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Characterization of operators of class C_0 and a formula for their minimal function

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Introduction

The class of operators C_0 was introduced in 1964 by SZ.-NAGY and C. FOIAŞ [11] and consists of all completely non unitary contractions on a complex Hilbert space which are annihilated by a non-identically zero function in H^∞ of the unit disc. Among the annihilating H^∞ functions of a contraction of class C_0, there exists an inner function which divides (in H^∞) all others ([11] or [12, p. 124]). This inner function is determined up to a constant factor of modulus 1, and is called the minimal function of the contractions. For results concerning the structure of contractions of class C_0 we refer to [3], [13] and [14].

The first characterization of contractions of class C_0 was given by Sz.-Nagy and Foiaş in terms of an algebraic condition on the characteristic operator function [12, p. 265]. Using this characterization, J. DAZORD [4] obtained a characterization of C_0 operators in terms of a growth condition on their resolvent, which however is of an implicit form and is difficult to verify. (See Corollary 4.2.)

In this paper we give a characterization of C_0 operators in terms of an explicit growth condition on their resolvent, and establish a formula for the associated minimal function, also in terms of the resolvent (Theorem 1.1). A similar characterization of C_0 operators whose spectrum is a thin set in a certain sense, is given in [2]. (See Section 7.)

The above mentioned characterization and formula for the minimal function can also be expressed in terms of the characteristic operator function (Theorem 5.3). The interest in obtaining such a result was pointed out by R. G. DOUGLAS [3, p. 190].

Our exposition is self contained in the sense that the concepts from operator

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theory that are used in the proofs of the theorems, are essentially those which appear in their statement. So for example, except for Section 5, we do not use in our proofs the characteristic operator functions or functional model of Sz.-Nagy and Foiaş. Although resulting in longer proofs, this approach seems to be of interest and also leads to new proofs of the characterizations of operators of class C_0 given in [12, p. 265] and [4]. (See Corollary 4.2 and Corollary 5.4.) We also obtain a new proof of the existence of a minimal function for C_0 operators (Theorem 1.1).

The contents of the paper are as follows: In Section 1 we introduce the concept of meromorphic vector function of bounded α-characteristic and state our main result. In Section 2 we prove some preliminary results which are needed for the proof of the main result. Section 3, which is the principal part of the paper, is devoted to the study of contractions with resolvent of bounded 1-characteristic. To every such contraction T we associate a function φ_T in H^∞, which is expressed in terms of the resolvent of T, and is a minimal function of T in the case that T is of class C_0. We also characterize in this section the resolvents of operators in this class, and prove the invariance of the class under certain Möbius transformations. In Section 4 we present the proof of our main result and obtain as a Corollary the result of Dazord [4]. In Section 5 we characterize contractions T whose characteristic function θ_T has a scalar multiple, and express our main result in terms of θ_T. In Section 6 we characterize contractions of class D_0, that is, contractions which are annihilated by a non-identically zero function in the disc algebra, and give the general form of an annihilating function of such a contraction. Finally in Section 7, we consider contractions with resolvent of bounded α-characteristic for some $0 \leqq \alpha < 1$, and prove that they are of class D_0, and have (in a certain sense) a thin spectrum.

The basic notions and facts concerning the Banach algebra H^∞ and the functional calculus of Sz.-Nagy and Foiaş for completely non unitary contractions, will be used freely in the sequel without giving always an explicit reference. For H^∞ we refer to [7] or [9] and for the functional calculus to [12, Chapter III].

1. Definitions and main result

Throughout this paper, $\mathscr{H}$ will denote a complex Hilbert space and $\mathscr{L}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. For an operator T in $\mathscr{L}(\mathscr{H})$ we shall denote by $\sigma(T)$ its spectrum, by $\varrho(T)$ its resolvent set, and by $R_T(\lambda)$ its resolvent, $(\lambda I-T)^{-1}$, $\lambda\in\varrho(T)$. We shall also denote by L_T, the operator function defined by: $L_T(\lambda)=(I-\bar{\lambda}T)R_T(\lambda)$, $\lambda\in\varrho(T)$. The term contraction will mean in the sequel an operator T in $\mathscr{L}(\mathscr{H})$ such that $\|T\|\leqq 1$.

The open unit disc $\{\lambda\in\mathbf{C}\colon |\lambda|<1\}$ will be denoted by D and the unit circle $\{\lambda\in\mathbf{C}\colon |\lambda|=1\}$ by Γ.

If X is a complex Banach space and F is an X-valued meromorphic function on D, we shall denote for every $0 \leqq t < 1$ by $n(t, F)$ the number of poles of F in the disc $\{\lambda \in \mathbf{C}: |\lambda| \leqq t\}$ (counting multiplicity), and for every $\alpha \geqq 0$ and $0 \leqq r < 1$ we set

$$N(r, F) = \int_0^r ((n(t, F) - n(0, F))/t)\, dt + n(0, F) \log r$$

and

$$m_\alpha(r, F) = (1/2\pi) \int_0^{2\pi} \log^+ \|(1-r)^\alpha F(re^{i\theta})\|\, d\theta$$

(where for $a \geqq 0$, $\log^+ a = \max\{\log a, 0\}$).

We define the *α-characteristic* of an X-valued meromorphic function on D to be the function

$$T_\alpha(F, r) = m_\alpha(F, r) + N(F, r), \quad 0 \leqq r < 1.$$

If $\sup\limits_{0 \leqq r < 1} T_\alpha(F, r) < \infty$, then we say that F is of *bounded α-characteristic.*

The set of all X-valued meromorphic functions on D of bounded α-characteristic, will be denoted by $N_\alpha(X)$. The elements in $N_0(X)$ are called functions of *bounded characteristic.* For $X = \mathbf{C}$ this is the classical definition of R. Nevanlinna [15]. Vector valued functions of bounded characteristic are considered in [2].

To simplify notations we shall denote in the sequel the set $N_\alpha(\mathscr{L}(\mathscr{H}))$ by N_α; and if T is a contraction such that the operator function $\lambda \to R_T(\lambda)$, $\lambda \in \varrho(T) \cap D$, is meromorphic on D and is in N_α, we shall say briefly that R_T is in N_α.

We recall that a contraction T is said to be of class $C_{0\cdot}$, if $T^n x \to 0$ as $n \to \infty$, for every x in $\mathscr{H}$ [12, p. 72].

Our main result is the following:

Theorem 1.1. *A contraction T is of class C_0, if and only if, T is of class $C_{0\cdot}$, and R_T is in N_1. Furthermore, if the last two conditions are satisfied, and $\{\lambda_1, \lambda_2, \ldots\}$ is the sequence of poles of R_T in D repeated according to multiplicity, with k of the λ_n being equal to zero, then T has a minimal function given by*

$$m_T(z) = z^k \prod_{\lambda_j \neq 0} (\bar{\lambda}_j/|\lambda_j|)((\lambda_j - z)/(1 - \bar{\lambda}_j z)) \exp(-w(z)), \quad z \in D,$$

where

$$w(z) = \lim_{\varrho \to 1-} (1/2\pi) \int_0^{2\pi} ((e^{it} + z)/(e^{it} - z)) \log \|L_T(\varrho e^{it})\|\, dt, \quad z \in D,$$

or alternatively,

$$w(z) = \int_\Gamma ((e^{it} + z)/(e^{it} - z))\, d\mu(t), \quad z \in D,$$

where μ is a positive measure on Γ which is the weak star limit as $\varrho \to 1-$; of the measures $(1/2\pi) \log \|L_T(\varrho e^{it})\|\, dt$, $0 < \varrho < 1$.

Remark 1. As will be shown in Lemma 2.2, the assumption that R_T is in N_1 implies that $\sum_n (1-|\lambda_n|)<\infty$ and therefore (cf. [7, p. 54]) the above product converges uniformly on compact subsets of D to an inner H^∞ function. The existence of the limits which define the function w and the measure μ will be established in Proposition 3.1.

