*Acta Sci. Math.*, 50 (1986), 213—223

# The distance between unitary orbits of normal operators

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The problem of computing the distance between unitary orbits of operators is an important and difficult problem, even in the finite dimensional case. These problems have a long history, and we will mention some of the important results for the operator norm only.

In 1912, WEYL [21] proved that given two Hermitian matrices $A$ and $B$ with spectrum $a_1 \leqq a_2 \leqq \ldots \leqq a_n$ and $b_1 \leqq b_2 \leqq \ldots \leqq b_n$ (repeated according to multiplicity) respectively, then

$$\operatorname{dist}\left(\mathcal{U}(A), \mathcal{U}(B)\right) = \max_{1 \leqq i \leqq n} |a_i - b_i|.$$

This distance is clearly attained by a commuting pair of diagonal matrices in $\mathcal{U}(A)$ and $\mathcal{U}(B)$, respectively.

The normal case has received much attention, but the final answer is still not known. However, the natural analogue for the right hand side is obtained by looking at *commuting* pairs. If $A$ and $B$ commute, they are simultaneously diagonalizable which results in a pairing of eigenvalues $\{a_i, b_i\}$, $1 \leqq i \leqq n$, and $\|A-B\| = \max |a_i - b_i|$. This is minimized if the pairing is optimal. This suggests the *spectral distance*

$$\delta(A, B) = \min_{\pi} \max_{1 \leqq i \leqq n} |a_i - b_{\pi(i)}|$$

where $\pi$ runs over all permutations. Recently, it has been shown [5] that there is a universal constant $c$ independent of dimension such that

$$\delta(A, B) \geqq \operatorname{dist}\left(\mathcal{U}(A), \mathcal{U}(B)\right) \geqq c^{-1} \delta(A, B).$$

A number of cases in which equality (i.e. $c=1$) exists are known: unitaries [4], self-adjoint and skew-adjoint [18], and scalar multiples of unitaries [6].

In infinite dimensions, unitary orbits have received much attention. In this case, $\mathcal{U}(A)$ is rarely closed. However, for normals, there is a nice description of

Received February 27, 1984.

[1]) Research partially supported by Natural Sciences and Engineering Research Council of Canada (NSERC).

$\overline{\mathcal{U}(A)}$ in terms of a crude multiplicity function [10]. This, in turn, can be interpreted in terms of the spectrum of $A$ modulo various (closed 2-sided) ideals of $\mathcal{B}(H)$ [8]. For more general operators, the invariants for $\overline{\mathcal{U}(A)}$ can be very complicated, and on this there is a large literature ([16], [9], [19], [11]). An important problem in operator theory and $C^*$ algebras has been the relation between the unitary orbit and ideal perturbations. The grandfather result is the Weyl—von Neumann Theorem and it has many important successors [3], [8], [19], [20], [12].

There is less information known about distances between unitary orbits. Some of this information has been obtained in an effort to give quantitative estimates in the study of compact perturbations following [7]. For example, BERG [3] gave concrete estimates for the distance between unitary orbits of direct sums of normals and weighted shifts. Later, HERRERO [15] gave very good estimates of the distance between unitary orbits of power partial isometries by improving on Berg's technique. Finally, the problem for pairs of self-adjoint operators has recently been solved [1]. They define a spectral distance in terms of the crude multiplicity function, and show that this is exactly the distance between the unitary orbits. Furthermore, this distance is achieved by commuting diagonal operators in the closure of the orbits.

This is the starting point for the work of this paper. We show that the same spectral distance is the right one for normal operators. This distance is the infimum of $\|A-B\|$ for commuting pairs in the closed orbits, and this distance is attained by a pair of commuting diagonal operators. When $A$ has no isolated eigenvalues of finite multiplicity, this is exactly the distance between orbits. In general, we obtain that

$$\operatorname{dist}(\mathcal{U}(A), \mathcal{U}(B)) \geqq c^{-1}\delta(A, B)$$

where $c$ is the same constant as in [5]. The general problem of determining if $c=1$ reduces to the separable case with finite spectra. We do not know if a positive answer in the finite dimensional case would imply the same in the separable case. However, we believe that any proof would almost surely generalize.

