\pi n) \iiint_{-\pi}^{\pi} G_n^{|Q_m|^2}(\lambda, t) G_n^{|Q_m|^2}(t, \mu) \times \\
&\times |Q_m(t)|^2[1/h(t; \theta_0)-1/h(t; \theta_1)] Z^f(d\lambda)\overline{Z^f(d\mu)}.
\end{aligned} \tag{25}$$

Hence

$$\begin{aligned}
\mathrm{E}(W_n) &= (1/4\pi) \int_{-\pi}^{\pi} (\ln h(t; \theta_0)/h(t; \theta_1))\, dt + \\
&+ (1/4\pi n) \iint_{-\pi}^{\pi} |G_n^{|Q_m|^2}(\lambda, t)|^2 |Q_m(t)|^2[1/h(t; \theta_0)-1/h(t; \theta_1)] f(\lambda; \theta_0)\, d\lambda\, dt.
\end{aligned}$$

Now using Lemma 1 we obtain

(26)

$$\mathrm{E}(W_n) = (1/4\pi) \int_{-\pi}^{\pi} \ln (h(t; \theta_0)/h(t; \theta_1))\, dt + (1/4\pi) \int_{-\pi}^{\pi} [1-h(t; \theta_0)/h(t; \theta_1)]\, dt + o(1).$$

By the obvious inequality

$$\ln (h(t;\theta_0)/h(t;\theta_1)) < (h(t;\theta_0)/h(t;\theta_1))-1 \tag{27}$$

(here by assumption A2 we have strict inequality) from (26) we obtain

$$\lim_{n\to\infty} \mathrm{E}(W_n) \overset{\text{def}}{=} -l(\theta_0, \theta_1), \quad \text{say, where} \quad l(\theta_0, \theta_1) > 0.$$

Also, from (25) we find

$$D(\sqrt{n}\,W_n) = (1/4\pi n) \iint_{-\pi}^{\pi} \Big| \int_{-\pi}^{\pi} G_n^{|Q_m|^2}(\lambda, t) G_n^{|Q_m|^2}(t, \mu) \times$$
$$\times |Q_m(t)|^2 [1/h(t;\ \theta_0) - 1/h(t;\ \theta_1)]\, dt \Big|^2 f(\lambda;\ \theta_0) f(\mu;\ \theta_0)\, d\lambda\, d\mu.$$

Hence by Lemma 2 we get

$$\lim_{n\to\infty} D(\sqrt{n}W_n) = (1/4\pi) \int_{-\pi}^{\pi} [1 - h(t;\ \theta_0)/h(t;\ \theta_1)]^2\, dt < \infty.$$

The desired result (24) then follows by a simple application of Chebyshev's inequality: $K(\theta_0, \theta_1)$ can be any constant less than $l(\theta_0, \theta_1)$.

Lemma 5. *Let $\theta_1 \in \Theta$ and $\theta_2 \in S$ be chosen such that $|\theta_2 - \theta_1| < \delta$ (δ possibly depending on θ_1). Then there exists a number $n_0 > 0$ such that for $n \geqq n_0$*

(28) $$|U_n(\theta_1; X) - U_n(\theta_2; X)| \leqq H_{\delta,n}(\theta_1; X),$$

where $H_{\delta,n} = H_{\delta,n}(\theta_1; X)$ is a random variable such that

(29) $$\lim_{\delta\to 0} E(H_{\delta,n}) = 0 \quad \textit{uniformly in} \quad n \geqq n_0,$$

and

(30) $$\lim_{n\to\infty} D(H_{\delta,n}) = 0.$$

Proof. From (25), using inequality (27), we obtain

(31) $$|U_n(\theta_1) - U_n(\theta_2)| \leqq (1/4\pi) \int_{-\pi}^{\pi} [1 + \tilde{I}_n(\lambda)/h(\lambda;\ \theta_1)] \big|\ln\big(h(\lambda;\ \theta_1)/h(\lambda;\ \theta_2)\big)\big|\, d\lambda.$$