Remark 2. It is readily verified that the above formula for m_T can also be written in the form

$$m_T(z)=\lim_{\varrho\to 1-} z^k \prod_{\lambda_j\neq 0} \frac{\bar\lambda_j}{|\lambda_j|}\frac{\lambda_j-z}{1-\bar\lambda_j z}\exp\int_0^{2\pi}\frac{z+e^{it}}{z-e^{it}}\log\|\varrho+(1-\varrho)^2e^{it}R_T(\varrho e^{it})\|\frac{dt}{2\pi}.$$

2. Preliminary results

In this section we present some preliminary results which are needed for the proof of Theorem 1.1. In the sequel, T will denote a fixed contraction in $\mathcal{L}(\mathcal{H})$. Following [12] we associate with T the self-adjoint operators

$$D_T=(I-T^*T)^{1/2} \quad\text{and}\quad D_{T^*}=(I-TT^*)^{1/2}$$

and set $\mathcal{D}_T=\overline{D_T\mathcal{H}}$ and $\mathcal{D}_{T^*}=\overline{D_{T^*}\mathcal{H}}$. In addition we denote by K_T the operator function defined by

$$K_T(\lambda)=D_TR_T(\lambda)(I-\lambda T^*),\quad \lambda\in\varrho(T),$$

and by U_T the set $\{x\in\mathcal{H}:\ \|D_{T^*}x\|\leqq 1\}$.

The first result of this section which will be needed in the proof of Theorem 1.1 appears in [12, p. 263], however it is expressed there in terms of the characteristic operator function, and one part of its proof depends on the functional model. In order to keep our exposition self-contained, we present below an equivalent formulation of this result, and give a proof which is similar to that in [12] but does not depend on the functional model and does not use explicitly the characteristic operator function.

Lemma 2.1. *For every λ in $D\cap\varrho(T)$*

$$\|L_T(\lambda)\|=\sup\{\|K_T(\lambda)x\|:\ x\in U_T\}.$$

Proof. We assume first that T is invertible, and prove the assertion for $\lambda=0$, that is, we show that

$$\|T^{-1}\|=\sup\{\|D_TT^{-1}x\|:\ x\in U_T\}.$$

Since $D_T T^{-1} = T^{-1} D_{T^*}$ [12, p. 7] this equality is equivalent to the equality $\|T^{-1}\| = \|J\|$ where J denotes the restriction of T^{-1} to $\mathscr{D}_{T^*}$. To show this, choose x in $\mathscr{H}$ and consider the orthogonal decomposition $x = x_1 + x_2$ where $x_1 \in \mathscr{D}_{T^*}$ and $x_2 \in \mathscr{D}_{T^*}^{\perp}$. Using the facts that T^{-1} maps $\mathscr{D}_{T^*}$ onto $\mathscr{D}_T$ and maps $\mathscr{D}_{T^*}^{\perp}$ isometrically onto $\mathscr{D}_T^{\perp}$ [12, p. 7], and that $\|J\| \geqq 1$ (since T is a contraction) we obtain that

$$\|T^{-1}x\|^2 = \|T^{-1}x_1\|^2 + \|T^{-1}x_2\|^2 \leqq \|J\|^2 \|x\|^2.$$

This shows that $\|T^{-1}\| \leqq \|J\|$, and since the reverse inequality is obvious, we conclude that $\|T^{-1}\| = \|J\|$. To prove the assertion in the general case, we assume that λ is in $D \cap \varrho(T)$, and consider the operator $T_\lambda = (\lambda I - T)(I - \bar{\lambda} T)^{-1}$, which is also a contraction [12, p. 14]. Since T_λ is invertible, we have by the assertion just proved that

$$\|L_T(\lambda)\| = \|T_\lambda^{-1}\| = \sup \{\|D_{T_\lambda} T_\lambda^{-1} x\| : x \in U_{T_\lambda}\}.$$

Setting $S = (1 - |\lambda|^2)^{1/2} (I - \bar{\lambda} T)^{-1}$, we obtain by a simple computation that

$$\|D_{T_\lambda} T_\lambda^{-1} x\|^2 = \langle D_{T_\lambda}^2 T_\lambda^{-1} x, T_\lambda^{-1} x \rangle = \|K_T(\lambda) S^* x\|^2$$

and noticing that $x \in U_{T_\lambda}$ if and only if $S^* x \in U_T$, we obtain the desired conclusion.

Lemma 2.2. *If F is a meromorphic function on D with values in some complex Banach space, with poles $\{\lambda_1, \lambda_2, \ldots\}$ in D repeated according to multiplicity, then the following conditions are equivalent:*

(a) $\sup N(r, F) < \infty$,

(b) $\sum_n (1 - |\lambda_n|) < \infty$.

Proof. We assume that $n(0, F) = 0$. The general case can be reduced to this one by an obvious argument. We also assume that $|\lambda_n| \leqq |\lambda_{n+1}|$, $n = 1, 2, \ldots$, and set $\nu(t) = n(t, F)$, $0 \leqq t < 1$. Integrating by parts and taking into account the assumption that $\nu(0) = 0$, we obtain that for every $0 \leqq r < 1$

$$\sum_{n=1}^{\nu(r)} \log r/|\lambda_n| = \int_0^r (\log r/t)\, d\nu(t) = \int_0^r (\nu(t)/t)\, dt = N(r, F).$$

This shows that condition (a) is equivalent to the condition $\sum_{n=1}^{\infty} \log 1/|\lambda_n| < \infty$ which is clearly equivalent to condition (b). This completes the proof.

From Lemma 2.2 we obtain an equivalent definition of the class $N_a(X)$:

Corollary 2.3. *If X is a complex Banach space and F is an X-valued meromorphic function on D, with poles $\{\lambda_1, \lambda_2, \ldots\}$ repeated according to multiplicity,* *then F is in $N_\alpha(X)$ for some $\alpha \geqq 0$, if and only if* $\sup_{0\leqq r<1} m_\alpha(r, F) < \infty$ *and* $\sum_n (1-|\lambda_n|) < \infty$.

Proof. This is an immediate consequence of Lemma 2.2 and the definition of the class $N_\alpha(X)$.

We recall that if $\{\lambda_1, \lambda_2, \ldots\}$ is a sequence in D which satisfies condition (b) of Lemma 2.2, and if k is the number of λ_n equal to zero, then the Blaschke product

$$B(z) = z^k \prod_{\lambda_n \neq 0} (\bar{\lambda}_n/|\lambda_n|)((\lambda_n - z)/(1-\bar{\lambda}_n z))$$

converges uniformly on compact subsets of D, and B is an inner function in H^∞ whose zeros in D are precisely the points λ_n, and each zero has multiplicity equal to the number of times it occurs in the sequence [7, p. 54].

If T is a contraction such that R_T is in N_α for some $\alpha \geqq 0$, then by Corollary 2.3 the sequence of poles of R_T in D (repeated according to multiplicity) satisfies condition (b) of Lemma 2.2, and therefore by the above observation the Blaschke product associated with this sequence is a well defined inner function in H^∞. We shall denote in the sequel this function by B_T.

Lemma 2.4. *If T is a contraction with resolvent in N_α for some $\alpha \geqq 0$, then the function* $\log \|B_T(z) L_T(z)\|$ *is subharmonic in D.*

Proof. Since the zeros of B_T coincide with the poles of R_T in D, including multiplicity, the operator function $B_T(z) K_T(z)$ is holomorphic in D, and therefore [7, p. 34], for every $x, y \in \mathscr{H}$, the function $\log |\langle B_T(z) K_T(z) x, y\rangle|$ is subharmonic in D. Hence, since by Lemma 2.1,

$$\log \|B_T(z) L_T(z)\| = \sup \{\log |\langle B_T(z) K_T(z) x, y\rangle| : x \in U_T,\ \|y\| \leqq 1\}$$

for all $z \in D \cap \varrho(T)$, it follows that the function $\log \|B_T(z) L_T(z)\|$ is subharmonic in $D \cap \varrho(T)$, and therefore since it is continuous in D and $D \setminus \varrho(T)$ is a discrete set, it follows by simple argument that it is also subharmonic in D.

