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Orthonormal systems of polynomials in the divergence theorems for double orthogonal series

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1. Introduction. Let $(X, \mathscr{F}, \mu)$ be a positive measure space and $\{\varphi_{ik}(x): i, k= =0, 1, \ldots\}$ an orthonormal system (in abbreviation: ONS) defined on X. We will consider the double orthogonal series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}\varphi_{ik}(x) \tag{1.1}$$

where $\{a_{ik}\}$ is a double sequence of real numbers for which

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

The rectangular partial sums and (C, α, β)-means of series (1.1) are defined by

$$s_{mn}(x) = \sum_{i=0}^{m}\sum_{k=0}^{n} a_{ik}\varphi_{ik}(x)$$

and

$$\sigma_{mn}^{\alpha\beta}(x) = (1/A_m^\alpha A_n^\beta) \sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta-1} s_{ik}(x),$$

respectively, where

$$A_m^\alpha = \binom{m+\alpha}{m} \quad (\alpha > -1, \beta > -1;\ m, n = 0, 1, \ldots).$$

2. Preliminary results: Convergence theorems. The extension of the Rademacher—Menšov theorem proved by a number of authors (see, e.g. [1], [5, Corollary 2], etc.) reads as follows.

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Theorem A. *If*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log(i+2)]^2[\log(k+2)]^2 < \infty, \tag{2.1}$$

then series (1.1) *regularly converges a.e.*

In this paper the logarithms are to the base 2.

The convergence behavior improves when considering $\sigma_{mn}^{\alpha\beta}(x)$ with $\alpha \geqq 0$ and $\beta \geqq 0$ instead of $s_{mn}(x)$. The following two extensions of the Menšov—Kaczmarz ($\alpha=1$) and Zygmund ($\alpha>0$) theorems were proved in [7].

Theorem *B. If* $\alpha>0$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log\log(i+4)]^2[\log(k+2)]^2 < \infty, \tag{2.2}$$

then series (1.1) *is regularly* $(C, \alpha, 0)$*-summable a.e.*

Theorem C. *If* $\alpha>0$, $\beta>0$, *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log\log(i+4)]^2[\log\log(k+4)]^2 < \infty, \tag{2.3}$$

then series (1.1) *is regularly* (C, α, β)*-summable a.e.*

The next three theorems give information on the order of magnitude of $s_{mn}(x)$ and $\sigma_{mn}^{\alpha\beta}(x)$, respectively, in the more general setting of (1.2).

Theorem D [6, Corollary 2]. *If condition* (1.2) *is satisfied, then*

$$s_{mn}(x) = o_x\{\log(m+2)\log(n+2)\} \quad \text{a.e. as} \quad \max(m, n) \to \infty. \tag{2.4}$$

Theorem E [9, Theorem 1]. *If* $\alpha>0$ *and condition* (1.2) *is satisfied, then*

$$\sigma_{mn}^{\alpha 0}(x) = o_x\{\log\log(m+4)\log(n+2)\} \quad \text{a.e. as} \quad \max(m, n) \to \infty. \tag{2.5}$$

Theorem F [9, Theorem 2]. *If* $\alpha>0$, $\beta>0$, *and condition* (1.2) *is satisfied, then*

$$\sigma_{mn}^{\alpha\beta}(x) = o_x\{\log\log(m+4)\log\log(n+4)\} \quad \text{a.e. as} \quad \max(m, n) \to \infty. \tag{2.6}$$

3. Preliminary results: Divergence theorems. The conditions (2.1)—(2.3) and statements (2.4)—(2.6) are the best possible.

To see this, from here on let $(X, \mathscr{F}, \mu)$ be the unit square $X=[0, 1]\times[0, 1]$ in $\mathbf{R}^2$, $\mathscr{F}$ the σ-algebra of the Borel measurable subsets, and μ the Lebesgue measure. In the sequel, the unit interval $[0, 1]$ will be denoted by I, the unit square $I\times I$ by S, and the plane Lebesgue measure by $|\cdot|$.

Theorem A′ [10, Theorem 4]. *If* $\{a_{ik}\}$ *is a double sequence of numbers for which*

$$|a_{ik}| \geqq \max\{|a_{i+1,k}|,\ |a_{i,k+1}|\} \quad (i, k = 0, 1, \ldots) \tag{3.1}$$

and

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log(i+2)]^2[\log(k+2)]^2 = \infty, \tag{3.2}$$

then there exists a uniformly bounded double ONS $\{\varphi_{ik}(x_1, x_2)\}$ *of step functions on* S *such that*

$$\limsup |s_{mn}(x_1, x_2)| = \infty \quad \textit{a.e. as} \quad \min(m, n) \to \infty. \tag{3.3}$$

A function φ is said to be a step function on S if S can be represented as a finite union of disjoint rectangles with sides parallel to the coordinate axes, and φ is constant on each of these rectangles.

Remark 1. As a matter of fact, (3.3) was proved in [10] under (3.1) and the stronger condition that for all pairs of i_0 and k_0

$$\sum_{i=i_0}^{\infty}\sum_{k=k_0}^{\infty} a_{ik}^2[\log(i+2)]^2[\log(k+2)]^2 = \infty. \tag{3.2′}$$

Assume (3.2′) is not satisfied for a certain pair of i_0 and k_0, but (3.2) is. Then either

$$\sum_{i=0}^{\infty}\sum_{k=0}^{k_0-1} a_{ik}^2[\log(i+2)]^2 = \infty$$

or

$$\sum_{i=0}^{i_0-1}\sum_{k=0}^{\infty} a_{ik}^2[\log(k+2)]^2 = \infty.$$

Now, it is a routine to construct an ONS $\{\varphi_{ik}(x_1, x_2)\}$ such that

$$\limsup s_{mn}(x_1, x_2) = \infty \quad \text{a.e. as} \quad m \to \infty \quad \text{and} \quad n = k_0 - 1, \quad \text{or} \quad m = i_0 - 1 \quad \text{and} \quad n \to \infty. \tag{3.3′}$$

On the other hand, since (3.2′) is not satisfied, by Theorem A the truncated series

$$\sum_{i=i_0}^{\infty}\sum_{k=k_0}^{\infty} a_{ik}\varphi_{ik}(x_1, x_2) \tag{1.1′}$$

regularly converges a.e. Consequently, the divergence expressed in (3.3′) cannot be spoilt as $\min(m, n) \to \infty$ and this is (3.3) to be shown.