Let us denote by

$$H_{\theta_1,\delta(\theta_1)} = \sum_{k=1}^{p} \sup_{-\pi\leqq\lambda\leqq\pi}\ \sup_{|\theta_1-\theta|<\delta(\theta_1)} |\partial \ln h(\lambda;\ \theta)/\partial\theta_k|,$$

where $\delta(\theta_1) > \delta$ is chosen so that the set $\{\theta;\ |\theta_1 - \theta| \leqq \delta(\theta_1)\}$ is contained in S. Then, by the mean value theorem, from (31) we get

$$|U_n(\theta_1) - U_n(\theta_2)| \leqq H_{\delta,n}(\theta_1; X),$$

where

(32) $$H_{\delta,n}(\theta_1;\ X) = (\delta/4\pi) H_{\theta_1,\delta(\theta_1)} \int_{-\pi}^{\pi} [1 + \tilde{I}_n(t)/h(t;\ \theta_1)]\, dt.$$

We now show that the random variable $H_{\delta,n}(\theta_1; X)$ satisfies the conditions (29) and (30). From (32) we have

(33) $$E(H_{\delta,n}) = (\delta/2) H_{\theta_1,\delta(\theta_1)} + (\delta/4\pi n) H_{\theta_1,\delta(\theta_1)} \times$$
$$\times \iint_{-\pi}^{\pi} |G_n^{|Q_m|^2}(\lambda, t) Q_m(t)|^2 \big(f(\lambda;\ \theta_1)/h(t;\ \theta_1)\big)\, d\lambda\, dt$$

and

$$(34)\qquad D(H_{\delta,n}) = ((\delta/4\pi n)H_{\theta_1,\delta(\theta_1)})^2 \int\int_{-\pi}^{\pi} \Big| \int_{-\pi}^{\pi} (G_n^{|Q_m|^2}(\lambda;\, t)G_n^{|Q_m|^2}(t,\, \mu)\times$$

$$\times |Q_m(t)|^2/h(t;\, \theta_1))\, dt\Big|^2 f(\lambda;\, \theta_1) f(\mu;\, \theta_1)\, d\lambda\, d\mu.$$

So it is easy to see that (29) follows from (33) and Lemma 1 while (30) follows from (34) and Lemma 2.

Lemma 6 (WALKER [11]). *Let the random variable $U_n(\theta)$ satisfy the relation* (28) *for all $\theta_1 \in \Theta$; $\theta_2 \in S$ such that $|\theta_1 - \theta_2| < \delta$ (δ possibly depending on θ_1) and $H_{\delta,n}(\theta_1; X)$ satisfies the relations* (29) *and* (30). *Then*

$$\mathrm{P}\lim_{n\to\infty} \hat{\theta}_n = \theta_0, \quad \textit{the true value of } \theta.$$

The proof of Theorem 1 now immediately follows from Lemmas 4, 5 and 6.

2. Asymptotic normality and asymptotic efficiency of AEML $\hat{\theta}_n$. Having established the consistency of AEML $\hat{\theta}_n$, we can go on to obtain the limiting distribution of the vector $\sqrt{n}(\hat{\theta}_n - \theta_0)$ in the usual way by applying the mean value theorem to $U_n^{(i)}(\hat{\theta}_n) - U_n^{(i)}(\theta_0)$, $i = \overline{1, p}$, where $U_n^{(i)}$ denotes the partial derivative $\partial U_n(\theta)/\partial\theta_i$, $(U_n^{(i)}(\theta_0) = \partial U_n(\theta)/\partial\theta_i|_{\theta=\theta_0})$, and θ_0 is the true value of θ. Of course, further conditions must be imposed on SD to ensure that the second order partial derivatives $U_n^{(ij)} = \partial^2 U_n/\partial\theta_i\partial\theta_j$ satisfy a suitable continuity condition and that a central limit theorem can be applied to give the limiting joint distribution of $\sqrt{n}\, U_n^{(i)}(\theta_0)$, $i = \overline{1, p}$.

