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Birkhoff quadrature formulas based on Tchebycheff nodes

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1. In 1974 P. Turán [6] raised the following problems on Birkhoff quadrature. If in the n^{th} row of a matrix A, there are n interpolation points $1 \geqq x_{1n} > x_{2n} > \ldots > x_{nn} \geqq -1$ then A is called "very good" if for an arbitrary set of numbers y_{kn} and y''_{kn} there is a uniquely determined polynomial $D_n(f;\, A) = D_n(f)$ of degree at most $2n-1$ for which

$$D_n(f;\, A)_{x=x_{kn}} = y_{kn} = f(x_{kn}), \quad \left(\frac{d^2}{dx^2} D_n(f;\, A)\right)_{x=x_{kn}} = y''_{kn}, \quad k = 1, 2, \ldots, n.$$

In that case $D_n(f;\, A)$ can be uniquely written as

$$D_n(f;\, A) = \sum_{i=1}^{n} f(x_{in})\gamma_{in}(x;\, A) + \sum_{i=1}^{n} y''_{in}\varrho_{in}(x;\, A)$$

where $\gamma_{in}(x;\, A)$, $\varrho_{in}(x;\, A)$ are fundamental functions of the first and second kind, respectively.

Problem XXXVI. What is the best class of functions for which the integrals of the polynomials

$$\sum_{i=1}^{n} f(x_{in})\gamma_{in}(x, \Pi) \quad (n \text{ even})$$

tend to $\int_{-1}^{1} f(x)\, dx$?

Here the n^{th} row of the Π matrix is referred to the zeros of $\Pi_n(x) = \int_{-1}^{x} P_{n-1}(t)\, dt$ where $P_n(x)$ is the Legendre polynomial of degree n.

Problem XXXVIII. Does there exist a matrix A satisfying

$$\int_{-1}^{1} \gamma_{in}(x;\, A)\, dx \geqq 0, \quad i = 1, 2, \ldots, n;\ n \geqq n_0?$$

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Problem XXXIX. Determine the "good" matrices for which

$$\sum_{i=1}^{n}\left|\int_{-1}^{1}\gamma_{in}(x;\ A)\,dx\right|$$

is minimal.

In 1982 the author [7] was able to answer the above problems. This can be summarized by the following quadrature formula exact for polynomials of degree $\leqq 2n-1$:

$$\int_{-1}^{1} f(x)\,dx = \frac{3((f(1)+f(-1)))}{n(2n-1)}+\frac{2(2n-3)}{n(n-2)(2n-1)}\sum_{i=2}^{n-1}\frac{f(x_{in})}{P_{n-1}^2(x_{in})}+$$

$$+\frac{1}{n(n-1)(n-2)(2n-1)}\sum_{i=2}^{n-1}\frac{(1-x_{in}^2)f''(x_{in})}{P_{n-1}^2(x_{in})}.$$

In the above formula x_{in}'s are chosen to be the zeros of $\Pi_n(x)$.

In 1961, the author [9] extended the results of Saxena and Sharma [6] of (0, 1, 3) interpolation to Tchebycheff abscissas. We proved that if n is even, then for preassigned values y_{i0}, y_{i1}, y_{i3} $(i=1, 2, \ldots, n)$ there exists a uniquely determined polynomial $f_n(x)$ of degree $\leqq 3n-1$ such that

$$f_n(x_{in}) = y_{i0},\quad f_n'(x_{in}) = y_{i1},\quad f_n'''(x_{in}) = y_{i3},\quad i = 1, 2, \ldots, n \tag{1.1}$$

where x_{in}'s are the zeros of $T_n(x)$.

The object of this paper is to obtain new quadrature formulas based on $f(x_{in})$, $f'(x_{in})$, $f'''(x_{in})$ where x_{in}'s are the zeros of $T_n(x)$. We now state the main theorems of this paper.