Remark 2. In particular, it follows from Theorem A′ that $\log(n+2)$ cannot be replaced in condition (2.1) by any sequence $\varrho(n)$ tending to ∞ slower than $\log(n+2)$ as $n \to \infty$. Similar observation pertains to Theorems B′ and C′ below.

Theorem B′ [11, Theorem 2]. *Set*

$$A_{-2,k}=|a_{0k}|,\quad A_{-1,k}=|a_{1k}|,\quad A_{pk}=\Big\{\sum_{i=2^p+1}^{2^{p+1}} a_{ik}^2\Big\}^{1/2}\quad (p,k=0,1,\ldots).$$

If

$$A_{pk}\geqq \max\{A_{p+1,k}, A_{p,k+1}\}\quad (p=-2,-1,0,\ldots;\ k=0,1,\ldots)$$

and

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log\log(i+4)]^2[\log(k+2)]^2=\infty,$$

then there exists a double ONS $\{\varphi_{ik}(x_1,x_2)\}$ *of step functions on* S *such that*

$$\limsup|\sigma_{mn}^{10}(x_1,x_2)|=\infty\quad a.e.\ as\quad \min(m,n)\to\infty.$$

Making the convention that for $p=-2$ and -1 by 2^p we mean -1 and 0, respectively, the definition of the A_{pk} can be unified as

$$A_{pk}=\Big\{\sum_{i=2^p+1}^{2^{p+1}} a_{ik}^2\Big\}^{1/2}\quad (p=-2,-1,0,\ldots;\ k=0,1,\ldots).$$

Theorem C′ [12, Theorem 1]. *Set*

$$A_{pq}^*=\Big\{\sum_{i=2^p+1}^{2^{p+1}}\sum_{k=2^q+1}^{2^{q+1}} a_{ik}^2\Big\}^{1/2}\quad (p,q=-2,-1,0,\ldots).$$

If

$$A_{pk}^*\geqq \max\{A_{p+1,k}^*, A_{p,k+1}^*\}\quad (p=-2,-1,0,\ldots;\ k=0,1,\ldots)$$

and

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log\log(i+4)]^2[\log\log(k+4)]^2=\infty,$$

then there exists a double ONS $\{\varphi_{ik}(x_1,x_2)\}$ *of step functions on* S *such that*

$$\limsup|\sigma_{mn}^{11}(x_1,x_2)|=\infty\quad a.e.\ as\quad \min(m,n)\to\infty.$$

The divergence theorems corresponding to Theorems D, E and F will be stated in Section 5.

4. Main results. Following the arguments due to Menšov [4] and Leindler [3], we can conclude an approximation theorem for double ONS of L^2-functions by double orthonormal systems of polynomials (in abbreviation: ONSP) in x_1 and x_2. This theorem can be considered an extension of [3, Theorem 1] from single to double ONS.

Theorem 1. *Let* $\{\varphi_{ik}(x_1,x_2): i,k=0,1,\ldots\}$ *be a double* ONS *on* S, $\{\varepsilon_{rs}: r,s=1,2,\ldots\}$ *a double sequence of positive numbers,* $\{M_r: r=1,2,\ldots\}$ *and* $\{N_s: s=1,2,\ldots\}$ *two strictly increasing sequences of nonnegative integers. Then there exist a double* ONSP $\{P_{ik}(x_1,x_2): i,k=0,1,\ldots\}$ *on* S *and a double sequence*

$\{E_{rs}: r, s=1,2,\ldots\}$ *of measurable subsets of S such that the following properties are satisfied:*

(i) $|E_{rs}| \leqq \varepsilon_{rs}$ $(r, s=1,2,\ldots)$;

(ii) *For every* $(x_1, x_2) \in S \setminus E_{rs}$, *and for every* $M_{r-1} < i \leqq M_r$ *and* $N_{s-1} < k \leqq N_s$,

$$|\varphi_{ik}(x_1, x_2) - (-1)^{j_{rs}(x_1, x_2)} P_{ik}(x_1, x_2)| \leqq \varepsilon_{rs} \tag{4.1}$$

$$(r, s = 1, 2, \ldots;\ M_0 = N_0 = -1).$$

where $j_{rs}(x_1, x_2)$ *equals* 0 *or* 1 *depending on* r, s, x_1, *and* x_2, *but not depending on* i *and* k;

(iii) *If the functions* φ_{ik} *are (not necessarily uniformly) bounded on* S, *then*

$$\max_{(x_1, x_2) \in S} |P_{ik}(x_1, x_2)| \leqq 2\Big\{\sup_{(x_1, x_2) \in S} |\varphi_{ik}(x_1, x_2)| + 1\Big\} \quad (i, k = 0, 1, \ldots).$$

Remark 3. If the φ_{ik} are bounded, in particular, step functions on S, then it suffices to require that the functions φ_{ik} are orthonormal only in each block $M_{r-1} < i \leqq M_r$ and $N_{s-1} < k \leqq N_s$ $(r, s=1,2,\ldots)$, but not altogether.