We shall also need in the sequel the following elementary result which appears in [12, p. 263]. For the sake of completeness, we include the proof.

Lemma 2.5. *If T is a contraction then for every λ in $D \cap \varrho(T)$*

$$(1-|\lambda|)\|R_T(\lambda)\| \leqq \|L_T(\lambda)\| \leqq 1 + 2(1-|\lambda|)\|R_T(\lambda)\|.$$

Proof. Assume that λ is in $D \cap \varrho(T)$. Since $R_T(\lambda) = (I - \bar{\lambda} T)^{-1} L_T(\lambda)$ we have that $\|R_T(\lambda)\| \leqq \|(I-\bar{\lambda}T)^{-1}\| \|L_T(\lambda)\|$, and since T is a contraction,

$$\|(I-\bar{\lambda}T)^{-1}\| = \Big\|\sum_{n=0}^{\infty} \bar{\lambda}^n T^n\Big\| \leqq \sum_{n=0}^{\infty} |\lambda|^n = 1/(1-|\lambda|),$$

and consequently $(1-|\lambda|)\|R_T(\lambda)\|\leqq\|L_T(\lambda)\|$. The second inequality is an immediate consequence of the identity $L_T(\lambda)=\bar\lambda I+(1-|\lambda|^2)R_T(\lambda)$ which follows by a simple computation.

We shall also need in the sequel the fact that the class $C_0.$ is invariant under certain Möbius transformation. This is given by:

Lemma 2.6. *If T is a contraction of class $C_0.$ and $\alpha\in D$, then the operator $T_\alpha=(\alpha I-T)(I-\bar\alpha T)^{-1}$ is also of class $C_0.$.*

Proof. Since T is a contraction T_α is also a contraction, and therefore the sequence of self-adjoint operators $T_\alpha^{*n}T_\alpha^n$ is decreasing, hence converges strongly to some self-adjoint operator L. The assertion that T_α is of class $C_0.$ is clearly equivalent to the assertion that $L=0$. To prove this, notice that $T_\alpha^* LT_\alpha=L$, hence $(\bar\alpha I-T^*)L(\alpha I-T)=(I-\alpha T^*)L(I-\bar\alpha T)$ and therefore $T^*LT=L$. This implies that $T^{*n}LT^n=L$ for every positive integer n, so that for every $x\in\mathscr{H}$ we have that

$$\|Lx\|\leqq\|L\|\,\|T^nx\|,\quad n=0,1,2,\dots,$$

and consequently, since T is of class $C_0.$, we conclude that $L=0$. This completes the proof.

3. Contractions with resolvent in N_1

We begin by showing that the limits in the definitions of the function w and the measure μ in the statement of Theorem 1.1, actually exist for every contraction with resolvent in N_1. This enables us to associate with every such contraction T a function φ_T in H^∞, which by virtue of Theorem 1.1, is a minimal function when T is of class C_0.

Proposition 3.1. *If T is a contraction with resolvent in N_1, then the measures $(1/2\pi)\log\|L_T(\varrho e^{it})\|\,dt$, $0\leqq\varrho<1$, converge as $\varrho\to1-$, in the weak star topology to a positive measure μ on Γ. Furthermore, if w and φ_T are the holomorphic functions on D defined by*

$$w(z)=\int_\Gamma((e^{it}+z)/(e^{it}-z))\,d\mu(t),\quad z\in D$$

and

$$\varphi_T(z)=B_T(z)\exp(-w(z)),\quad z\in D$$

then

$$w(z)=\lim_{\varrho\to1-}(1/2\pi)\int_0^{2\pi}((e^{it}+z)/(e^{it}-z))\log\|L_T(\varrho e^{it})\|\,dt,\quad z\in D$$

and

$$\|\varphi_T(z)L_T(z)\|\leqq1,\quad z\in D.$$

In addition, the function φ_T has the following minimality property: If f is a function in H^∞ which satisfies the condition

$$\|f(z)L_T(z)\| \leqq 1, \quad z \in D$$

then there exists a function h in H^∞ such that $\|h\|_\infty \leqq 1$ and $f = h\varphi_T$.

Proof. Since B_T is in H^∞ and $B_T \not\equiv 0$, we have that

$$\int_0^{2\pi} \left|\log |B_T(\varrho e^{i\theta})|\right| d\theta, \quad 0 < \varrho < 1,$$

is bounded as $\varrho \to 1$ (cf. [10, p. 90]) and therefore noticing that $\|L_T(\lambda)\| \geqq 1$, $\lambda \in D \cap \cap \varrho(T)$ (since T is a contraction), we obtain from the assumption that R_T is in N_1 and Lemma 2.5, that

$$\int_0^{2\pi} \left|\log \|B_T(\varrho e^{i\theta}) L_T(\varrho e^{i\theta})\|\right| d\theta, \quad 0 < \varrho < 1,$$

is also bounded as $\varrho \to 1$. Combining this with the fact that by Lemma 2.4 the function $\log \|B_T(z)L_T(z)\|$ is subharmonic in D, we infer (cf. [6] or [7, p. 38]) that the measures

$$(1/2\pi) \log \|B_T(\varrho e^{it}) L_T(\varrho e^{it})\| \, dt, \quad 0 < \varrho < 1,$$

converge as $\varrho \to 1-$, in the weak star topology to a measure μ on Γ, and the function

$$u(z) = \int_\Gamma \operatorname{Re}\left((e^{it}+z)/(e^{it}-z)\right) d\mu(t), \quad z \in D$$

is the least harmonic majorant of the function $\log \|B_T(z)L_T(z)\|$ in D. Since B_T is a Blaschke product, we have that [7, p. 56]

$$\lim_{\varrho \to 1-} \int_0^{2\pi} \left|\log |B_T(\varrho e^{i\theta})|\right| d\theta = \lim_{\varrho \to 1-} \left(-\int_0^{2\pi} \log |B_T(\varrho e^{i\theta})| \, d\theta\right) = 0$$

and therefore μ is also the weak star limit as $\varrho \to 1-$, of the measures $(1/2\pi) \log \|L_T(\varrho e^{it})\| \, dt$, $0 \leqq \varrho < 1$. Thus remembering that $\|L_T(\lambda)\| \geqq 1$, $\lambda \in D \cap \varrho(T)$, we obtain that μ is a positive measure, and since for every $z \in D$ the function $e^{it} \to \to (e^{it}+z)/(e^{it}-z)$ is continuous on Γ, we also have that

$$w(z) = \lim_{\varrho \to 1-} (1/2\pi) \int_0^{2\pi} \left((e^{it}+z)/(e^{it}-z)\right) \log \|L_T(\varrho e^{it})\| \, dt, \quad z \in D.$$