Theorem 2. *Let the functions $\partial \ln h(\lambda; \theta)/\partial\theta_k$, $k = \overline{1, p}$, be continuous in (λ, θ) for $\lambda \in [-\pi, \pi]$, $\theta \in S$, and the functions $\partial^2 \ln h(\lambda; \theta)/\partial\theta_j\partial\theta_k$, $\partial^3 \ln h(\lambda; \theta)/\partial\theta_l\partial\theta_j\partial\theta_k$, $k, j, l = \overline{1, p}$, be continuous in (λ, θ) for $\lambda \in [-\pi, \pi]$, $\theta \in N_\delta(\theta_0)$, where $N_\delta(\theta_0) = \{\theta;\, |\theta - \theta_0| < \delta\}$ is some neighbourhood of θ_0, and let the matrix $\Gamma_0 = \|\gamma_{ij}(\theta_0)\|_{i,j=\overline{1,p}}$ with*

$$\gamma_{ij}(\theta_0) = (1/4\pi) \int_{-\pi}^{\pi} (\partial \ln h(\lambda;\, \theta)/\partial\theta_i)_{\theta=\theta_0} (\partial \ln h(\lambda;\, \theta)/\partial\theta_j)_{\theta=\theta_0}\, d\lambda$$

be non-singular. Then the limiting distribution of the vector $\sqrt{n}(\hat{\theta}_n - \theta_0)$ when $n \to \infty$ is $N(0, \Gamma_0^{-1})$, and Γ_0 is the limit $(n \to \infty)$ of the Fisher information matrix.

To prove this theorem we need two lemmas.

Lemma 7. *Under the conditions of Theorem* 2,

$$\mathrm{P}\lim_{n\to\infty} U_n^{(ij)}(\theta_n^*) = \lim_{n\to\infty} \mathrm{E}(U_n^{(ij)}(\theta_0)) = \gamma_{ij}(\theta_0)$$

where $\theta_n^* = \omega\hat{\theta}_n + (1-\omega)\theta_0 \in N_\delta(\theta_0)$, $0 \le \omega \le 1$.

Proof is similar to that of Lemma 9 in [5], and so is omitted.

Lemma 8. *Under the conditions of Theorem* 2 *the limiting distribution of the vector* $(-\sqrt{n}\,U_n^{(1)}(\theta_0), \dots, -\sqrt{n}\,U_n^{(p)}(\theta_0))'$ *is* $N(0, \Gamma_0)$.

Proof. To prove this lemma it is sufficient to show that for any non-zero vector $v=(v_1, \dots, v_p)'$ the random variable

(35)
$$\Delta_n(\theta_0) \stackrel{\text{def}}{=} -\sqrt{n} \sum_{k=1}^{p} v_k U_n^{(k)}(\theta_0) =$$
$$= \frac{\sqrt{n}}{4\pi} \sum_{k=1}^{p} v_k \int_{-\pi}^{\pi} \left[\frac{\tilde{I}_n(t)}{h(t;\,\theta_0)} - 1\right] \left(\frac{\partial}{\partial\theta_k} \ln h(t;\,\theta)\right)_{\theta=\theta_0} dt =$$
$$= \frac{1}{4\pi\sqrt{n}} \iiint_{-\pi}^{\pi} G_n^{|Q_m|^2}(\lambda, t) G_n^{|Q_m|^2}(t, \mu) |Q_m(t)|^2 \frac{a(t;\,\theta_0)}{h(t;\,\theta_0)}\, dt\, Z^f(d\lambda)\overline{Z^f(d\mu)} -$$
$$- \frac{\sqrt{n}}{4\pi} \int_{-\pi}^{\pi} a(t;\,\theta_0)\, dt,$$

where

$$a(t;\,\theta_0) = \sum_{k=1}^{p} v_k (\partial \ln h(t;\,\theta)/\partial\theta_k)_{\theta=\theta_0},$$

has the limiting distribution $N(0, \gamma^2/2)$, where

$$\gamma^2 = (1/2\pi) \int_{-\pi}^{\pi} a^2(t;\,\theta_0)\, dt = 2v'\Gamma_0 v.$$

Let us denote

(36)
$$\Psi_n(\lambda, \mu;\,\theta_0) = \int_{-\pi}^{\pi} G_n^{|Q_m|^2}(\lambda, t) G_n^{|Q_m|^2}(t, \mu) \frac{|Q_m(t)|^2 a(t;\,\theta_0)}{h(t;\,\theta_0)}\, dt.$$