## 1. Preliminaries

Let $\mathfrak{H}$ be a Hilbert space, and let $\mathcal{B}(\mathfrak{H})$ denote the algebra of bounded linear operators on $\mathfrak{H}$. Given an operator $A$, $\mathcal{U}(A)$ denotes the set $\{UAU^*\colon U \text{ unitary in } \mathcal{B}(\mathfrak{H})\}$ and $\overline{\mathcal{U}(A)}$ denotes its closure. For $\mathfrak{M}$ a closed subspace of $\mathfrak{H}$, $\dim \mathfrak{M}$ is the cardinality of an orthonormal basis for $\mathfrak{M}$. Let $h$ denote the dimension of $\mathfrak{H}$. For each infinite cardinal $\alpha \leqq h$, let $\mathcal{I}_\alpha$ denote the closed two sided ideal generated by $\{T\in\mathcal{B}(\mathfrak{H})\colon \dim \overline{\operatorname{Ran} T} < \alpha\}$. Let $\sigma_\alpha(A)$ denote $\sigma(A+\mathcal{I}_\alpha)$ as an element of the quotient $C^*$ algebra $\mathcal{B}(\mathfrak{H})/\mathcal{I}_\alpha$. In particular, $\mathcal{I}_{\aleph_0}$ is the set of compact operators and $\sigma_{\aleph_0}(A)=\sigma_e(A)$, the essential spectrum. Also, let $\sigma_0(A)$ denote the isolated points of finite multiplicity in $\sigma(A)$, known as the normal eigenvalues of $A$. For convenience of notation, we write $\sigma_1(A)$ for $\sigma(A)$.

Let $A$ be a normal operator, and let $E_A(\cdot)$ denote its spectral measure. For $r>0$, $\lambda$ in $\mathbf{C}$, the disc of radius $r$ about $\lambda$ is denoted $D_r(\lambda)$. In [10], a *crude multiplicity function* is defined for normal operators by

$$\alpha(\lambda)=\inf\{\operatorname{rank} E_A(D_r(\lambda)):\ r>0\}.$$

It is shown there that two normal operators on a separable space have the same closed unitary orbit if and only if they have the same crude multiplicity function. This is easily generalized to Hilbert spaces of arbitrary cardinality, and is a special case of HADWIN's Theorem 3.14 [12]. It is not difficult to see (cf. [8]) that if $\alpha$ is an infinite cardinal, then $\{\lambda:\ \alpha(\lambda)\geqq\alpha\}$ equals $\sigma_\alpha(A)$. For $\alpha(\lambda)=n$ to be finite, non-zero, $\lambda$ must be an isolated eigenvalue of multiplicity $n$. Thus the theorem of Gellar—Page and Hadwin can be formulated:

Proposition 1.1. *Two normal operators A and B have the same closed unitary orbits if and only if* $\sigma_\alpha(A)=\sigma_\alpha(B)$ *for each infinite cardinal, and* $\sigma_0(A)=\sigma_0(B)$ *including multiplicity.*

We are now ready to discuss the spectral distance formula of AZOFF and DAVIS [1]. In the finite case, one might also define a spectral distance

$$\varrho(A,B)=\sup_{F \text{ finite}}\inf\{r:\ \operatorname{rank} E_A(F)\leqq\operatorname{rank} E_B(F_r)\ \text{ and }\ \operatorname{rank} E_B(F)\leqq\operatorname{rank} E_A(F_r)\}$$

where $F_r=\{\lambda:\ \operatorname{dist}(\lambda,F)\leqq r\}$. A simple application of the Marrige Lemma [14] shows that indeed $\varrho=\delta$. So we define our spectral distance as follows:

Definition 1.2. $\delta(A,B)$ is the infimum of real numbers $r>0$ such that

$$\operatorname{rank} E_A(F)\leqq\operatorname{rank} E_B(F_r)\quad\text{and}\quad\operatorname{rank} E_B(F)\leqq\operatorname{rank} E_A(F_r)$$

for every compact subset $F$ of $\mathbf{C}$.

We wish to relate this formula to the various spectra. Let $d_H(X,Y)$ denote the Hausdorff distance

$$d_H(X,Y)=\max\{\sup_{x\in X}\operatorname{dist}(x,Y),\ \sup_{y\in Y}\operatorname{dist}(y,X)\}.$$

Let $\operatorname{Fin}(X)$ denote the collection of finite subsets of $X$. Let $\delta_f(A,B)$ denote the maximum of

$$\sup_{F\in\operatorname{Fin}\sigma_0(A)}\inf\{r:\ \operatorname{rank} E_A(F)\leqq\operatorname{rank} E_B(F_r)\}$$

and the corresponding term with $A$ and $B$ interchanged. (Should $\sigma_0(A)$ be empty, this term is defined to be zero.) We obtain:

Proposition 1.3. *Let A and B be normal operators. Then*

$$\delta(A,B)=\max\{d_f(A,B),\ \sup_{\aleph_0\le\alpha\le h} d_H(\sigma_\alpha(A),\sigma_\alpha(B))\}.$$