It is also clear that $u(z) = \operatorname{Re} w(z)$, $z \in D$, and therefore from the above mentioned majorant property of u, we obtain that

$$\log \|B_T(z) L_T(z)\| \leqq \operatorname{Re} w(z), \quad z \in D$$

which is equivalent to the desired inequality,

$$\|\varphi_T(z)L_T(z)\| \leqq 1, \quad z\in D.$$

To prove the last assertion, assume that f is a function in H^∞ that satisfies the condition $\|f(z)L_T(z)\|\leqq 1$, $z\in D\cap\varrho(T)$. We may clearly assume that $f\not\equiv 0$. Since $\|L_T(z)\|\geqq 1$, $z\in D\cap\varrho(T)$, it follows by continuity that $|f(z)|\leqq 1$, $z\in D$. Consider the factorization $f=B\cdot g$, where B is the Blaschke product formed by the zeros of f in D. Then g is in H^∞ and $0<|g(z)|\leqq 1$, $z\in D$, [9, p. 66]. Using the hypothesis on f and Lemma 2.5 we obtain that

$$(1-|z|)\,|f(z)|\,\|R_T(z)\| \leqq 1, \quad z\in D\cap\varrho(T).$$

This implies that every pole of R_T in D, is a zero of f whose multiplicity is not less than the order of the pole. Thus $B=B_1\cdot B_T$, where B_1 is also a Blaschke product. Using again the hypothesis on f we obtain that

$$\log\|B(z)L_T(z)\| \leqq -\log|g(z)|, \quad z\in D\cap\varrho(T)$$

and by continuity this inequality also holds for all $z\in D$. Since $g(z)\neq 0$, $\forall z\in D$, the function $-\log|g(z)|$ is harmonic in D, hence is a harmonic majorant of the function $\log\|B(z)L_T(z)\|$ in D. But by Lemma 2.4 and the above factorization of B, this function is subharmonic in D, and therefore (by [7, p. 38] or [6]) its least harmonic majorant u_1 in D is given by

$$u_1(z) = \lim_{\varrho\to 1-}(1/2\pi)\int_0^{2\pi}\operatorname{Re}\left((e^{it}+z)/(e^{it}-z)\right)\log\|B(\varrho e^{it})L_T(\varrho e^{it})\|\,dt, \quad z\in D$$

and since B is a Blaschke product it follows by the argument already used in the proof of the first part of the proposition that

$$u_1(z) = \lim_{\varrho\to 1-}(1/2\pi)\int_0^{2\pi}\operatorname{Re}\left((e^{it}+z)/(e^{it}-z)\right)\log\|L_T(\varrho e^{it})\|\,dt = \operatorname{Re} w(z)$$

for all $z\in D$. Combining all these facts we obtain that

$$\operatorname{Re} w(z) \leqq -\log|g(z)|, \quad z\in D$$

hence the holomorphic function

$$h(z) = B_1(z)\,g(z)\exp\left(w(z)\right), \quad z\in D$$

satisfies the conditions $|h(z)|\leqq 1$, $z\in D$ and $f=h\varphi_T$. This concludes the proof of the proposition.

Remark. It is clear that $\lim_{\varrho\to 1-}\|L_T(\varrho e^{it})\|=1$, uniformly on every compact subset of $\Gamma\setminus\sigma(T)$, and therefore the closed support of the measure μ defined in

Proposition 3.1 is contained in $\Gamma \cap \sigma(T)$. Hence in particular, if the set $\Gamma \cap \sigma(T)$ has linear measure zero, μ is a singular measure, and φ_T is an inner function.

We can now characterize contractions with resolvent in N_1.

Theorem 3.2. *If T is a contraction then the following conditions are equivalent:*

(a) *R_T is in N_1.*

(b) *R_T is a meromorphic operator function on D which admits a representation of the form*

$$R_T(\lambda) = G(\lambda)/\varphi(\lambda), \quad \lambda \in D \cap \varrho(T)$$

where φ is a function in H^∞ whose zeros in D coincide with the poles of R_T (including multiplicity) and G is a holomorphic operator function on D, which satisfies the condition

$$\sup_{\lambda \in D} (1-|\lambda|)\|G(\lambda)\| < \infty.$$

(c) *The set $D \cap \varrho(T)$ is not empty, and there exists a function $f \not\equiv 0$ in H^∞ such that*

$$\sup_{\lambda \in D \cap \varrho(T)} (1-|\lambda|)|f(\lambda)|\,\|R_T(\lambda)\| < \infty.$$

Proof. (a)$\Rightarrow$(b): If R_T is in N_1, then the zeros of the function φ_T (associated with T by Proposition 3.1) coincide with the poles of R_T, including multiplicity, and therefore the meromorphic function $\varphi_T R_T$ extends to a holomorphic operator function on D, which we denote by G. It follows from Proposition 3.1 and Lemma 2.5 that $(1-|\lambda|)\|G(\lambda)\| \leqq 1$ for $\lambda \in D \cap \varrho(T)$, and by continuity this inequality holds also for all $\lambda \in D$. Hence condition (b) is satisfied with $\varphi = \varphi_T$ and $G = \varphi_T R_T$.

(b)$\Rightarrow$(c): This is obvious.

(c)$\Rightarrow$(a): Assume that condition (c) holds for some function $f \not\equiv 0$ in H^∞, and denote for every $\lambda \in \varrho(T)$ by $d(\lambda)$ the distance of λ from $\sigma(T)$. Since for every $\lambda \in \varrho(T)$ we have the inequality $(d(\lambda))^{-1} \leqq \|R_T(\lambda)\|$, (cf. [5, p. 567]) it follows from the assumption on f that for some constant $M>0$

$$|f(\lambda)| \leqq Md(\lambda)/(1-|\lambda|), \quad \lambda \in D \cap \varrho(T)$$

and therefore by continuity, f vanishes on $D \cap \sigma(T)$. Consequently, since f is holomorphic and $f \not\equiv 0$, the set $D \cap \sigma(T)$ is discrete, and therefore by condition (c), all the singularities of R_T in D are poles, and the order of each pole does not exceed its multiplicity as a zero of f. Thus R_T is meromorphic on D, and by the Blaschke condition satisfied by the zeros of a function in H^∞ [7, p. 53], we obtain that the sequence of poles of R_T in D satisfies condition (b) of Lemma 2.2. Condition (c) also implies that there exists a constant $K>0$ such that

$$\int_0^{2\pi} \log^+ \|(1-r)R_T(re^{i\theta})\|\, d\theta \leqq \int_0^{2\pi} \left|\log |f(re^{i\theta})|\right| d\theta + K$$

for all $0\leqq r<1$. Since $f\in H^\infty$ and $f\not\equiv 0$, the expression on the right hand side of the above inequality is dominated by a positive constant which does not depend on r [10, p. 90]. Thus by Corollary 2.3 we conclude that R_T is in N_1, and the proof of the theorem is complete.

We conclude this section with a result that describes the action of certain Möbius transformations on contractions with resolvent in N_1. This result will be required for the proof of Theorem 1.1.

Proposition 3.3. *Let T be a contraction with resolvent in N_1. Fix $\alpha\in D$ and consider the function $q(z)=(\alpha-z)/(1-\bar{\alpha}z)$, $z\in D$. Then the contraction $T_\alpha=q(T)$ has also resolvent in N_1, and there exists a constant c of modulus* 1, *such that*

$$\varphi_{T_\alpha}(z) = c\varphi_T(q(z)), \quad z\in D.$$

Proof. A simple computation shows that $\lambda\in\varrho(T_\alpha)$ if and only if $q(\lambda)\in\varrho(T)$ and

$$R_{T_\alpha}(\lambda) = (\bar{\alpha}\lambda-1)^{-1}(I-\bar{\alpha}T)R_T(q(\lambda)), \quad \lambda\in\varrho(T_\alpha).$$

Thus, using the representation of R_T given by part (b) of Theorem 3.2, we obtain that

$$R_{T_\alpha}(\lambda) = G_1(\lambda)/\varphi_1(\lambda), \quad \lambda\in D\cap\varrho(T_\alpha)$$

where $\varphi_1(\lambda)=(\bar{\alpha}\lambda-1)^{-1}\varphi(q(\lambda))$ and $G_1(\lambda)=(1-\bar{\alpha}T)G(q(\lambda))$ for every $\lambda\in D$. Hence remembering that

$$\sup_{\lambda\in D}(1-|\lambda|)\|G(\lambda)\| < \infty$$

and using the estimate

$$(1-|\lambda|)/(1-|q(\lambda)|) \leqq 4/(1-|\alpha|), \quad \lambda\in D$$

(see [7, p. 3, formula 1.5]) we obtain that

$$\sup_{\lambda\in D}(1-|\lambda|)\|G_1(\lambda)\| < \infty$$

and therefore by Theorem 3.2, R_{T_α} is in N_1.