Since the function $\Psi_n(\lambda, \mu;\,\theta_0)$ is Hermitian-symmetric in (λ, μ) and belongs to L_h^2, by Schmidt's theorem (see [10]) we get

(37)
$$\Psi_n(\lambda, \mu;\,\theta_0) = \sum_{j=1}^{n} \nu_j(\theta_0) \varphi_j(\lambda;\,\theta_0) \overline{\varphi_j(\mu;\,\theta_0)},$$

where $\nu_j(\theta_0)$, $j=\overline{1,n}$, is the sequence of the eigen-values and $\varphi_j(\lambda;\,\theta_0)$, $j=\overline{1,n}$, is the sequence of the orthonormal eigen-functions of the operator $\Phi_n(a_0/h_0)\Phi_n h_0$. The latter is an integral operator in L_h^2 generated by the kernel $\Psi_n(\lambda, \mu;\,\theta_0)$. Now from (36) and (37) we have

(38)
$$\eta_n(\theta_0) \stackrel{\text{def}}{=} (1/4\pi\sqrt{n}) \iint_{-\pi}^{\pi} \Psi_n(\lambda, \mu;\,\theta_0) Z^f(d\lambda)\overline{Z^f(d\mu)} =$$
$$= (1/4\pi\sqrt{n}) \sum_{j=1}^{n} \nu_j(\theta_0) \iint_{-\pi}^{\pi} \varphi_j(\lambda;\,\theta_0)\overline{\varphi_j(\mu;\,\theta_0)} Z^f(d\lambda)\overline{Z^f(d\mu)} =$$
$$= (1/4\pi\sqrt{n}) \sum_{j=1}^{n} \nu_j(\theta_0) \left|\int_{-\pi}^{\pi} \varphi_j(\lambda;\,\theta_0) Z^f(d\lambda)\right|^2 = (1/4\pi\sqrt{n}) \sum_{j=1}^{n} \nu_j(\theta_0) y_j^2(\theta_0),$$

where $y_j(\theta_0)=\int_{-\pi}^{\pi}\varphi_j(\lambda;\theta_0)Z^f(d\lambda)$, $j=\overline{1,n}$, is a sequence of independent identically $N(0,1)$ distributed random variables.

It is well known that the characteristic function $\varphi_{\eta_n}(\alpha)$ of the random variable $\eta_n(\theta_0)$ has the form (see [7], [8])

(39)
$$\varphi_{\eta_n}(\alpha)=\prod_{j=1}^{n}\left(1-\left(i\alpha v_j(\theta_0)/4\pi\sqrt{n}\right)\right)^{-1/2}.$$

Therefore from (35), (38) and (39) it follows that the characteristic function $\varphi_{\Delta_n}(\alpha)$ of the random variable $\Delta_n(\theta_0)$ has the form

$$\varphi_{\Delta_n}(\alpha)=\exp\left\{-\left(i\alpha\sqrt{n}/4\pi\right)\int_{-\pi}^{\pi}a(t;\theta_0)\,dt\right\}\prod_{j=1}^{n}\left(1-\left(i\alpha v_j(\theta_0)/4\pi\sqrt{n}\right)\right)^{-1/2}$$

and hence

(40)
$$\ln\varphi_{\Delta_n}(\alpha)=-(1/2)\sum_{j=1}^{n}\ln\left(1-\left(i\alpha v_j(\theta_0)/4\pi\sqrt{n}\right)\right)-\left(i\alpha\sqrt{n}/4\pi\right)\int_{-\pi}^{\pi}a(t;\theta_0)\,dt.$$

Using the inequalities (16), (17) and Lemma 1 in [5] it is easy to show that

$$(1/\sqrt{n})|\Phi_n(a_0/h_0)\Phi_n h_0|_h\to 0,\quad n\to\infty$$
$$\sup_n\,(1/\sqrt{n})\|\Phi_n(a_0/h_0)\Phi_n h_0\|_h<\infty.$$