Proof. Let $\alpha$ be an infinite cardinal, and suppose $\lambda$ belongs to $\sigma_\alpha(A)$. Then rank $E_A(D_\varepsilon(\lambda))\ge\alpha$ for all $\varepsilon>0$. Thus if $r=\delta(A,B)$, we have rank $E_B(D_{r+\varepsilon}(\lambda))\ge\alpha$ for all $\varepsilon>0$. Hence $\sigma_\alpha(B)$ intersects $\overline{D_r(\lambda)}$, so $\operatorname{dist}(\lambda,\sigma_\alpha(B))\le r$. Letting $\alpha$ and $\lambda$ run over all possibilities, and then interchanging the role of $A$ and $B$, we obtain

$$\delta(A,B)\ge\sup_{\aleph_0\le\alpha\le h} d_H(\sigma_\alpha(A),\sigma_\alpha(B)).$$

By definition, $\delta(A,B)\ge\delta_f(A,B)$, so $\delta(A,B)$ is greater than the right hand side, say $s$.

Conversely, let $F$ be a compact subset of $\mathbf{C}$, and let $\alpha=\operatorname{rank}E_A(F)$. If $\alpha$ is infinite, then $\sigma_\alpha(A)$ intersects $F$. Thus $\sigma_\alpha(B)$ intersects $F_s$, and thus rank $E_B(F_{s+\varepsilon})\ge\alpha$ for all $\varepsilon>0$. If $\alpha$ is finite and $F$ is contained in $\sigma_0(A)$, the definition of $\delta_f(A,B)$ gives $\alpha\le\operatorname{rank}E_B(F_{s+\varepsilon})$ for $\varepsilon>0$. Finally, if $\alpha$ is finite but $F$ is not contained in $\sigma_0(A)$, then $F$ intersects $\sigma_e(A)$. So as in the infinite case, $\alpha\le\aleph_0\le\operatorname{rank}E_B(F_{s+\varepsilon})$ for all $\varepsilon>0$. It follows that $\delta(A,B)\le s$, and equality is obtained.

Remark 1.4. If $\sigma(A)=\sigma_e(A)$, then

$$\delta(A,B)=\sup_{1\le\alpha\le h} d_H(\sigma_\alpha(A),\sigma_\alpha(B)).$$

Here $\alpha$ runs over the infinite cardinals and $\alpha=1$, where $\sigma_1(B)=\sigma(B)$ by definition. To see this, note that if $F$ is a finite subset of $\sigma_0(B)$, then rank $E_A(F_r)$ equals zero or is infinite. So $\inf\{r:\ \operatorname{rank}E_A(F_r)\ge\operatorname{rank}E_B(F)\}=\inf\{r:\ F_r\cap\sigma(A)\ne\emptyset\}$. Hence $\delta_f(A,B)=\sup_{\lambda\in\sigma_0(B)}\operatorname{dist}(\lambda,\sigma(A))$.

## 2. Lower bounds

Proposition 2.1. *If A and B are normal operators on $\mathfrak{H}$, then*

$$\operatorname{dist}(\mathcal{U}(A),\mathcal{U}(B))\ge\sup_{1\le\alpha\le h} d_H(\sigma_\alpha(A),\sigma_\alpha(B)).$$

Proof. Let $\alpha=1$ or some infinite cardinal, and let $\lambda$ belong to $\sigma_\alpha(A)$. Then rank $E_A(D_\varepsilon(\lambda))\ge\alpha$ for all $\varepsilon>0$. If $B'$ belongs to $\mathcal{U}(B)$ and $\|A-B'\|=s$, we will show that rank $E_B(D_{s+\varepsilon}(\lambda))\ge\alpha$ for all $\varepsilon>0$. Otherwise for some $\varepsilon>0$, there is a unit vector $x$ in $\operatorname{Ran}E_A(D_{\varepsilon/2}(\lambda))\cap\operatorname{Ran}E_{B'}(D_{s+\varepsilon}(\lambda))^\perp$. This gives

$$\|A-B'\|\ge\|(A-B')x\|\ge\|(B'-\lambda)x\|-\|(A-\lambda)x\|\ge s+\varepsilon-\varepsilon/2=s+\varepsilon/2.$$

Hence $\sigma_\alpha(B)$ intersects $\overline{D_s(\lambda)}$, and thus $\operatorname{dist}(\lambda, \sigma_\alpha(B)) \leqq s$. By symmetry, $d_H(\sigma_\alpha(A), \sigma_\alpha(B)) \leqq \operatorname{dist}(\mathcal{U}(A), \mathcal{U}(B))$ for all $\alpha$.