We turn now to the proof of the second assertion. A direct computation shows that for every $\lambda\in D\cap\varrho(T_\alpha)$

$$L_{T_\alpha}(\lambda) = (1-\alpha\bar{\lambda})(\bar{\alpha}\lambda-1)^{-1}L_T(q(\lambda))$$

and therefore $\|L_{T_\alpha}(\lambda)\|=\|L_T(q(\lambda))\|$. Hence using the fact that by Proposition 3.1,

$$\|\varphi_T(\lambda)L_T(\lambda)\| \leqq 1, \quad \lambda\in D\cap\varrho(T)$$

we obtain that also

$$\|\varphi_T(q(\lambda)L_{T_\alpha}(\lambda)\| \leqq 1, \quad \lambda\in D\cap\varrho(T_\alpha)$$

and therefore by the minimality property of the function φ_{T_α}, there exists a function g in H^∞, such that $\|g\|_\infty \leqq 1$ and

$$\varphi_{T_\alpha}(\lambda) = g(\lambda)\varphi_T(q(\lambda)), \quad \lambda\in D.$$

Changing the roles of T_α and T and noticing that $q(q(\lambda))=\lambda$, $\forall\lambda\in D$, we obtain in the same way, that there exists a function h in H^∞, such that $\|h\|_\infty \leqq 1$ and

$$\varphi_T(\lambda) = h(\lambda)\varphi_{T_\alpha}(q(\lambda)), \quad \lambda\in D$$

and therefore

$$\varphi_T(q(\lambda)) = h(q(\lambda))\varphi_{T_\alpha}(\lambda), \quad \lambda\in D.$$

Hence by the maximum principle $g\equiv c$, where c is a constant of modulus 1. This concludes the proof.

4. Proof of the main result

For the proof of Theorem 1.1 we require one more preliminary result.

Lemma 4.1. *Let T be a completely non unitary contraction such that R_T is in N_1. Then setting $\varphi=\varphi_T$, we have that*

$$(1-|\lambda|)\|\varphi(T)R_T(\lambda)\| \leqq 3, \quad \lambda\in D\cap\varrho(T).$$

Proof. For every $\lambda\in D$ consider the holomorphic function h_λ on D defined by

$$h_\lambda(z) = (\varphi(z)-\varphi(\lambda))(z-\lambda)^{-1}, \quad z\in D, \quad z\neq\lambda.$$

It is easily verified that $h_\lambda\in H^\infty$ and $\|h_\lambda\|_\infty \leqq 2/(1-|\lambda|)$, and therefore also $\|h_\lambda(T)\| \leqq 2/(1-|\lambda|)$. Since

$$\varphi(T)-\varphi(\lambda)I = (T-\lambda I)h_\lambda(T), \quad \lambda\in D$$

it follows that

$$(\varphi(T)-\varphi(\lambda)I)R_T(\lambda) = -h_\lambda(T), \quad \lambda\in D\cap\varrho(T)$$

and consequently

$$\|(\varphi(T)-\varphi(\lambda)I)R_T(\lambda)\| \leqq 2/(1-|\lambda|), \quad \lambda\in D\cap\varrho(T).$$

This implies the desired conclusion by virtue of Lemma 2.5 and Proposition 3.1.

Proof of Theorem 1.1. We assume first that T is an invertible contraction of class C_0 with resolvent in N_1, and prove that T is annihilated by φ_T. For this, we set $\varphi=\varphi_T$, and consider the operator function

$$F(\lambda)=T\varphi(T)R_T(\lambda),\quad \lambda\in\varrho(T).$$

Observe that since T is of class $C_{0\cdot}$, it is completely non unitary, and therefore $\varphi(T)$ is well defined. The singularities of F in D, which are the poles of R_T, are removable, since by Lemma 4.1 we have for every $0<r<1$; that

$$\sup\{\|F(\lambda)\|:\ \lambda\in\varrho(T),\ |\lambda|<r\}<\infty.$$

Thus F is holomorphic in D, and therefore using the assumption that T is invertible we obtain from the Taylor expansion of R_T around $z=0$, that

$$F(\lambda)=-\sum_{n=0}^{\infty}\varphi(T)T^{-n}\lambda^n,\quad \lambda\in D$$

the series converging in the operator norm. Combining this with the Laurent expansion of R_T for $|z|>1$, we obtain that for every $re^{i\theta}\in D$,

$$F(re^{i\theta})-F(r^{-1}e^{i\theta})=-\sum_{n=-\infty}^{\infty}r^{|n|}\varphi(T)T^{-n}e^{in\theta}$$

the series converging again in the operator norm. On the other hand, using the resolvent identity

$$R_T(\lambda)-R_T(\lambda')=(\lambda'-\lambda)R_T(\lambda)R_T(\lambda'),\quad \lambda,\lambda'\in\varrho(T)$$

we obtain that for every $re^{i\theta}\in D$

$$F(re^{i\theta})-F(r^{-1}e^{i\theta})=e^{i\theta}r^{-1}(1-r^2)F(re^{i\theta})R_T(r^{-1}e^{i\theta})$$

and therefore by Lemma 4.1, we obtain that for every $x\in\mathscr{H}$ and $re^{i\theta}\in D$

$$\left\|\left(F(re^{i\theta})-F(r^{-1}e^{i\theta})\right)x\right\|\leqq 6r^{-1}\|R_T(r^{-1}e^{i\theta})x\|.$$

Hence applying the Parseval identity for Hilbert space valued functions on Γ, we obtain that for every $x\in\mathscr{H}$

$$\sum_{n=-\infty}^{\infty}r^{2|n|}\|\varphi(T)T^{-n}x\|^2=(1/2\pi)\int_0^{2\pi}\left\|\left(F(re^{i\theta})-F(r^{-1}e^{i\theta})\right)x\right\|^2d\theta\leqq$$

$$\leqq 36r^{-2}(1/2\pi)\int_0^{2\pi}\|R_T(r^{-1}e^{i\theta})x\|^2d\theta=36\sum_{n=0}^{\infty}r^{2n}\|T^nx\|^2.$$

(The proof of this inequality was inspired by the methods in [16].) Since T is a contraction

$$\|T^{-n}\varphi(T)x\|\leqq\|T^{-n-1}\varphi(T)x\|,\quad n=0,1,2,\ldots,$$

and therefore

$$\|\varphi(T)x\|^2 \leqq (1-r^2) \sum_{n=-\infty}^{\infty} r^{2|n|} \|T^{-n}\varphi(T)x\|^2.$$

Combining this with the preceeding estimate we obtain that

$$\|\varphi(T)x\|^2 \leqq 36(1-r^2) \sum_{n=0}^{\infty} r^{2n} \|T^n x\|^2.$$

But the assumption that T is of class $C_{0\cdot}$ implies that the expression on the right hand side of the above inequality tends to zero as $r \to 1-$, and consequently, $\varphi(T)x=0$. Since this holds for every $x \in \mathscr{H}$, we conclude that $\varphi(T)=0$. To prove the same result for T not necessarily invertible, assume again that T is of class $C_{0\cdot}$ and that R_T is in N_1. Choose $\alpha \in D \cap \varrho(T)$, and consider the function $q(z)= =(\alpha-z)/(1-\bar{\alpha}z)$, $z \in D$, and the invertible contraction $T_\alpha = q(T)$. By Lemma 2.6 and Proposition 3.3, T_α is also of class $C_{0\cdot}$ and has resolvent in N_1, and therefore by what has just been proved, we have that $\varphi_\alpha(T)=0$ where φ_α denotes the function φ_{T_α}. But by Proposition 3.3, $\varphi_\alpha = c\varphi \circ q$ where c is a constant of modulus 1, and therefore using the fact that $q \circ q(\lambda)=\lambda$, $\forall \lambda \in D$, we obtain that $\varphi = c^{-1}\varphi_\alpha \circ q$ and consequently $\varphi(T)=c^{-1}\varphi_\alpha(T_\alpha)=0$. This establishes the assertion in the general case.