Therefore by Lemma 3 we have

(41)
$$\lim_{n\to\infty}\left[\sum_{j=1}^{n}\left\{\ln\left(1-\frac{i\alpha v_j(\theta_0)}{4\pi\sqrt{n}}\right)+\frac{i\alpha v_j(\theta_0)}{4\pi\sqrt{n}}-\frac{1}{2}\left(\frac{i\alpha v_j(\theta_0)}{4\pi\sqrt{n}}\right)^2\right\}\right]=$$
$$=\lim_{n\to\infty}\left[\ln\det\left(E_n-\Phi_n\frac{i\alpha a_0}{4\pi\sqrt{n}\,h_0}\Phi_n h_0\right)+\right.$$
$$\left.+\operatorname{tr}\left(\Phi_n\frac{i\alpha a_0}{4\pi h_0\sqrt{n}}\Phi_n h_0\right)-\frac{1}{2}\left\|\Phi_n\frac{i\alpha a_0}{4\pi h_0\sqrt{n}}\Phi_n h_0\right\|_h^2\right]=0.$$

Further, by Lemma 1 we have

$$\lim_{n\to\infty}\left[\operatorname{tr}\left(\Phi_n\frac{i\alpha a_0}{4\pi h_0\sqrt{n}}\Phi_n h_0\right)-\frac{i\alpha\sqrt{n}}{4\pi}\int_{-\pi}^{\pi}a(t;\theta_0)\,dt\right]=0,$$

and by Lemma 2,

$$\lim_{n\to\infty}\left[\frac{1}{2}\left\|\Phi_n\frac{i\alpha a_0}{4\pi h_0\sqrt{n}}\Phi_n h_0\right\|_h^2-\frac{\alpha^2}{4\pi}\int_{-\pi}^{\pi}a^2(t;\theta_0)\,dt\right]=0.$$

Therefore from (40) and (41) we obtain

$$\lim_{n\to\infty} \ln \varphi_{\Delta_n}(\alpha) = -(\alpha^2/4\pi) \int_{-\pi}^{\pi} a^2(t;\ \theta_0)\, dt.$$

Thus the random variable $\Delta_n(\theta_0)$ has the limiting distribution $N(0, \gamma^2/2)$.

Proof of Theorem 2. Since $U_n^{(j)}(\hat{\theta}_n)=0$, $j=\overline{1,p}$, by the mean value theorem we have

$$0 = U_n^{(j)}(\hat{\theta}_n) = U_n^{(j)}(\theta_0) + \sum_{i=1}^{p} (\hat{\theta}_{ni} - \theta_{0i})\, U_n^{(ij)}(\theta_n^*), \tag{42}$$

where $\theta_n^* = \omega\hat{\theta}_n + (1-\omega)\theta_0 \in N_\delta(\theta_0)$, $0 \leqq \omega \leqq 1$. The relation (42) may be rewritten as

$$-\sqrt{n}\, U_n^{(j)}(\theta_0) = \sum_{i=1}^{p} \sqrt{n}\, (\hat{\theta}_{ni} - \theta_{0i}) U_n^{(ij)}(\theta_n^*). \tag{43}$$

Now by Lemma 7

$$\mathrm{P}\lim_{n\to\infty} U_n^{(ij)}(\theta_n^*) = \gamma_{ij}(\theta_0),$$

and by Lemma 8 the random vector $(-\sqrt{n}\, U_n^{(1)}(\theta_0), \ldots, -\sqrt{n}\, U_n^{(p)}(\theta_0))'$ has the limiting distribution $N(0, \Gamma_0)$. Therefore (43) implies that the vector $\sqrt{n}\, (\hat{\theta}_n - \theta_0)$ has the limiting distribution $N(0, \Gamma_0^{-1})$.