Remark 4. If in each block $M_{r-1} < i \leqq M_r$ and $N_{s-1} < k \leqq N_s$ the φ_{ik} can be represented in a product form, i.e.

$$\varphi_{ik}(x_1, x_2) = \varphi_i^{(1,r)}(x_1)\, \varphi_k^{(2,s)}(x_2) \tag{4.2}$$

where both $\{\varphi_i^{(1,r)}(x_1): M_{r-1} < i \leqq M_r\}$ and $\{\varphi_k^{(2,s)}(x_2): N_{s-1} < k \leqq N_s\}$ are bounded orthonormal functions on I $(r, s=1,2,\ldots)$, then the resulting P_{ik} can also be taken in the product form

$$P_{ik}(x_1, x_2) = P_i^{(1,r)}(x_1)\, P_k^{(2,s)}(x_2) \tag{4.3}$$

in each block $M_{r-1} < i \leqq M_r$ and $N_{s-1} < k \leqq N_s$, where both $\{P_i^{(1,r)}(x_1): M_{r-1} < i \leqq M_r\}$ and $\{P_k^{(2,s)}(x_2): N_{s-1} < k \leqq N_s\}$ are orthonormal polynomials on I $(r, s=1,2,\ldots)$.

The above approximation theorem enables us to strengthen Theorems A′, B′, and C′ in the same sense as it was done by Leindler [3, Theorems A and G] in the case of single ONS. Namely, if there exists a double ONS for which such and such a series or sequence diverges a.e., then there exists a double ONSP which exhibits this divergence phenomenon.

Theorem 2. *In each of Theorems* A′, B′ *and* C′ *the double* ONS $\{\varphi_{ik}(x_1, x_2)\}$ *can be replaced by a double* ONSP $\{P_{ik}(x_1, x_2)\}$ *of the form* (4.3).

5. Immediate consequences of Leindler's results. Here we cite the main lemma of Leindler [3, Lemma 3] in the form of the following

Theorem G. *Let* $\{\varepsilon_r: r=1,2,\ldots\}$ *be a sequence of positive numbers,* $\{M_r: r=1,2,\ldots\}$ *a strictly increasing sequence of nonnegative integers, and*

$\{\varphi_i(x_1): i=0, 1, \ldots\}$ *a system of bounded functions such that the* φ_i *are orthonormal on* I *in each block* $M_{r-1}<i\leqq M_r$ $(r=1, 2, \ldots;\ M_0=-1)$. *Then there exist an* ONSP $\{P_i(x_1): i=0, 1, \ldots\}$ *on* I *and a sequence* $\{E_r: r=1, 2, \ldots\}$ *of measurable subsets of* I *such that the following properties are satisfied:*

$\overline{\text{(i)}}$ $|E_r|\leqq\varepsilon_r$ $(r=1, 2, \ldots;$ *here* $|\cdot|$ *means the linear Lebesgue measure);*

$\overline{\text{(ii)}}$ *For every* $x_1\in I\setminus E_r$ *and for every* $M_{r-1}<i\leqq M_r$,

$$|\varphi_i(x_1)-(-1)^{j_r(x_1)}P_i(x_1)| \leqq \varepsilon_r \quad (r = 1, 2, \ldots),$$

where $j_r(x_1)$ *equals* 0 *or* 1 *depending on* r *and* x_1, *but not on* i;

$\overline{\text{(iii)}}$ $$\max_{x_1\in I}|P_i(x_1)| \leqq 2\left\{\sup_{x_1\in I}|\varphi_i(x_1)|+1\right\} \quad (i = 0, 1, \ldots).$$

Assume we have two sequences $\{\varepsilon_r^{(1)}: r=1, 2, \ldots\}$ and $\{\varepsilon_s^{(2)}: s=1, 2, \ldots\}$ of positive numbers, two strictly increasing sequences $\{M_r: r=1, 2, \ldots\}$ and $\{N_s: s=1, 2, \ldots\}$ of nonnegative integers, and two systems $\{\varphi_i^{(1)}(x_1): i=0, 1, \ldots\}$ and $\{\varphi_k^{(2)}(x_2): k=0, 1, \ldots\}$ of bounded functions on I with bounds $B_i^{(1)}$ and $B_k^{(2)}$, respectively, such that the $\varphi_i^{(1)}$ are orthonormal on I in each block $M_{r-1}<i\leqq M_r$ $(r=1, 2, \ldots; M_0=-1)$ and the $\varphi_k^{(2)}$ are orthonormal on I in each block $N_{s-1}<k\leqq N_s$ $(s=1, 2, \ldots;\ N_0=-1)$. Applying Theorem G separately to both cases yields two ONSP $\{P_i^{(1)}(x_1): i=0, 1, \ldots\}$ and $\{P_k^{(2)}(x_2): k=0, 1, \ldots\}$, two sequences $\{E_r^{(1)}: r=1, 2, \ldots\}$ and $\{E_r^{(2)}: s=1, 2, \ldots\}$ of measurable subsets of I so that properties $\overline{\text{(i)}}$—$\overline{\text{(iii)}}$ are satisfied, respectively.

It is not hard to verify that the product ONSP given by

$$P_{ik}(x_1, x_2) = P_i^{(1)}(x_1)P_k^{(2)}(x_2) \quad (i, k = 0, 1, \ldots)$$

provides an approximation to the product ONS $\{\varphi_{ik}(x_1, x_2)=\varphi_i^{(1)}(x_1)\varphi_k^{(2)}(x_2): i, k=0, 1, \ldots\}$ with the following properties:

$\overline{\overline{\text{(i)}}}$ Setting $E_{rs}=(E_r^{(1)}\times I)\cup(I\times E_s^{(2)})$,

$$|E_{rs}| \leqq \varepsilon_r^{(1)}+\varepsilon_s^{(2)} \quad (r, s = 1, 2, \ldots); \tag{5.1}$$