We show next that φ_T is a minimal function of T. For this assume that f is a function in H^∞ such that $\|f\|_\infty \leqq 1$ and $f(T)=0$. To prove that φ_T divides f in H^∞, consider for every $\lambda \in D$, the holomorphic function g_λ on D defined by

$$g_\lambda(z) = \big(f(z)-f(\lambda)\big)\big(1-\overline{f(\lambda)}f(z)\big)^{-1}(z-\lambda)^{-1}(1-\bar{\lambda}z), \quad z \in D, \quad z \neq \lambda.$$

Since $\|f\|_\infty \leqq 1$ also $\|g_\lambda\|_\infty \leqq 1$ and therefore also $\|g_\lambda(T)\| \leqq 1$. Using the identity

$$g_\lambda(z)\big(1-\overline{f(\lambda)}f(z)\big) = \big(f(z)-f(\lambda)\big)(1-\bar{\lambda}z)(z-\lambda)^{-1}$$

and the assumption that $f(T)=0$, we obtain that for every $\lambda \in D \cap \varrho(T)$,

$$\|f(\lambda)L_T(\lambda)\| = \|g_\lambda(T)\| \leqq 1.$$

Consequently, by Proposition 3.1, there exists a function $h \in H^\infty$ such that $\|h\|_\infty \leqq 1$ and $f = h\varphi_T$. Hence φ_T divides every function in H^∞ which annihilates T. We show now that φ_T is an inner function. For this, set again $\varphi = \varphi_T$ and consider the canonical factorization $\varphi = \varphi_1 \cdot \varphi_2$ where φ_1 and φ_2 are the outer and inner factors of φ, respectively. Then $\varphi_1(T)\varphi_2(T)=0$, and therefore using the fact that $\varphi_1(T)$ has zero kernel since φ_1 is outer [12, p. 118], we obtain that also $\varphi_2(T)=0$. Hence by the result just proved, we have that

$$|\varphi_1(\lambda)|^{-1} = |\varphi_2(\lambda) \cdot \varphi(\lambda)^{-1}| \leqq 1, \quad \text{for all} \quad \lambda \in D.$$

On the other hand since $\|\varphi\|_\infty \leqq 1$ also $\|\varphi_1\|_\infty \leqq 1$, and therefore by the maximum principle $\varphi_1 \equiv c$, where c is a constant of modulus 1. Thus φ_T is an inner function, and consequently is a minimal function of T. (Since $\varphi(0)$ is real we actually have that $c=1$, so that $\varphi=\varphi_2$.)

Finally to prove the remaining assertion of the theorem, assume that T is a contraction of class C_0. Then by [12, p. 123] T is of class $C_{0\cdot}$. (For a proof of this fact which is independent of dilation theory see [8].) To show that R_T is in N_1, consider a function $f \not\equiv 0$ in H^∞ such that $f(T)=0$. For every $\lambda \in D$, consider the holomorphic function f_λ on D defined by

$$f_\lambda(z) = (f(z)-f(\lambda))(z-\lambda)^{-1}, \quad z \in D, \quad z \neq \lambda.$$

Then $\|f_\lambda\|_\infty \leqq 2\|f\|_\infty/(1-|\lambda|)$ and therefore also $\|f_\lambda(T)\| \leqq 2\|f\|_\infty/(1-|\lambda|)$. Since $f(T)=0$ we have that $f_\lambda(T)(\lambda I-T)=f(\lambda)I$, hence if $f(\lambda)\neq 0$ then $\lambda \in D \cap \varrho(T)$, and therefore since $f \not\equiv 0$, the set $D \cap \varrho(T)$ is not empty. It also follows from the preceeding facts that

$$\|f(\lambda)R_T(\lambda)\| = \|f_\lambda(T)\| \leqq 2\|f\|_\infty/(1-|\lambda|),$$

for all λ in $D \cap \varrho(T)$. Thus R_T satisfies condition (c) of Theorem 3.2 and therefore by that theorem R_T is in N_1. This concludes the proof of Theorem 1.1.

Remark 1. Observe that we obtained above also a proof of the existence of a minimal function for a contraction of class C_0 which is different from the proof of this fact given in [12, p. 124]. Still another proof of this fact appears in [3, p. 188].

Remark 2. It follows from Proposition 3.1 and the proof of Theorem 1.1, that if T is a contraction of class C_0 then a function f in H^∞ annihilates T, if and only if, it satisfies the condition

$$\sup_{\lambda \in D \cap \varrho(T)} |f(\lambda)|\,\|L_T(\lambda)\| < \infty$$

which by virtue of Lemma 2.5 is also equivalent to the condition

$$\sup_{\lambda \in D \cap \varrho(T)} (1-|\lambda|)\,|f(\lambda)|\,\|R_T(\lambda)\| < \infty.$$

An immediate consequence of Theorem 1.1 and Theorem 3.2 is the following:

Corollary 4.2 (Dazord [4]). *A contraction T is of class C_0, if and only if, T is of class $C_{0\cdot}$, and there exists a function $f \not\equiv 0$ in H^∞ such that*

$$\sup_{\lambda \in D \cap \varrho(T)} (1-|\lambda|)\,|f(\lambda)|\,\|R_T(\lambda)\| < \infty.$$

5. Contractions whose characteristic function has a scalar multiple

In this section we express the preceeding results in terms of the characteristic operator function associated with a contraction. We recall that [12, Chapter VI] if T is a contraction then its characteristic function is the holomorphic operator function θ_T on D, whose value at every $\lambda\in D$ is the bounded linear operator $\theta_T(\lambda)$ from the Hilbert space $\mathcal{D}_T$ to the Hilbert space $\mathcal{D}_{T^*}$, which is defined by the relation

$$\theta_T(\lambda)D_T = D_{T^*}(I-\lambda T^*)^{-1}(\lambda I-T).$$

For every $\lambda\in\varrho(T)$ the operator $\theta_T(\lambda)$ is invertible and its inverse $\theta_T(\lambda)^{-1}$ is the bounded linear operator from $\mathcal{D}_{T^*}$ to $\mathcal{D}_T$ which satisfies the equality

$$\theta_T(\lambda)^{-1}D_{T^*} = D_T(\lambda I-T)^{-1}(I-\lambda T^*) = K_T(\lambda)$$

(K_T is the operator function defined in Section 2). Thus from Lemma 2.1 we obtain for every $\lambda\in\varrho(T)$ the equality $\|L_T(\lambda)\| = \|\theta_T(\lambda)^{-1}\|$, which is also proved in [12, p. 264]. This implies by Lemma 2.5, that R_T is meromorphic in D if and only if θ_T^{-1} is meromorphic in D, and in this case these two functions have the same poles in D with the same orders (see also [12, p. 264]). Thus using Lemma 2.5 we obtain:

Proposition 5.1. *If T is a contraction then R_T is in N_1, if and only if, θ_T^{-1} is of bounded characteristic (as a function from $D\cap\varrho(T)$ into the Banach space of all bounded linear operators from $\mathcal{D}_{T^*}$ to $\mathcal{D}_T$).*

We recall [12, p. 264] that a function $f\not\equiv 0$ in H^∞ is called a scalar multiple of θ_T, if there exists a holomorphic operator function Ω on D whose values are bounded linear operators from $\mathcal{D}_{T^*}$ to $\mathcal{D}_T$ with norm not exceeding 1, such that for every $\lambda\in D$

$$\Omega(\lambda)\theta_T(\lambda) = f(\lambda)I_1 \quad \text{and} \quad \theta_T(\lambda)\Omega(\lambda) = f(\lambda)I_2$$

where I_1 and I_2 are the identity operators on $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ respectively.