Corollary 2.2. *If $A$ and $B$ are normal, and $\sigma(A)=\sigma_e(A)$, then*

$$\operatorname{dist}(\mathcal{U}(A), \mathcal{U}(B)) \geqq \delta(A, B).$$

Proposition 2.3. *If $A$ and $B$ are commuting normal operators, then*

$$\|A-B\| \geqq \delta(A, B).$$

Proof. By Propositions 1.3 and 2.1, we need only show that $\|A-B\| \geqq \delta_f(A, B)$. Suppose $F$ is a finite subset of $\sigma_0(A)$, $s>0$, and $\operatorname{rank} E_A(F) > \operatorname{rank} E_B(F_s)$. It suffices to show that $\|A-B\| \geqq s$. Now $E_A(F)$ belongs to $W^*(A) = \{A\}''$, and thus commutes with $B$. So the restrictions $A_0$ and $B_0$ of $A$ and $B$ to $E_A(F)\mathfrak{H}$ are commuting normal operators. Hence $A_0$ and $B_0$ are simultaneously diagonalizable. The spectrum of $A_0$ lies in $F$, but $B_0$ has at least one eigenvalue outside $F_s$, which is paired with some eigenvalue of $A_0$. Thus

$$\|A-B\| \geqq \|A_0-B_0\| \geqq s.$$

The main result of this section is an easy corollary of the result of [5].

Theorem 2.4. *There is a universal constant $c>0$ so that for every pair of normal operators $A$ and $B$ acting on the same space,*

$$\operatorname{dist}(\mathcal{U}(A), \mathcal{U}(B)) \geqq c^{-1}\delta(A, B).$$

Proof. By Propositions 1.3 and 2.1, it suffices to show that $\|A-B\| \geqq c^{-1}\delta_f(A, B)$. So let $F$ be a finite subset of $\sigma_0(A)$, $s>0$, and $\operatorname{rank} E_A(F) > \operatorname{rank} E_B(F_s)$. Let $\mathfrak{K}=E_A(F)$, $\mathfrak{L}=E_B(F_s)^\perp\mathfrak{H}$, $\tilde{A}=A|\mathfrak{K}$ and $\tilde{B}=B|\mathfrak{L}$. Let $\mathbf{C}=E_A(F)E_B(F_s)^\perp$ be thought of as an operator from $\mathfrak{L}$ to $\mathfrak{K}$. Then

$$\|\tilde{A}Q-Q\tilde{B}\| = \|E_A(F)(A-B)E_B(F_s)^\perp\| \leqq \|A-B\|.$$

By Theorem 4.2 of [5], we get

$$\|A-B\| \geqq sc^{-1}\|Q\|.$$

However, since $\operatorname{codim} \mathfrak{L} = \operatorname{rank} E_B(F_s) < \dim \mathfrak{K}$, $\mathfrak{L}$ and $\mathfrak{K}$ intersect, and thus $\|Q\|=1$. This completes the proof.

## 3. Best commuting approximants

Now we construct closest possible diagonal operators in the unitary orbits. There is a technical matching problem that has to be solved, and this will be left to the next section.

In the following proof, we make use of this fact: If $X$ is a compact subset of the plane, there is a Borel function $f$ mapping $\mathbf{C}$ to $X$ so that $|z-f(z)|=d(z)\equiv \operatorname{dist}(z, X)$. To obtain such an $f$, let $F(z)=\{x\in X\colon |z-x|=d(z)\}$. Let

$$\theta(z) = \min\{\operatorname{Arg}(x-z)\colon x\in F(z)\}$$

where $\operatorname{Arg}(w)$ belongs to $[0, 2\pi)$ such that $w=|w|\exp(i\operatorname{Arg} w)$. Let $f(z)=z+d(z)\exp(i\theta(z))$. It is readily verified that $f$ is Borel as required.

