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The arity of minimal clones

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Clones play a central role in universal algebra and in multiple-valued logic. A set of finitary operations is a *clone of operations* if it is closed under composition and contains all projections. The clones on a fixed set form a complete lattice with respect to inclusion. If the set is finite, then the lattice of clones is atomic and coatomic. The coatoms, i.e. the maximal clones, were classified in IVO ROSENBERG's profound paper [4]. On the contrary, quite little is known about the minimal clones. Recently BÉLA CSÁKÁNY [1], [2] determined all minimal clones on the three-element set.

By definition, the *arity of a minimal clone* of operations is the minimum of arities of the nontrivial operations in the clone. Identifying any two variables in such an operation turns this operation into a projection. As observed by ŚWIERCZKOWSKI [5], if such an operation has at least four variables then it is a *semiprojection*, i.e. there is an index i $(1 \leqq i \leqq k)$ such that $f(a_1, \ldots, a_k) = a_i$ whenever $|\{a_1, \ldots, a_k\}| < k$. Since the arity of any nontrivial semiprojection does not exceed the cardinality of the underlying set, it follows that the arity k of a minimal clone of operations on an n-element set must satisfy

$$k \leqq \begin{cases} 3, & \text{if} \quad n = 2, \\ n, & \text{if} \quad n \geqq 3 \end{cases}$$

(see [3, 4.4.7]). It is easy to find unary, binary and ternary minimal clones, for example those generated by a constant function, a semilattice operation, or a median operation of a lattice, respectively (see [3, pp. 114—115]). If $n > 2$, then any minimal clone contained in the clone generated by an arbitrary nontrivial n-ary semiprojection has arity n. But for $3 < k < n$ the existence of a k-ary minimal clone of operations on an n-element set was not known.

Theorem. *There exists a k-ary minimal clone of operations on an n-element set* $(n \geqq 3)$ *if, and only if,* $1 \leqq k \leqq n$.

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Proof. By the preceding remarks it is enough to point out a k-ary minimal clone for $3 \leqq k < n$. Fix different elements $b_1, \ldots, b_k, b_{k+1} \in A$, where $|A| = n$. Define a k-ary nontrivial operation f by

$$f(a_1, \ldots, a_k) = \begin{cases} b_{k+1}, & \text{if } a_1 = b_1 \text{ and } \{a_2, \ldots, a_k\} = \{b_2, \ldots, b_k\}, \\ a_1, & \text{otherwise.} \end{cases}$$

Since f is a semiprojection, the clone $[f]$ generated by f is k-ary. We are going to prove that $[f]$ is a minimal clone. For any term t representing a function in $[f]$ (in short, $t \in [f]$) we denote the first (from the left) variable of t by $\sigma(t)$, for example $\sigma(f(f(x_2, x_1, x_3), x_1, x_1)) = x_2$.

First we show that f satisfies the following identities:

(1) $f(f, t_2, \ldots, t_k) = f$ for any k-ary $t_2, \ldots, t_k \in [f]$,

(2) $f(x_1, t_2, \ldots, t_k) = f$ if the k-ary terms $t_2, \ldots, t_k \in [f]$ are such that $\sigma(t_i) = x_i$ $(i = 2, \ldots, k)$,

(3) $f(x_1, t_2, \ldots, t_k) = x_1$ if $t_2, \ldots, t_k \in [f]$ and for some $i, 2 \leqq i \leqq k$, $\sigma(t_i) = x_1$,

(4) $f(x_1, t_2, \ldots, t_k) = x_1$ if $t_2, \ldots, t_k \in [f]$ and for some i and j, $2 \leqq i < j \leqq k$, $\sigma(t_i) = \sigma(t_j)$.

Indeed, substitute $a_1, a_2, \ldots \in A$ for the variables $x_1, x_2, \ldots$. Observe that, by the definition of f, if $t \in [f]$ and $\sigma(t) = x_i$, then at the given valuation t takes on a_i or b_{k+1} and the latter can occur only if $a_i = b_1$. Hence if $a_1 \neq b_1$ then (1)—(4) obviously hold. Suppose $a_1 = b_1$. In (3) t_i takes on b_1 or b_{k+1}, in both cases the left hand side is b_1. Similarly, in (4) either t_i and t_j have equal values or one of them takes on b_{k+1}, again forcing the left hand side to be b_1. If $\{a_2, \ldots, a_k\} = \{b_2, \ldots, b_k\}$ then f takes on b_{k+1} and we have equality in (1), moreover, in (2) the set of values of $t_2, \ldots, t_k$ is also $\{b_2, \ldots, b_k\}$ hence we have equality here as well. If $\{a_2, \ldots, a_k\} \neq \{b_2, \ldots, b_k\}$ then some b_i $(2 \leqq i \leqq k)$ cannot occur as the value of any k-ary term hence in (1) and (2) both sides take on b_1.

Now the minimality of $[f]$ will be derived from the identities (1)—(4). We prove by induction on the length that any term is either equal to a projection or it turns into f by suitable identification and permutation of variables. Take a term $t = f(t_1, \ldots, t_k)$. By the inductive hypothesis, if t_1 is not a projection then suitable identification and permutation of variables yields a term $t' = f(f, t_2', \ldots, t_k')$, where $t_2', \ldots, t_k'$ are k-ary. By (1) we have $t' = f$. Now let t_1 be a projection. Without loss of generality we may assume that $t_1 = x_1$. If there is a t_i $(2 \leqq i \leqq k)$ with $\sigma(t_i) = x_1$ or there are t_i, t_j $(2 \leqq i < j \leqq k)$ such that $\sigma(t_i) = \sigma(t_j)$ then $t = x_1$ by (3) or (4), respectively. Otherwise, by permuting the variables we may assume that $\sigma(t_i) = x_i$ for $i = 2, \ldots, k$. Then identifying $x_{k+1}, \ldots$ with x_1 we obtain f by (2). Hence for any nontrivial $t \in [f]$ we have $f \in [t]$, therefore $[f]$ is a minimal clone. Thus the proof is complete.

Note that by our proof any nontrivial k-ary operation satisfying the identities (1)—(4) generates a minimal clone. Indeed, the minimality of a clone is an inner property, hence it can be advantageous to consider clones abstractly. Following W. TAYLOR [6, pp. 360—361], an *abstract clone* T is a heterogeneous algebra on a series of base sets $T_1, T_2, \ldots$ equipped with composition operations $C^r_m\colon T_r \times T^r_m \to T_m$ $(m, r=1, 2, \ldots)$ and constants (that correspond to the projections) $e^n_i \in T_n$ $(i=1, \ldots, n;\ n=1, 2, \ldots)$ satisfying the identities

$$C^r_m(t, C^n_m(u_1, v_1, \ldots, v_n), \ldots, C^n_m(u_r, v_1, \ldots, v_n)) =$$
$$= C^n_m(C^r_n(t, u_1, \ldots, u_r), v_1, \ldots, v_n) \quad (m, n, r = 1, 2, \ldots),$$
$$C^n_m(e^n_i, t_1, \ldots, t_n) = t_i \quad (m, n = 1, 2, \ldots;\ i = 1, \ldots, n),$$
$$C^n_n(t, e^n_1, \ldots, e^n_n) = t \quad (n = 1, 2, \ldots).$$

Subclones, homomorphisms, etc. are defined in the natural manner. An abstract clone is minimal if it is generated by any of its nontrivial members. Any homomorphism of a minimal abstract clone onto a nontrivial clone of operations on a set yields a minimal clone of operations. We will pursue this line of research in a forthcoming paper.

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