$\overline{\overline{\text{(ii)}}}$ For every $(x_1, x_2)\in S\setminus E_{rs}$, and for every $M_{r-1}<i\leqq M_r$ and $N_{s-1}<k\leqq N_s$,

$$|\varphi_i^{(1)}(x_1)\varphi_k^{(2)}(x_2)-(-1)^{j_r^{(1)}(x_1)+j_s^{(2)}(x_2)}P_i^{(1)}(x_1)P_k^{(2)}(x_2)| \leqq \tag{5.2}$$
$$\leqq \Big(\max_{N_{s-1}<k\leqq N_r} B_k^{(2)}\Big)\varepsilon_r^{(1)}+\Big(\max_{M_{r-1}<i\leqq M_s} B_i^{(1)}\Big)\varepsilon_s^{(2)}+\varepsilon_r^{(1)}\varepsilon_s^{(2)} \quad (r, s = 1, 2, \ldots),$$

where both $j_r^{(1)}(x_1)$ and $j_s^{(2)}(x_2)$ equal 0 or 1;

$\overline{\overline{\text{(iii)}}}$ $$\max_{(x_1, x_2)\in S}|P_i^{(1)}(x_1)P_k^{(2)}(x_2)| \leqq$$
$$\leqq 4\left\{\sup_{x_1\in I}|\varphi_i^{(1)}(x_1)|+1\right\}\left\{\sup_{x_2\in I}|\varphi_k^{(2)}(x_2)|+1\right\} \quad (i, k = 0, 1, \ldots).$$

Relation (5.2) immediately follows via the identity

$$\varphi^{(1)}\varphi^{(2)}-(-1)^{j^{(1)}+j^{(2)}}P^{(1)}P^{(2)}=\varphi^{(2)}[\varphi^{(1)}-(-1)^{j^{(1)}}P^{(1)}]+\varphi^{(1)}[\varphi^{(2)}-(-1)^{j^{(2)}}P^{(2)}]-$$
$$-[\varphi^{(1)}-(-1)^{j^{(1)}}P^{(1)}][\varphi^{(2)}-(-1)^{j^{(2)}}P^{(2)}].$$

The main trouble is that the right-hand sides in (5.1) and (5.2) are of $O\{\varepsilon_r^{(1)}+\varepsilon_s^{(2)}\}$ and thus do not tend to 0 as $\max(r,s)\to\infty$. In spite of this disadvantage, the approximation result just obtained is enough to state, for instance, that the double ONS can be replaced by double ONSP in the divergence theorems showing the exactness of Theorems D, E, and F. This is due to the fact that in these cases r and s can be chosen so as to depend on a single parameter l, say: $r=r_l$ and $s=s_l$, while both $r_l\to\infty$ and $s_l\to\infty$ as $l\to\infty$.

However, we can proceed another way. Starting with the strengthened versions of [3, Theorems D and E], the following three theorems can be deduced simply by forming the product system of two appropriate single ONSP as well as the product system $\{a_{ik}=a_i^{(1)}a_k^{(2)}\colon i,k=0,1,\ldots\}$ of the corresponding single sequences $\{a_i^{(1)}\}$ and $\{a_k^{(2)}\}$ of coefficients.

Theorems D′. *If $\{\varrho(n)\colon n=0,1,\ldots\}$ is a nondecreasing sequence of positive numbers for which*

$$\varrho(n)=o\{\log(n+2)\}\quad as\quad n\to\infty, \tag{5.3}$$

then there exist a double ONSP $\{P_{ik}(x_1,x_2)=P_i^{(1)}(x_1)P_k^{(2)}(x_2)\colon i,k=0,1,\ldots\}$ *on S and a double sequence $\{a_{ik}\}$ of coefficients such that condition* (1.2) *is satisfied and*

$$\limsup|S_{mn}(x_1,x_2)|/\varrho(m)\varrho(n)=\infty\quad a.e.\ as\quad \min(m,n)\to\infty \tag{5.4}$$

where

$$S_{mn}(x_1,x_2)=\sum_{i=0}^{m}\sum_{k=0}^{n}a_{ik}P_{ik}(x_1,x_2)\quad (m,n=0,1,\ldots).$$

Using a double ONS $\{\varphi_{ik}(x_1,x_2)=\varphi_i^{(1)}(x_1)\varphi_k^{(2)}(x_2)\}$ in the counterexample, this theorem was proved in [11, Theorem 2].

Theorem E′. If $\{\varrho(n)\colon n=0,1,\ldots\}$ and $\{\tau(m)\colon m=0,1,\ldots\}$ *are two nondecreasing sequences of positive numbers, ϱ satisfying* (5.3) *and τ satisfying*

$$\tau(m)=o\{\log\log(m+4)\}\quad as\quad m\to\infty, \tag{5.5}$$

then there exist a double ONSP $\{P_{ik}(x_1,x_2)=P_i^{(1)}(x_1)P_k^{(2)}(x_2)\}$ *on S and a double sequence $\{a_{ik}\}$ of coefficients such that* (1.2) *is satisfied and for every* $\alpha>0$

$$\limsup|\Sigma_{mn}^{\alpha 0}(x_1,x_2)|/\tau(m)\varrho(m)=\infty\quad a.e.\ as\quad \min(m,n)\to\infty,$$

where

$$\Sigma_{mn}^{\alpha 0}(x_1,x_2)=(1/A_m^{\alpha})\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{\alpha-1}S_{ik}(x_1,x_2)\quad (m,n=0,1,\ldots).$$

Theorem F′. *If* $\{\tau(m)\}$ *is a nondecreasing sequence of positive numbers satisfying* (5.5), *then there exist a double* ONSP $\{P_{ik}(x_1, x_2) = P_i^{(1)}(x_1)P_k^{(2)}(x_2)\}$ *and a double sequence* $\{a_{ik}\}$ *of coefficients such that* (1.2) *is satisfied and for every* $\alpha > 0$ *and* $\beta > 0$

(5.6) $$\limsup |\Sigma_{mn}^{\alpha\beta}(x_1, x_2)|/\tau(m)\tau(n) = \infty \quad a.e.\ as \quad \min(m, n) \to \infty$$

where

$$\Sigma_{mn}^{\alpha\beta}(x_1, x_2) = (1/A_m^\alpha A_n^\beta) \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta-1} S_{ik}(x_1, x_2) \quad (m, n = 0, 1, \ldots).$$

Using double ONS $\{\varphi_{ik}(x_1, x_2) = \varphi_i^{(1)}(x_1)\varphi_k^{(2)}(x_2)\}$ in the counterexamples, Theorems E′ and F′ were included in [8, Section 5].