Theorem 3.1. *Suppose $A$ and $B$ are normal operators on $\mathfrak{H}$. There are commuting diagonal operators $A'$ and $B'$ in $\overline{\mathcal{U}(A)}$ and $\overline{\mathcal{U}(B)}$ respectively such that $\|A'-B'\|=\delta(A, B)$.*

Proof. Let $r=\delta(A, B)$. For each infinite cardinal $\alpha\leqq h$, let $A_\alpha$ be a diagonal normal operator on a Hilbert space $\mathfrak{H}_\alpha$ of cardinality $\alpha$ with $\sigma(A_\alpha)=\sigma_\alpha(A_\alpha)=\sigma_\alpha(A)$. Similarly, let $B_\alpha$ be a diagonal normal operator on $\mathfrak{H}_\alpha$ with $\sigma(B_\alpha)=\sigma_\alpha(B_\alpha)=\sigma_\alpha(B)$. Let $f_\alpha$ be a Borel map of $\sigma_\alpha(A)$ into $\sigma_\alpha(B)$ such that $|f_\alpha(a)-a|\leqq r$ for all $a$ in $\sigma_\alpha(A)$. Similarly, let $g_\alpha$ be a Borel map of $\sigma_\alpha(B)$ into $\sigma_\alpha(A)$ such that $|g_\alpha(b)-b|\leqq r$. These maps exist since $d_H(\sigma_\alpha(A), \sigma_\alpha(B))\leqq r$. Consider the diagonal normal operators $\hat{A}_\alpha=A_\alpha\oplus g_\alpha(B_\alpha)$ and $\hat{B}_\alpha=f_\alpha(A_\alpha)\oplus B_\alpha$ acting on $\mathfrak{H}_\alpha\oplus\mathfrak{H}_\alpha$. This construction guarantees that $\dim(\mathfrak{H}_\alpha\oplus\mathfrak{H}_\alpha)=\alpha$, $\sigma(\hat{A}_\alpha)=\sigma_\alpha(\hat{A}_\alpha)=\sigma_\alpha(A)$, $\sigma(\hat{B}_\alpha)=\sigma_\alpha(\hat{B}_\alpha)=\sigma_\alpha(B)$, and $\|\hat{A}_\alpha-\hat{B}_\alpha\|\leqq r$.

Consider $\mathfrak{H}=\sum_{\aleph_0\leqq\alpha\leqq h}(\mathfrak{H}_\alpha\oplus\mathfrak{H}_\alpha)$, and the diagonal normal operators $\hat{A}=\oplus\sum_{\aleph_0\leqq\alpha\leqq h}\hat{A}_\alpha$ and $\hat{B}=\oplus\sum_{\aleph_0\leqq\alpha\leqq h}\hat{B}_\alpha$. These are commuting diagonal operators such that $\|\hat{A}-\hat{B}\|\leqq r$, $\sigma_\alpha(\hat{A})=\sigma_\alpha(A)$ and $\sigma_\alpha(\hat{B})=\sigma_\alpha(B)$ for all infinite cardinals, and $\sigma_0(\hat{A})=\emptyset=\sigma_0(\hat{B})$.

To obtain the desired operators, we need to add on summands to give the isolated eigenvalues of finite multiplicity. We will produce commuting diagonal normal operators $A_f$ and $B_f$ on a separable space such that $\|A_f-B_f\|\leqq r$, $\sigma_e(A_f)\subseteq\sigma_e(A)$ (and $\sigma_e(B_f)\subseteq\sigma_e(B)$) and $\sigma_0(A_f)\setminus\sigma_e(A)$ and $\sigma_0(A)$ (correspondingly $\sigma_0(B_f)\setminus\sigma_e(B)$ and $\sigma_0(B)$) coincide including multiplicity. Once this is accomplished, let $A'=\hat{A}\oplus A_f$ and $B'=\hat{B}\oplus B_f$. These are commuting diagonal normals such that $\|A'-B'\|\leqq r$. Furthermore, $\sigma_\alpha(A')=\sigma_\alpha(A)$ for infinite cardinals and $\sigma_0(A')=\sigma_0(A)$ including multiplicity, so by Proposition 1.1, $A'$ belongs to $\overline{\mathcal{U}(A)}$. Similarly, $B'$ belongs to $\overline{\mathcal{U}(B)}$.

To produce $A_f$ and $B_f$, we must find a matching of the points in $\sigma_0(A)$ to points in $\sigma(B)$ within distance $r$. Points close to $\sigma_e(B)$ can be "absorbed", but points further from $\sigma_e(B)$ must match up including multiplicity with points in $\sigma_0(B)$; and vice versa. To do this, we need a curious Marriage Lemma type theorem. This theorem will be stated and proved in the next section. First we obtain the appropriate setting for our problem, and finish the proof modulo this combinatorical theorem.