The above equalities are clearly equivalent to the equality

$$\theta_T(\lambda)^{-1} = \Omega(\lambda)/f(\lambda)$$

for every $\lambda\in D$ such that $f(\lambda)\neq 0$. Thus from [2, Th. 2.1] and Proposition 5.1 we obtain

Theorem 5.2. *If T is a contraction then the following conditions are equivalent:*
(a) *R_T is in N_1,*
(b) *θ_T^{-1} is of bounded characteristic,*
(c) *θ_T has a scalar multiple.*

Remark. It follows from Theorem 5.2, Proposition 3.1 and the identity $\|\theta_T(\lambda)^{-1}\|=\|L_T(\lambda)\|$, $\lambda\in D\cap\varrho(T)$, that if θ_T has a scalar multiple f, then φ_T is also a scalar multiple of θ_T, and there exists a function h in H^∞ such that $\|h\|_\infty\leqq 1$ and $f=h\varphi_T$. Thus φ_T divides every other scalar multiple of θ_T and therefore can be called a minimal scalar multiple of θ_T.

Finally, from Theorem 1.1, Theorem 5.2 and the equality $\|\theta_T(\lambda)^{-1}\|=\|L_T(\lambda)\|$, $\lambda\in D\cap\varrho(T)$, we obtain a characterization of operators of class C_0 and a formula for their minimal function expressed in terms of the characteristic function:

Theorem 5.3. *A contraction T is of class C_0 if and only if T is of class $C_{0\cdot}$ and one of the three equivalent conditions of Theorem 5.2 is satisfied. Furthermore, if T is of class C_0 it has a minimal function given by*

$$m_T(z) = B_T(z)\exp\big(-w(z)\big),\quad z\in D$$

where

$$w(z) = \lim_{\varrho\to 1-}(1/2\pi)\int_0^{2\pi}\big((e^{it}+z)/(e^{it}-z)\big)\log\|\theta_T(\varrho e^{it})^{-1}\|\,dt,\quad z\in D$$

or alternatively

$$w(z) = \int_\Gamma\big((e^{it}+z)/(e^{it}-z)\big)\,d\mu(t),\quad z\in D$$

where

$$\mu = w^*\lim_{\varrho\to 1-}(1/2\pi)\log\|\theta_T(\varrho e^{it})^{-1}\|\,dt.$$

An immediate consequence of Theorem 5.3 is

Corollary 5.4 (Sz.-Nagy and Foiaş [12, p. 265]). *A contraction T is of class C_0 if and only if T is of class $C_{0\cdot}$ and θ_T has a scalar multiple.*

6. Contractions which are annihilated by functions in the disc algebra

Let A denote the disc algebra, that is the Banach algebra of all continuous functions on the closed unit disc $\bar D$ which are holomorphic in D, equipped with the supremum norm. In view of the von Neumann inequality [12, p. 32] and the fact that the polynomials are dense in A, there exists for every contraction T a norm continuous multiplicative homomorphism of the Banach algebra A into the Banach algebra $\mathscr{L}(\mathscr{H})$, which extends the mapping $p\to p(T)$ where p is a polynomial (see also [3, p. 167] and [8]). It is easily verified that if $f(z)=\sum_{n=0}^\infty a_nz^n$ is a function in A, then the operator $f(T)$ which corresponds to f by this homomorphism, is given by $f(T)=\lim_{r\to 1-}\sum_{n=0}^\infty a_nr^nT^n$ (where the convergence of the series and the limit are in the operator norm). It is also clear that if T is completely non unitary, then this homomorphism is the restriction to A of the homomorphism from H^∞ into $\mathscr{L}(\mathscr{H})$ given by [12, p. 117, Theorem 2.3].

Following [8] we shall say that a contraction T is of class D_0 if there exists a function $f\not\equiv 0$ in A such that $f(T)=0$.

The characterization of contractions of class D_0 is given by:

Theorem 6.1. *A contraction T is of class D_0, if and only if R_T is in N_1 and the set $\sigma(T)\cap\Gamma$ has linear measure zero.*

Proof. We assume first that T is a contraction such that R_T is in N_1 and $\sigma(T)\cap\Gamma$ has linear measure zero, and show that T is of class D_0. Since the set $\sigma(T)\cap\Gamma$ has linear measure zero, there exists by a Theorem of Fatou [9, p. 80] a function $g\not\equiv 0$ in A such that $g=0$ on $\sigma(T)\cap\Gamma$. (One can also choose by that theorem, g to be outer and so that it vanishes only on $\sigma(T)\cap\Gamma$.) Consider the function $f=g\varphi_T$. We claim that f is in A and that $f(T)=0$. By the remark following the proof of Propositon 3.1, the closed support of the measure μ which is associated with φ_T, is contained in $\sigma(T)\cap\Gamma$, and therefore since this set has linear measure zero (by assumption) μ is a singular measure and φ_T is an inner function. Since the accumulation points of the zeros of B_T are also contained in $\sigma(T)\cap\Gamma$; it follows [9, p. 68] that φ_T is continuous on $\bar{D}\setminus(\sigma(T)\cap\Gamma)$. Therefore, since $g=0$ on $\sigma(T)\cap\Gamma$, the function f extends to a continuous function on $\bar{D}$ which vanishes on $\sigma(T)\cap\Gamma$. Thus f is in A.

To show that $f(T)=0$, consider the canonical decomposition $T=T_0\oplus T_1$ [12, p. 9] where T_0 and T_1 are the unitary and completely non unitary parts of T respectively. Since $f(T)=f(T_0)\oplus f(T_1)$, we have to show that $f(T_0)=0$ and $f(T_1)=0$. The fact that T_0 is unitary implies that $\sigma(T_0)\subset\sigma(T)\cap\Gamma$, and therefore, since $f=0$ on $\sigma(T)\cap\Gamma$, it follows from the spectral theorem for unitary operators that $f(T_0)=0$. To show that also $f(T_1)=0$, observe first that for every $\lambda\in\varrho(T)$

$$R_T(\lambda) = R_{T_0}(\lambda)\oplus R_{T_1}(\lambda) \quad \text{and} \quad \sigma(T)\cap D = \sigma(T_1)\cap D.$$

This implies that R_{T_1} is also in N_1 and $B_T=B_{T_1}$. Also, using the facts that for every $\lambda\in\varrho(T)$, $L_T(\lambda)=L_{T_0}(\lambda)\oplus L_{T_1}(\lambda)$ and $\|L_{T_0}(\lambda)\|=1$ (since T_0 is unitary) and remembering that $\|L_{T_1}(\lambda)\|\geqq 1$; $\lambda\in\varrho(T_1)\cap D$, we obtain that

$$\|L_T(\lambda)\| = \|L_{T_1}(\lambda)\|, \quad \lambda\in\varrho(T)\cap D = \varrho(T_1)\cap D,$$

and combining this with the equality $B_T=B_{T_1}$, we infer that $\varphi_T=\varphi_{T_1}$. Since the set $\sigma(T)\cap\Gamma$ has linear measure zero, the same is true for its subset $\sigma(T_1)\cap\Gamma$, and therefore, since T_1 is completely non unitary, it follows [12, p. 84, Proposition 6.7] that T_1 is of class $C_{0\cdot}$. Thus by Theorem 1.1, T_1 is annihilated by φ_{T_1} and since $\varphi_{T_1}=\varphi_T$ we also have that $f(T_1)=0$. This proves that T is of class D_0.

To prove the converse assume that T is a contraction of class D_0. Then by [1, Corollary 4] or [8], the set $\sigma(T)\cap\Gamma$ has linear measure zero. The proof that R_T is in N_1 is exactly the same as the proof of the last part of Theorem 1.1. The only additional fact to observe is, that if f is a function in A, then for every $\lambda\in D$ the holomorphic function on D defined by

$$f_\lambda(z)=\big(f(z)-f(\lambda)\big)(z-\lambda)^{-1},\quad z\in\bar D,\quad z\neq\lambda$$

is also in A. This concludes the proof of the theorem.