Remark 5. Examining the structure of the counterexamples in Theorems D′ and F′, the following slightly sharper result can be concluded: Estimates (5.4) and (5.6) remain true if $\varrho(m)\varrho(n)$ and $\tau(m)\tau(n)$ in the denominators are replaced by $\varrho^2(\min(m, n))$ and $\tau^2(\min(m, n))$, respectively, provided

$$\varrho(2n) \leqq C\varrho(n) \quad (n = 0, 1, \ldots) \quad \text{and} \quad \tau(m^2) \leqq C\tau(m) \quad (m = 0, 1, \ldots),$$

where C is a positive constant.

Remark 6. A couple of other divergence theorems can be strengthened by inserting double ONSP in the above way. For example, the corresponding two-dimensional versions of [3, Theorems B, C, and F] hold also true.

6. Proof of Theorem 1. It relies on the basic ideas of Menšov [4] and Leindler [3]. Here we reformulate four lemmas of their papers for the two-dimensional case. These lemmas were stated and proved in the one-dimensional case. However, the proofs of the two-dimensional reformulations closely follow the original proofs. Where is needed, we indicate the necessary modifications.

Lemma 1. *Let* $\{f_i(x_1, x_2)\colon 1 \leqq i \leqq N\}$ *be continuous functions, while* $\{g_j(x_1, x_2)\colon 1 \leqq j \leqq N'\}$ *step functions on* S *and let* $\varepsilon > 0$. *Then there exist functions* $\{G_j(x_1, x_2)\colon 1 \leqq j \leqq N'\}$ *and a measurable subset* E *of* S *with the following properties:*

(a) *The functions* G_j *are continuous on* S;

(b) $|E| \leqq \varepsilon$;

(c) *For every* $(x_1, x_2) \in S \setminus E$ *and for every* $1 \leqq j \leqq N'$,

$$G_j(x_1, x_2) = (-1)^{j(x_1, x_2)} g_j(x_1, x_2)$$

where $j(x_1, x_2)$ *equals* 0 *or* 1 *depending on* (x_1, x_2) *but not on* j;

(d) $$\max_{(x_1, x_2) \in S} |G_j(x_1, x_2)| \leqq \max_{(x_1, x_2) \in S} |g_j(x_1, x_2)| \quad (1 \leqq j \leqq N');$$

(e) $$\left|\int_0^1 \int_0^1 f_i(x_1, x_2) G_j(x_1, x_2)\, dx_1\, dx_2\right| \leqq \varepsilon \quad (1 \leqq i \leqq N, 1 \leqq j \leqq N').$$

This auxiliary result is the restatement of [4, Lemma 2, pp. 30—32] with a similar proof. The only thing to explain is that if g is a step function on S, then for every $\varepsilon>0$ there exists a continuous function G on S such that

$$|\{(x_1, x_2)\colon\ G(x_1, x_2) \neq g(x_1, x_2)\}| \leqq \varepsilon \tag{6.1}$$

and

$$\max_{(x_1, x_2)\in S} |G(x_1, x_2)| \leqq \max_{(x_1, x_2)\in S} |g(x_1, x_2)|. \tag{6.2}$$

This is clear if $g(x_1, x_2)=g^{(1)}(x_1)g^{(2)}(x_2)$ even with some $G(x_1, x_2)=G^{(1)}(x_1)G^{(2)}(x_2)$.

In the general case, it is enough to consider the characteristic function $g(x_1, x_2)=$ $=\chi_R(x_1, x_2)$ of a rectangle R inside S given by $R=\langle\alpha_1, \alpha_2\rangle\times\langle\beta_1, \beta_2\rangle$, where $\langle\alpha_1, \alpha_2\rangle$ denotes one of the intervals (α_1, α_2), $[\alpha_1, \alpha_2)$, $(\alpha_1, \alpha_2]$, $[\alpha_1, \alpha_2]$ and $\langle\beta_1 \beta_2\rangle$ has a similar meaning. Setting

$$\delta = \min\{\varepsilon/4, (\alpha_2-\alpha_1)/2, (\beta_2-\beta_1)/2\}$$

we define $G(x_1, x_2)$ to be $G^{(1)}(x_1)\, G^{(2)}(x_2)$ where

$$G^{(1)}(x_1) = \begin{cases} 1 & \text{for}\quad \alpha_1+\delta \leqq x_1 \leqq \alpha_2-\delta, \\ 0 & \text{for}\quad 0 \leqq x_1 \leqq \alpha_1 \quad\text{and}\quad \alpha_2 \leqq x_1 \leqq 1, \\ \text{linear} & \text{for}\quad \alpha_1 < x_1 < \alpha_1+\delta \quad\text{and}\quad \alpha_2-\delta < x_1 < \alpha_2 \end{cases}$$

in such a way that $G^{(1)}$ is continuous on I; and $G^{(2)}$ is defined in an analogous manner. It is easy to check that this G meets the conditions (6.1) and (6.2).