Let $\mathscr{A}_0$ denote the points of $\sigma_0(A)$ repeated according to multiplicity which are distance greater than $r$ from $\sigma_e(B)$. Similarly let $\mathscr{B}_0$ be the corresponding subset of $\sigma_0(B)$. Define a relation $R$ on $\sigma_0(A)\times\sigma_0(B)$ by $aRb$ if and only if $|a-b|\leqq r$. For each $\mathscr{A}$ in $\mathrm{Fin}(\mathscr{A}_0)$, the set $F(\mathscr{A})=\{b\in\sigma_0(B)\colon aRb \text{ for some } a \text{ in } \mathscr{A}\}$ is finite. And since $\delta(A,B)\geqq\delta_f(A,B)$, we have $|\mathscr{A}|\leqq|F(\mathscr{A})|<\infty$. Similarly, set $G(\mathscr{B})=$ $=\{a\in\sigma_0(A)\colon aRb \text{ for some } b \text{ in } \mathscr{B}\}$ for each $\mathscr{B}$ in $\mathrm{Fin}(\mathscr{B}_0)$. Again, $|\mathscr{B}|\leqq|G(\mathscr{B})|<\infty$.

This relation satisfies the hypotheses of the Combinatorial Lemma 4.1. Thus there are sets $\mathscr{A}_1$ and $\mathscr{B}_1$ such that $\mathscr{A}_0\subseteq\mathscr{A}_1\subseteq\sigma_0(A)$ and $\mathscr{B}_0\subseteq\mathscr{B}_1\subseteq\sigma_0(B)$, and a *bijection* $f\colon \mathscr{A}_1\to\mathscr{B}_1$ such that $|a-f(a)|\leqq r$ for every $a$ in $\mathscr{A}_1$. Let $\mathscr{A}_2=\sigma_0(A)\setminus\mathscr{A}_1$ and choose a function $f_2\colon \mathscr{A}_2\to\sigma_e(B)$ such that $|f_2(a)-a|\leqq r$ for all $a$ in $\mathscr{A}_2$. Similarly, define $\mathscr{B}_2$ and $g_2\colon \mathscr{B}_2\to\sigma_e(A)$.

Define diagonal normal operators as follows:

$$
\begin{aligned}
A_1 &= \mathrm{diag}\{a_n\colon a_n\in\mathscr{A}_1\}, & B_1 &= \mathrm{diag}\{f(a_n)\colon a_n\in\mathscr{A}_1\},\\
A_2 &= \mathrm{diag}\{a_n\colon a_n\in\mathscr{A}_2\}, & B_\infty &= \mathrm{diag}\{f_2(a_n)\colon a_n\in\mathscr{A}_2\},\\
A_\infty &= \mathrm{diag}\{g_2(b_n)\colon b_n\in\mathscr{B}_2\}, & B_2 &= \mathrm{diag}\{b_n\colon b_n\in\mathscr{B}_2\},\\
A_f &= A_1\oplus A_2\oplus A_\infty, & B_f &= B_1\oplus B_\infty\oplus B_2.
\end{aligned}
$$

The properties of $f$, $f_2$ and $g_2$ show that $\|A_f-B_f\|\leqq r$, $\sigma_0(A_1\oplus A_2)$ agrees with $\sigma_0(A)$ including multiplicity, and $\sigma_e(A_f)\cup\sigma_0(A_\infty)\subseteq\sigma_e(A)$. The corresponding statements for $B_f$ hold also. So $A_f$ and $B_f$ are commuting diagonal operators with the required properties. This completes the proof.

Corollary 3.2. *If $A$ and $B$ are normal operators, then*

$$\mathrm{dist}\,(\mathscr{U}(A),\mathscr{U}(B))\leqq\delta(A,B).$$

Corollary 3.3. *If $A$ and $B$ are normal operators, and $\sigma(A)=\sigma_e(A)$, then*

$$\mathrm{dist}\,(\mathscr{U}(A),\mathscr{U}(B))=\delta(A,B).$$

Remark 3.4. The equality $\mathrm{dist}\,(\mathscr{U}(A),\mathscr{U}(B))=\delta(A,B)$ is readily verified for several classes of normals: the self adjoint case [1] can also be proven using the technique of [6], as can the case of scalar multiples of unitaries. The technique of [5] works for a self-adjoint $A$ and skew-adjoint $B$. Just remember that one needs only worry about $\delta_f(A,B)$.

## 4. The combinatorial lemma

The result that we need is an infinite analogue of the Marriage Lemma [14]. There are a number of infinite versions of this theorem, notably [13] and [17], and our proof is very similar. However, the set up we require seems sufficiently peculiar that no known theorem applies directly.

Let $\mathscr{A}$ and $\mathscr{B}$ be sets, and let $R$ be a relation on $\mathscr{A}\times\mathscr{B}$. For $A$ a subset of $\mathscr{A}$, define $F(A)=\{b\in\mathscr{B}: aRb$ for some $a$ in $A\}$. Similarly, for $B$ a subset of $\mathscr{B}$, define $G(B)=\{a\in\mathscr{A}: aRb$ for some $b$ in $B\}$.