The preceeding proof shows that if T is a contraction of class D_0, then for every function g in A that vanishes on $\sigma(T)\cap\Gamma$, the function $f=g\varphi_T$ is also in A and $f(T)=0$. We show next that this is the general form of a function in A which annihilates T.

Proposition 6.2. *If T is a contraction of class D_0 and f is a function in A, then $f(T)=0$, if and only if there exists a function g in A that vanishes on $\sigma(T)\cap\Gamma$ such that $f=g\varphi_T$.*

Proof. In view of the preceeding observation it remains to show that every function in A which annihilates T, is of the above form. To show this assume that f is a function in A such that $f(T)=0$. Then also $f(T_1)=0$; where T_1 denotes as before the completely non unitary part of T. Therefore by Theorem 1.1, there exists a function g in H^∞ such that $f=g\varphi_{T_1}$, and since by the proof of Theorem 6.1, $\varphi_{T_1}=\varphi_T$, we have that $f=g\varphi_T$. Since φ_T is an inner function which (as observed in the proof of Theorem 6.1) is continuous on $\bar D\setminus(\sigma(T)\cap\Gamma)$; and since as shown in the proof of [1, Th. 4], $f=0$ on $\sigma(T)$, it follows that g extends to a continuous function on $\bar D$, which vanishes on $\sigma(T)\cap\Gamma$. This completes the proof of the proposition.

Remark. As observed in [8], it follows from the characterization of closed ideals in the algebra A [9, p. 85] that every contraction T of class D_0 determines uniquely a closed set $K\subset\Gamma$ of linear measure zero, and an inner function φ, such that a function f in A annihilates T, if and only if $f=g\varphi$, where g is a function in A that vanishes on K. Proposition 6.2 gives an independent proof of this fact and also provides the more precise information that $K=\sigma(T)\cap\Gamma$ and $\varphi=\varphi_T$.

7. Contractions with resolvent in N_α for some $0\leq\alpha<1$

According to [2, Theorem 1.2] a contraction T has resolvent of bounded characteristic, if and only if, T is of class D_0 and $\sigma(T)$ is a thin set, that is, in addition to the Blaschke condition satisfied by the countable set $\sigma(T)\cap D$, also the condition

$$\int_0^{2\pi} (\log 1/d(e^{i\theta}, \sigma(T)))\, d\theta < \infty$$

holds (where for $\lambda \in \mathbf{C}$, $d(\lambda, \sigma(T))$ denotes the distance of λ from $\sigma(T)$).

For contractions with resolvent in N_α for some $0<\alpha<1$, we only have a partial result:

Theorem 7.1. *Assume that T is a contraction such that R_T is in N_α for some $0<\alpha<1$. Then T is of class D_0 and*

$$\int_0^{2\pi} (\log^+ 1/d(e^{i\theta}, \sigma(T)))^{1-\delta}\, d\theta < \infty$$

for every $\delta>0$.

Proof. We prove first the second assertion of the theorem. To simplify notations we set $d(\lambda)=d(\lambda, \sigma(T))$, for every $\lambda \in \mathbf{C}$. Remembering that [5, p. 567]

$$(d(\lambda))^{-1} \leqq \|R_T(\lambda)\|, \quad \lambda \in \varrho(T)$$

and using the assumption that R_T is in N_α, we obtain that there exists a constant $M>0$ such that

$$\int_0^{2\pi} (\log (1-r)^\alpha / d(re^{i\theta}))\, d\theta \leqq M, \quad 0 \leqq r < 1.$$

For every $t>0$, consider the set

$$E_t = \{\theta \in [0, 2\pi)\colon\ d(e^{i\theta}) \leqq t\}$$

and denote its Lebesgue measure by $m(t)$. Thus m is the distribution function of the function $\theta \to d(e^{i\theta})$, $\theta \in [0, 2\pi)$. Noticing that

$$d(re^{i\theta}) \leqq (1-r)+d(e^{i\theta}), \quad e^{i\theta} \in \Gamma, \quad 0 \leqq r < 1,$$

we obtain from the preceeding inequality that for every $0<t\leqq 1$,

$$m(t) \log 1/2t^{1-\alpha} \leqq \int_{E_t} (\log t^\alpha / d((1-t)e^{i\theta}))\, d\theta \leqq M$$

and therefore since $\alpha<1$ we deduce that there exists a positive constant c such that

$$m(t) \leqq c(2+\log 1/t)^{-1}, \quad 0 < t < 2\pi.$$

It is also clear that $m(t)=2\pi$ for $t \geqq 2\pi$. Thus using the well known properties of the distribution function (cf. [10, p. 65]) we obtain by integrating by parts and using the estimate above, that for every $\delta > 0$

$$\int_0^{2\pi} (\log^+ 1/d(e^{i\theta}))^{1-\delta}\, d\theta \leqq \int_0^{2\pi} (2+\log 1/d(e^{i\theta}))^{1-\delta}\, d\theta =$$

$$= \int_0^{2\pi} (2+\log 1/t)^{1-\delta}\, dm(t) \leqq 2\pi + c(1-\delta) \int_0^{2\pi} (2+\log 1/t)^{-1-\delta}(1/t)\, dt.$$

Since $\delta > 0$, the last integral converges, and the assertion is established.

To prove that T is of class D_0, denote for every $e^{i\theta} \in \Gamma$ by $d_1(e^{i\theta})$ the distance of $e^{i\theta}$ from the set $\Gamma \cap \sigma(T)$, and fix $0 < \delta < 1$. Then by the assertion just proved,

$$\int_0^{2\pi} (\log^+ 1/d_1(e^{i\theta}))^{1-\delta}\, d\theta < \infty$$

and this clearly implies that the set $\Gamma \cap \sigma(T)$ has linear measure zero. Thus by Theorem 6.1, T is of class D_0.

References

[1] A. Atzmon, Operators which are annihilated by analytic functions and invariant subspaces, *Acta Math.*, **144** (1980), 27—63.

[2] A. Atzmon, Operators with resolvent of bounded characteristic, *J. Integral Equations and Operator Theory*, **6** (1983), 779—803.

[3] R. G. Douglas, Canonical models, in: *Topics in operator theory*, Mathematical Surveys, no. 13, Amer. Math. Soc. (Providence, R. I., 1974); pp. 163—218.

[4] J. Dazord, The growth of the resolvent of a contraction of class C_0, *Rev. Roumaine Math. Pures Appl.*, **24** (1979), 213—218.

[5] N. Dunford and J. T. Schwartz, *Linear operators*, part I, Interscience (New York, 1968).

[6] L. Gårdind and L. Hörmander, Strongly subharmonic functions, *Math. Scand.*, **13** (1974), 93—96.

[7] J. B. Garnett, *Bounded analytic functions*, Academic Press (New York, 1981).

[8] P. Ghatage, Generalized algebraic operators, *Proc. Amer. Math. Soc.*, **52** (1975), 232—236.

[9] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall (Englewood Cliffs, N. J., 1962).

[10] Y. Katznelson, *An introduction to harmonic analysis*, Dover (New York, 1976).

[11] B. Sz.-Nagy and C. Foiaş, Sur les contractions de l'espace de Hilbert. VII, Triangulations canoniques. Fonction minimum, *Acta Sci. Math.*, **25** (1964), 12—37.

[12] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland (Amsterdam, 1970).

[13] B. Sz.-Nagy and C. Foiaş, Modéle de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[14] B. Sz.-Nagy and C. Foiaş, Compléments à l'étude des opérateurs de classe C_0, *Acta Sci. Math.*, **31** (1970), 287—296, and **33** (1972), 113—116.

[15] R. Nevanlinna, *Analytic functions*, Springer-Verlag (New York, 1972).

[16] J. A. van Casteren, A problem of Sz.-Nagy, *Acta Sci. Math.*, **42** (1980), 189—194.

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