Remark 7. If $N'=N_1'N_2'$ and the functions f_i and g_j are given in the forms

$$f_i(x_1, x_2) = f_i^{(1)}(x_1) f_i^{(2)}(x_2) \quad (1 \leqq i \leqq N)$$

and

$$g_{jl}(x_1, x_2) = g_j^{(1)}(x_1)\, g_l^{(2)}(x_2) \quad (1 \leqq j \leqq N_1',\ 1 \leqq l \leqq N_2'),$$

then the functions G_j can be also represented in the form

$$G_{jl}(x_1, x_2) = G_j^{(1)}(x_1)\, G_l^{(2)}(x_2) \quad (1 \leqq j \leqq N_1',\ 1 \leqq l \leqq N_2'). \tag{6.3}$$

In order to see this, apply the original Menšov's lemma [4, Lemma 2] separately to the following two systems $\{f_i^{(1)}(x_1), g_j^{(1)}(x_1)\colon 1\leqq i\leqq N,\ 1\leqq j\leqq N_1'\}$ and $\{f_i^{(2)}(x_2), g_l^{(2)}(x_2)\colon 1\leqq i\leqq N, 1\leqq l\leqq N_2'\}$ with $\varepsilon^{(1)}=\varepsilon^{(2)}=\varepsilon/2$ instead of $\varepsilon>0$. As a result, we obtain two systems $\{G_j^{(1)}(x_1)\colon 1\leqq j\leqq N_1'\}$ and $\{G_l^{(2)}(x_2)\colon 1\leqq l\leqq N_2'\}$ with corresponding sets $E^{(1)}$ and $E^{(2)}$, and corresponding exponents $j^{(1)}(x_1)$ and $j^{(2)}(x_2)$. Letting (6.3),

$$E = (E^{(1)}\times I)\cup(I\times E^{(2)}), \quad\text{and}\quad j(x_1, x_2) = j^{(1)}(x_1)+j^{(2)}(x_2),$$

properties (a), (b), (c), and (e) are obviously satisfied (in case (e) provided $\varepsilon\leqq 1$). To verify the fulfillment of (d), we have to take into account that the extreme values of the step functions $g_j^{(1)}$and $g_l^{(2)}$ are not altered during the linearization process. Thus, for every $(x_1, x_2)\in S$ there exists a pair $(x_1', x_2')\in S$ such that

$$|G_j^{(1)}(x_1)| \leqq |G_j^{(1)}(x_1')| = |g_j^{(1)}(x_1')|$$

and

$$|G_l^{(2)}(x_2)| \leqq |G_l^{(2)}(x_2')| = |g_l^{(2)}(x_2')|.$$

Consequently, (d) is also satisfied.

Lemma 2. *Let* $0<r<r'$, $\{\Pi_i(x_1, x_2): 1\leqq i\leqq r\}$ *and* $\{Q_k(x_1, x_2): r<k\leqq r'\}$ *be nonidentically vanishing polynomials in* x_1 *and* x_2,

$$\nu = \max_{i,k}\Big\{\max_{(x_1,x_2)\in S} |\Pi_i(x_1, x_2)|, \max_{(x_1,x_2)\in S} |Q_k(x_1, x_2)|\Big\},$$

$$\varkappa = \min_{i,k}\Big\{\int_0^1\int_0^1 \Pi_i^2(x_1, x_2)\,dx_1\,dx_2,\ \int_0^1\int_0^1 Q_k^2(x_1, x_2)\,dx_1\,dx_2\Big\},$$

$$\sigma = \max_{\substack{i,k,l\\k\neq l}}\Big\{\int_0^1\int_0^1 \Pi_i(x_1, x_2)Q_k(x_1, x_2)\,dx_1\,dx_2,\ \int_0^1\int_0^1 Q_k(x_1, x_2)Q_l(x_1, x_2)\,dx_1\,dx_2\Big\},$$

$$\gamma = \max\{4r', \nu, 1/\varkappa\}, \quad \text{and} \quad \lambda = \gamma^{6(r'-r+1)}.$$

If the polynomials $\{\Pi_i(x_1, x_2): 1\leqq i\leqq r\}$ *are orthogonal on* S *and* $\sigma\lambda<1$, *then there exist polynomials* $\{\Pi_k(x_1, x_2): r<k\leqq r'\}$ *in* x_1 *and* x_2 *such that the following properties are satisfied:*

($\bar{a}$) *The polynomials* $\{\Pi_i(x_1, x_2): 1\leqq i\leqq r'\}$ *are orthogonal on* S;

($\bar{b}$) $\max_{(x_1,x_2)\in S} |Q_k(x_1, x_2)-\Pi_k(x_1, x_2)| \leqq \sigma\lambda \quad (r < k \leqq r').$

The proof is essentially a repetition of the proof of the corresponding result of MENŠOV [4, Lemma 3, pp. 32—36].

Remark 8. If $r'=r_1'r_2'$ and the polynomials Π_i and Q_k are given in the forms

$$\Pi_i(x_1, x_2) = \Pi_i^{(1)}(x_1)\Pi_i^{(2)}(x_2) \quad (1 \leqq i \leqq r)$$

and

$$Q_{kl}(x_1, x_2) = Q_k^{(1)}(x_1)Q_l^{(2)}(x_2) \quad (r < k \leqq r_1',\ r < l \leqq r_2'),$$

then the polynomials Π_k can also be represented in the form

$$\Pi_{kl}(x_1, x_2) = \Pi_k^{(1)}(x_1)\Pi_l^{(2)}(x_2) \quad (r < k \leqq r_1',\ r < l \leqq r_2'). \tag{6.4}$$

Indeed, apply the original Menšov's lemma [4, Lemma 3] separately to the systems $\{\Pi_i^{(1)}(x_1), Q_k^{(1)}(x_1): 1\leqq i\leqq r<k\leqq r_1'\}$ and $\{\Pi_i^{(2)}(x_2), Q_l^{(2)}(x_2): 1\leqq i\leqq r<l\leqq r_2'\}$ with the corresponding notations $\nu^{(1)}, \varkappa^{(1)}, \sigma^{(1)}, \gamma^{(1)}, \lambda^{(1)}$ and $\nu^{(2)}, \ldots, \lambda^{(2)}$. If $\sigma^{(1)}\lambda^{(1)}<1$ and $\sigma^{(2)}\lambda^{(2)}<1$, then we can obtain polynomials $\Pi_k^{(1)}$ for $r<k\leqq r_1'$ and $\Pi_l^{(2)}$ for $r<$