Lemma 4.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be sets with distinguished subsets $\mathscr{A}_0$ and $\mathscr{B}_0$; and let $R$ be a relation on $\mathscr{A}\times\mathscr{B}$ with $F$ and $G$ defined as above. Suppose that*

1) $|A|\leqq|F(A)|<\infty$ *for every $A$ in* $\mathrm{Fin}\,(\mathscr{A}_0)$,

1′) $|B|\leqq|G(B)|<\infty$ *for every $B$ in* $\mathrm{Fin}\,(\mathscr{B}_0)$.

*Then there are sets $\mathscr{A}_1$ and $\mathscr{B}_1$ such that $\mathscr{A}_0\subseteqq\mathscr{A}_1\subseteqq\mathscr{A}$ and $\mathscr{B}_0\subseteqq\mathscr{B}_1\subseteqq\mathscr{B}$, and a bijection $f\colon\mathscr{A}_1\to\mathscr{B}_1$ such that $aRf(a)$ for every $a$ in $\mathscr{A}_1$.*

Proof. Call a non-empty set $A$ in $\mathrm{Fin}\,(\mathscr{A}_0)$ *strict* if $|F(A)|=|A|$. The restriction of $R$ to $A\times F(A)$ satisfies 1). So the Marriage Lemma [14] gives a bijection $f\colon A\to F(A)$ such that $aRf(a)$ for every $a$ in $A$.

Note also that when $A$ is strict, $\mathscr{A}\setminus A$ and $\mathscr{B}\setminus F(A)$ with distinguished sets $\mathscr{A}_0\setminus A$ and $\mathscr{B}_0\setminus F(A)$ still satisfy 1) and 1′). For if $A'$ is a finite subset of $\mathscr{A}_0\setminus A$, then

$$|F(A')\setminus F(A)| = |F(A'\cup A)\setminus F(A)| \geqq |A'\cup A|-|F(A)| = |A'|.$$

Also if $B'$ is a finite subset of $\mathscr{B}_0\setminus F(A)$, then $G(B)$ is disjoint from $A$, so $|B'|\leqq$ $\leqq|G(B')|=|G(B')\setminus A|$.

Similarly, if $B$ is a strict subset of $\mathscr{B}_0$, there is likewise a bijection $f\colon G(B)\to B$ such that $f(a)Ra$ for all $a$ in $G(B)$. And $\mathscr{A}\setminus G(B)$ and $\mathscr{B}\setminus B$ satisfy 1) and 1′).

Use the Axiom of Choice to well order $\mathscr{A}_0\cup\mathscr{B}_0$. Starting out of $(A, B)$ we define $(A_\alpha, B_\alpha)$ by transfinite induction. At stage $\alpha$, we have a collection $\{(\mathscr{A}_\beta, \mathscr{B}_\beta)\colon \beta<\alpha\}$ of pairs satisfying 1), 1′) and $\mathscr{A}_\beta\supset\mathscr{A}_{\beta'}$ and $\mathscr{B}_\beta\supset\mathscr{B}_{\beta'}$ if $\beta<\beta'<\alpha$. When $\alpha$ is a limit ordinal, set $\mathscr{A}_\alpha=\bigcap_{\beta<\alpha}\mathscr{A}_\beta$ and $\mathscr{B}_\alpha=\bigcap_{\beta<\alpha}\mathscr{B}_\beta$. Since 1) and 1′) deal with finite sets, it is easy to verify that the hold for the intersection.

If $\alpha=\beta+1$ is a successor ordinal, then

(a) If there are strict subsets of $\mathscr{A}_0\cap\mathscr{A}_\beta$, choose a strict $A$ and obtain a pair $(A, F(A))$.

(b) If there are no strict subsets of $\mathscr{A}_0\cap\mathscr{A}_\beta$, but there are strict subsets of $\mathscr{B}_0\cap\mathscr{B}_\beta$, choose a strict $B$ and form a pair $(G(B), B)$.

(c) If there are no strict subsets, let $a$ (or $b$) be the least element of the well ordering of $\mathscr{A}_0\cup\mathscr{B}_0$ which belong to $\mathscr{A}_\beta\cup\mathscr{B}_\beta$. Take any $b$ (or $a$) such that $aRb$, and form the pair $(\{a\}, \{b\})$.