Finally, let us show that the matrix Γ_0 is the limit $(n\to\infty)$ of the Fisher information matrix. This statement follows from the following relation

$$\lim_{n\to\infty} (1/n)\mathrm{D}\Big(\sum_{i=1}^{p} v_i \partial L_n(\theta)/\partial\theta_i\Big) = \lim_{n\to\infty} (1/n)\mathrm{D}\Big(\sum_{i=1}^{p} v_i U_n^{(i)}(\theta)\Big) =$$

$$= \lim_{n\to\infty} (1/n)\,\mathrm{D}\Big((1/4\pi) \iiint_{-\pi}^{\pi} G_n^{|Q_m|^2}(\lambda, t) G_n^{|Q_m|^2}(t, \mu) |Q_m(t)|^2 \times$$

$$\times (a(t;\ \theta)/h(t;\ \theta))\, dt\, Z^f(d\lambda)\overline{Z^f(d\mu)} - (1/4\pi) \int_{-\pi}^{\pi} a(t;\ \theta)\, dt\Big) =$$

$$= \lim_{n\to\infty} (1/4\pi n) \iint_{-\pi}^{\pi} \Big| \int_{-\pi}^{\pi} G_n^{|Q_m|^2}(\lambda, t) G_n^{|Q_m|^2}(t, \mu) |Q_m(t)|^2 \times$$

$$\times (a(t;\ \theta)/h(t;\ \theta))\, dt \Big|^2 f(\lambda;\ \theta) f(\mu;\ \theta)\, d\lambda\, d\mu = (1/4\pi) \int_{-\pi}^{\pi} a^2(t;\ \theta)\, dt,$$

where, as before, $v=(v_1, \ldots, v_p)'$ is a non-zero vector and

$$a(t;\ \theta) = \sum_{i=1}^{p} v_i \partial \ln h(t;\ \theta)/\partial\theta_i.$$

4. Confidence regions for the parameter θ

The further arguments are based on the following theorem.

Theorem 3. *Let* $\hat{\theta}_n$ *be an arbitrary consistent, asymptotically normal and asymptotically efficient estimate of the parameter* θ *and let the random matrix* $\Gamma_* = \|\gamma^*_{kj}\|_{kj=\overline{1,p}}$ *be an arbitrary consistent estimate of the limit* Γ_0 *of the Fisher information matrix. Then the limiting* $(n \to \infty)$ *distribution of the statistic*

(44) $$S_n^2 = n \sum_{ij=1}^{p} (\hat{\theta}_{ni} - \theta_{0i})(\hat{\theta}_{nj} - \theta_{0j})\gamma^*_{ij}$$

is the χ^2*-distribution with* p *degrees of freedom.*

Proof. It is easy to see that

$$\operatorname{P}\lim_{n\to\infty} n \sum_{ij=1}^{p} (\hat{\theta}_{ni} - \theta_{0i})(\hat{\theta}_{nj} - \theta_{0j})[\gamma^*_{ij} - \gamma_{ij}(\theta_0)] = 0.$$

Hence the limiting distribution of the statistic S_n^2 is the same as that of

(45) $$n \sum_{ij=1}^{p} (\hat{\theta}_{ni} - \theta_{0i})(\hat{\theta}_{nj} - \theta_{0j})\gamma_{ij}(\theta_0).$$

Transforming the vector $\sqrt{n}\,(\hat{\theta}_n - \theta_0)$ to a vector ξ via the unitary transformation V such that the matrix $V'\Gamma_0 V$ is diagonal, we obtain

$$n(\hat{\theta}_n - \theta_0)'\Gamma_0(\hat{\theta}_n - \theta_0) = \sum_{i=1}^{p} \xi_i^2/\sigma_i^2,$$

where ξ_i is the i-th component of ξ and σ_i^2 is its variance. The random variables ξ_i, $i=\overline{1,p}$, converge in probability to independent normal random variables with mean 0 and variance σ_i^2, $i=\overline{1,p}$. Therefore the random variable (45) and hence the statistic S_n^2 has the limiting χ^2-distribution with p degrees of freedom.

Thus we have shown that for every interval $[\alpha, \beta]$ the relation

(46) $$\alpha < n(\hat{\theta}_n - \theta_0)'\Gamma_*(\theta_n - \theta_0) < \beta$$

has limiting probability $\int_\alpha^\beta \chi_p^2(x)\,dx$. If α and β are chosen so that

$$\int_\alpha^\beta \chi_p^2(x)\,dx = 1 - \varepsilon, \quad \varepsilon > 0,$$

then the set of values of θ satisfying (46) will be a confidence region for θ_0 with asymptotic confidence level ε.

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