$<l\leqq r_2'$ such that the systems $\{\Pi_k^{(1)}(x_1)\colon 1\leqq k\leqq r_1'\}$ and $\{\Pi_l^{(2)}(x_2)\colon 1\leqq l\leqq r_2'\}$ are orthonormal on I, respectively, and

$$\max_{x_1\in I} |Q_k^{(1)}(x_1)-\Pi_k^{(1)}(x_1)| \leqq \sigma^{(1)}\lambda^{(1)} \quad (r<k\leqq r_1')$$

and

$$\max_{x_2\in I} |Q_l^{(2)}(x_2)-\Pi_l^{(2)}(x_2)| \leqq \sigma^{(2)}\lambda^{(2)} \quad (r<l\leqq r_2').$$

Now letting (6.4), the fulfillment of $(\bar{a})$ is clear, while a slightly modified version of $(\bar{b})$ follows via the elementary estimate

$$\max_{(x_1,x_2)\in S} |Q_k^{(1)}(x_1)Q_l^{(2)}(x_2)-\Pi_k^{(1)}(x_1)\Pi_l^{(2)}(x_2)| \leqq$$

$$\leqq \nu^{(2)}\max_{x_1\in I}|Q_k^{(1)}(x_1)-\Pi_k^{(1)}(x_1)|+\nu^{(1)}\max_{x_2\in I}|Q_l^{(2)}(x_2)-\Pi_l^{(2)}(x_2)| \leqq$$

$$\leqq \nu^{(2)}\sigma^{(1)}\lambda^{(1)}+\nu^{(1)}\sigma^{(2)}\lambda^{(2)}.$$

This form is still enough during the proof of Lemma 4 below (cf. [3, p. 26, formula (3.4)]).

Lemma 3. *Let* $\{\varphi_{ik}(x_1,x_2)\colon i,\, k=0,1,\dots\}$ *be a double* ONS *on* S, $\{\varepsilon_{rs}\colon r,s=1,2,\dots\}$ *a double sequence of positive numbers,* $\{M_r\colon r=1,2,\dots\}$ *and* $\{N_s\colon s=1,2,\dots\}$ *two strictly increasing sequences of nonnegative integers. Then there exist a double system* $\{\psi_{ik}(x_1,x_2)\colon i,\,k=0,1,\dots\}$ *of bounded functions on* S *and a double sequence* $\{E_{rs}\colon r,s=1,2,\dots\}$ *of measurable subsets of* S *such that the following properties are satisfied:*

(α) *The functions* ψ_{ik} *are orthonormal on* S *in each block* $M_{r-1}<i\leqq M_r$ *and* $N_{s-1}<k\leqq N_s$ $(r,s=1,2,\dots;\ M_0=N_0=-1)$;

(β) $|E_{rs}|\leqq\varepsilon_{rs}$ $(r,s=1,2,\dots)$;

(γ) *For every* $(x_1,x_2)\in S\setminus E_{rs}$, *and for every* $M_{r-1}<i\leqq M_r$ *and* $N_{s-1}<k\leqq N_s$,

$$|\varphi_{ik}(x_1,x_2)-\psi_{ik}(x_1,x_2)|\leqq\varepsilon_{rs}\quad (r,s=1,2,\dots).$$

This lemma is a straightforward extension of a lemma due to Leindler [3, Lemma 4, pp. 33—36]. The original proof works in the two-dimensional setting, since the blocks $\{\varphi_{ik}(x_1,x_2)\colon M_{r-1}<i\leqq M_r$ and $N_{s-1}<k\leqq N_s\}$ and the corresponding ε_{rs} $(r,s=1,2,\dots)$ can be treated in an arrangement similar to the Cantor diagonal process, i.e. in the following succession: $\varepsilon_{11},\varepsilon_{12},\varepsilon_{21},\varepsilon_{13},\varepsilon_{22},\varepsilon_{14},\dots$ and the blocks are taken accordingly.

Lemma 4. *Let* $\{\varepsilon_{rs}\colon r,s=1,2,\dots\}$ *be a double sequence of positive numbers,* $\{M_r\colon r=1,2,\dots\}$ *and* $\{N_s\colon s=1,2,\dots\}$ *two strictly increasing sequences of nonnegative integers, and* $\{\psi_{ik}(x_1,x_2)\colon i,\,k=0,1,\dots\}$ *a double system of bounded functions such that the* ψ_{ik} *are orthonormal on* S *in each block* $M_{r-1}<i\leqq M_r$ *and* $N_{s-1}<k\leqq N_s$ $(r,s=1,2,\dots;\ M_0=N_0=-1)$. *Then there exist a double* ONSP $\{P_{ik}(x_1,x_2)\colon i,\,k=$

$=0, 1, \ldots\}$ *on* S *and a double sequence* $\{\bar{E}_{rs}: r, s=1, 2, \ldots\}$ *of measurable subsets of* S *with the following properties:*

($\bar{\alpha}$) $|\bar{E}_{rs}| \leqq \varepsilon_{rs}$ $(r, s=1, 2, \ldots)$;

($\bar{\beta}$) *For every* $(x_1, x_2) \in S \setminus \bar{E}_{rs}$, *and for every* $M_{r-1} < i \leqq M_r$ *and* $N_{s-1} < k \leqq N_s$,

$$|\psi_{ik}(x_1, x_2) - (-1)^{j_{rs}(x_1, x_2)} P_{ik}(x_1, x_2)| \leqq \varepsilon_{rs} \quad (r, s = 1, 2, \ldots)$$

where $j_{rs}(x_1, x_2)$ *equals* 0 *or* 1;

($\bar{\gamma}$) $\max_{(x_1, x_2) \in S} |P_{ik}(x_1, x_2)| \leqq 2\{\sup_{(x_1, x_2) \in S} |\psi_{ik}(x_1, x_2)| + 1\}$ $(i, k = 0, 1, \ldots)$.