In each case we obtain a pair $(A, B)$ with $|A|=|B|<\infty$, and, following earlier remarks, a bijection $f$: $A\to B$ such that $aRf(a)$ for $a$ in $A$. Furthermore, we set $\mathscr{A}_\alpha=\mathscr{A}_\beta\setminus A$ and $\mathscr{B}_\alpha=\mathscr{B}_\beta\setminus B$. By the remarks at the beginning of the proof, $\mathscr{A}_\alpha$ and $\mathscr{B}_\alpha$ satisfy 1) and 1′) in cases (a) and (b). They also hold in case (c), for if $A$ is a finite subset of $\mathscr{A}_\alpha\cap\mathscr{A}_0$, then since $A$ is not strict,

$$|F(A)\setminus\{b\}| \geqq |F(A)|-1 \geqq |A|.$$

The same holds for finite subsets of $\mathscr{B}_\alpha\cap\mathscr{B}_0$.

This procedure terminates at some ordinal $\alpha$ with $|\alpha|\leqq|\mathscr{A}_0\cup\mathscr{B}_0|$. The result is a disjoint collection of finite pairs $(A_i, B_i)$ which exhaust $\mathscr{A}_0$ and $\mathscr{B}_0$. On these sets, functions $f=f_i$ have been constructed so that $aRf(a)$ for all $a$ in $A_i$. Set $\mathscr{A}_1=\bigcup A_i$ and $\mathscr{B}_1=\bigcup B_i$. The union $f$ of the $f_i$'s is the required bijection.

## 5. Further remarks

The results of section 2 show that the problem of determining if the constant $c$ equals 1 is basically a finite dimensional problem, for the difficulty lies solely in the $\delta_f(A, B)$ term. A quantitative way of phrasing the key issue is

Question. If $A$ and $B$ are normal operators (on a separable space), $F$ is a finite subset of $\sigma_0(A)$, $s>0$, and $\operatorname{rank} E_B(F_s)<\operatorname{rank} E_A(F)$, then is $\|A-B\|\geqq s$?

The reason one can reduce to the separable case is the following. Suppose $A$ and $B$ are normal with $\|A-B\|<\delta(A, B)$. Let $\mathfrak{M}$ be a separable reducing subspace of $A$ containing the spectral subspace $E_A\big(\sigma(A)\setminus\sigma_{\aleph_1}(A)\big)$ such that $\sigma_e(A|\mathfrak{M})=\sigma_e(A)$, and let $\mathfrak{N}$ be a corresponding subspace for $B$. Let $\mathfrak{K}$ be the smallest reducing subspace for $C^*(A, B)$ containing both $\mathfrak{M}$ and $\mathfrak{N}$. This subspace is separable. Let $A'=A|\mathfrak{K}$, $A''=A|\mathfrak{K}^\perp$, $B'=B|\mathfrak{K}$ and $B''=B|\mathfrak{K}^\perp$. Then $\sigma_0(A')=\sigma_0(A)$ and $\sigma_e(A')=\sigma_e(A)$; and $\sigma(A'')=\sigma_{\aleph_1}(A'')=\sigma_{\aleph_1}(A)$. Similar relations hold for $B$. Furthermore,

$$\delta(A'', B'') = \sup_{\aleph_1\leqq\alpha\leqq h} d_H\big(\sigma_\alpha(A''), \sigma_\alpha(B'')\big) = \sup_{\aleph_1\leqq\alpha\leqq h} d_H\big(\sigma_\alpha(A), \sigma_\alpha(B)\big) \leqq$$

$$\leqq \|A''-B''\| \leqq \|A-B\|$$

and

$$\delta(A', B') = \max\big\{\delta_f(A, B),\ d_H\big(\sigma_e(A), \sigma_e(B)\big)\big\}.$$

Thus $\delta(A, B)=\max\{\delta(A', B'), \delta(A'', B'')\}$ and $\|A-B\|=\max\{\|A'-B'\|, \|A''-B''\|\}$. So it must be the case that

$$\delta(A', B') = \delta(A, B) > \|A-B\| \geqq \|A'-B'\|.$$

This reasonsing also leads to the conclusion that any constant $c$ valid in Theorem 2.4 for separable spaces is valid in general.

Next, it is easy to approximate $A$ and $B$ arbitrarily well by normals of finite spectrum. So one may assume that $\sigma(A)$ and $\sigma(B)$ are finite. This looks almost finite dimensional now.

Question. Can one show that any constant $c$ valid in Theorem 2.4 for all finite rank normals works in general?

I am confident that any proof valid in the finite case will extend to the separable one, but knowing this in advance would be nice.

Finally, a special case subsuming much of what is known and is perhaps easier than the general case is the situation $A=A^*$ and arbitrary normal $B$. Does this case have $c=1$?

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