Remark 9. If in each block $M_{r-1} < i \leqq M_r$ and $N_{s-1} < k \leqq N_s$ the functions ψ_{ik} can be represented in a product form analogous to (4.2):

$$\psi_{ik}(x_1, x_2) = \psi_i^{(1, r)}(x_1) \psi_k^{(2, s)}(x_2)$$

where both $\{\psi_i^{(1,r)}(x_1): M_{r-1} < i \leqq M_r\}$ and $\{\psi_k^{(2,s)}(x_2): N_{s-1} < k \leqq N_s\}$ are bounded orthonormal functions on I $(r, s=1, 2, \ldots)$, then the P_{ik} can be also represented in the product form (4.3) where both

$$\{P_i^{(1,r)}(x_1):\ M_{r-1} < i \leqq M_r\}$$

and

$$\{P_k^{(2,s)}(x_2):\ N_{s-1} < k \leqq N_s\}$$

are orthonormal polynomials on I $(r, s=1, 2, \ldots)$.

The proof of Lemma 4 can be modelled on that of Leindler's basic lemma [3, Lemma 3, pp. 26—33]. We only note that both the Egorov theorem and the Weierstrass approximation theorem are valid in two-dimensional setting, as well.

The former one states that if $\psi(x_1, x_2)$ is a bounded measurable function on S, then for every $\varepsilon > 0$ there exist a measurable step function $\varphi(x_1, x_2)$ on S and a measurable subset E of S such that

(1) $|E| \leqq \varepsilon$;

(2) $|\psi(x_1, x_2) - \varphi(x_1, x_2)| \leqq \varepsilon$ for $(x_1, x_2) \in S \setminus E$;

(3) $\max_{(x_1, x_2) \in S} |\varphi(x_1, x_2)| \leqq \sup_{(x_1, x_2) \in S} |\psi(x_1, x_2)|$;

(4) If $\psi(x_1, x_2) = \psi^{(1)}(x_1) \psi^{(2)}(x_2)$, then φ can be chosen in the form of $\varphi(x_1, x_2) = \varphi^{(1)}(x_1) \varphi^{(2)}(x_2)$.

Concerning the two-dimensional version of the Weierstrass theorem, we refer to [2, pp. 89—90]. Choosing $T=S$ and $\mathfrak{A}$ to be the set of the polynomials $P(x_1, x_2)$ in x_1 and x_2, properties (i)—(iv) in the cited paper are obviously satisfied, thereby ensuring the uniform approximation of continuous functions on S by the elements of $\mathfrak{A}$.

Finally, the validity of Remark 9 can be verified by means of Remarks 7 and 8.

7. Proof of Theorem 2. We will present only the proof of the sharpening of Theorem A′. The sharpenings of Theorem B′ and C′ can be proved similarly.

So, assume the fulfillment of (3.1) and (3.2). In the proof of [10, Theorem 4] a double ONS $\{\varphi_{ik}(x_1, x_2): i, k=0,1, \ldots\}$ of step functions on S and a double sequence $\{H_{rs}: r, s=-1, 0, 1, \ldots\}$ of measurable subsets of S were constructed with the following properties:

(i) $|H|=1$ where $H=\limsup H_{rs}$ as $\max(r, s)\to\infty$;

(ii) For every $(x_1, x_2)\in H_{rs}$

(7.1)
$$\max_{2^{r-1}<m\leqq 2^r}\ \max_{2^{s-1}<n\leqq 2^s}\left|\sum_{i=2^{r-1}+1}^{m}\ \sum_{k=2^{s-1}+1}^{n} a_{ik}\varphi_{ik}(x_1, x_2)\right| \geqq C\eta_{\max(r,s)}\quad (r, s=-1, 0, 1, \ldots)$$

where C is a positive constant and $\{\eta_r: r=-1, 0, 1, \ldots\}$ is an increasing sequence of positive numbers tending to ∞ as $r\to\infty$;

(iii) In each block $2^{r-1}<i\leqq 2^r$ and $2^{s-1}<k\leqq 2^s$ the functions φ_{ik} can be represented in the product form (4.2).

Given any $\delta>0$, on the basis of Theorem 1 we can construct a double ONSP $\{P_{ik}(x_1, x_2)\}$ and a double sequence $\{\bar H_{rs}\}$ of measurable subsets of S such that

$(\bar{\text{i}})$ $|\bar H|=1$ where $\bar H=\limsup \bar H_{rs}$ as $\max(r, s)\to\infty$;

$(\bar{\text{ii}})$ For every $(x_1, x_2)\in\bar H_{rs}$

(7.2)
$$\max_{2^{r-1}<m\leqq 2^r}\ \max_{2^{s-1}<n\leqq 2^s}\left|\sum_{i=2^{r-1}+1}^{m}\ \sum_{k=2^{s-1}+1}^{n} a_{ik}P_{ik}(x_1, x_2)\right| \geqq (C-\delta)\eta_{\max(r,s)}\quad (r, s=-1, 0, 1, \ldots)$$

(cf. [3, Theorem 2, p. 21 and its proof on pp. 38—39]);

$(\overline{\text{iii}})$ In each block $2^{r-1}<i\leqq 2^r$ and $2^{s-1}<k\leqq 2^s$ the polynomials P_{ik} can be represented in the product form (4.3).

Relation (7.2) implies the a.e. divergence of the double series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}P_{ik}(x_1, x_2)$$

in the same way as (7.1) implies the a. e. divergence of the double series (1.1).

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