Theorem 1. *Let*

$$1 > x_{1n} > x_{2n} > \ldots > x_{nn} > -1 \tag{1.2}$$

be the zeros of $T_n(x)=\cos n\theta$, $\cos\theta=x$. *Let* $f(x)$ *be any polynomial of degree* $\leqq 3n-1$. *Then we have*

$$\int_{-1}^{1}\frac{f(x)}{(1-x^2)^{1/2}}\,dx = \sum_{i=1}^{n} f(x_{in})A_{in}+\sum_{i=1}^{n} f'(x_{in})B_{in}+\sum_{i=1}^{n} f'''(x_{in})C_{in} \tag{1.3}$$

where

$$C_{in} = \frac{(1-x_{in}^2)^2}{12n^2}\int_{-1}^{1}\frac{l_{in}'(x)}{(1-x^2)^{1/2}}\,dx = \frac{(1-x_{in}^2)^2\pi}{12n^3}\left(T_n'(x_{in})-\frac{x_{in}}{1-x_{in}^2}\right), \tag{1.4}$$

$$B_{in} = -\frac{\pi}{4n}(1-x_{in}^2)T_n'(x_{in})+\left(\frac{3+2(2n^2+1)(1-x_{in}^2)}{12n^2}\right)\int_{-1}^{1}\frac{l_{in}'(x)\,dx}{(1-x^2)^{1/2}} \tag{1.5}$$

and

$$(1.6)\quad A_{in}=\frac{\pi}{n}\left[1+\frac{1}{3(1-x_{in}^2)}\left(\frac{1}{n^2}-1\right)-\frac{1}{2n^2(1-x_{in}^2)^2}+\frac{x_i T_n'(x_i)}{4}+\frac{T_n''(x_{in})}{2n^2}\right].$$

Theorem 2. *Let* $f_0(x)=1-x^2$ *then*

$$(1.7)\qquad \int_{-1}^{1}\frac{f_0(x)}{(1-x^2)^{1/2}}dx-\sum_{i=1}^{n} f_0(x_{in})A_{in}=\frac{\pi}{3}-\frac{\pi}{3n^2}.$$

An interesting consequence of Theorem 2 is the following:

Corollary 1. *For* $f_0(x)=1-x^2$

$$(1.8)\qquad \lim_{n\to\infty}\sum_{i=1}^{n} f_0(x_{in})u_{in}(x)\neq 1-x^2 \quad \textit{at some} \quad x\in[-1,1],$$

where $u_{in}(x)$ *are the fundamental polynomials of the first kind* (0, 1, 3) *interpolation based on Tchebycheff nodes. The explicit representation of* $u_{in}(x)$ *is given in the next section.*

Theorem 3. *There exist positive constants* c_1 *and* c_2 *independent of* n *such that*

$$(1.9)\qquad c_1 n\ln n<\sum_{i=1}^{n}|A_{in}|\leqq c_2 n\ln n.$$

Theorems 1, 2, 3 reveal an important fact that the quadrature formula obtained by integrating the Birkhoff interpolation polynomials of (0, 1, 3) interpolation based on Tchebycheff nodes is essentially very different from those obtained by integrating Lagrange or Hermite interpolation.

2. Explicit representation of the interpolatory polynomials; (0, 1, 3) case. In an earlier work [9] we obtained the explicit form of the polynomial $R_n(x)$ (n even positive integer) of degree $\leqq 3n-1$ satisfying

$$(2.1)\qquad R_n(x_{kn})=f_{kn},\quad R_n'(x_{kn})=g_{kn},\quad R_n'''(x_{kn})=h_{kn},\quad k=1,2,\dots,n.$$

It is given by

$$(2.2)\qquad R_n(x)=\sum_{i=1}^{n} f_{in}u_{in}(x)+\sum_{i=1}^{n} g_{in}v_{in}(x)+\sum_{i=1}^{n} h_{in}w_{in}(x),$$

where the polynomials $u_{in}(x)$, $v_{in}(x)$, $w_{in}(x)$ are uniquely determined by the following conditions:

$$(2.3)\qquad u_{in}'(x_{kn})=u_{in}'''(x_{kn})=0,\quad u_{in}(x_{kn})=\begin{cases}1 & \text{for } i=k\\ 0 & \text{for } i\neq k,\end{cases}\quad k=1,2,\dots,n,$$

$$(2.4)\qquad v_{in}(x_{kn})=v_{in}'''(x_{kn})=0,\quad v_{in}'(x_{kn})=\begin{cases}1 & \text{for } i=k\\ 0 & \text{for } i\neq k,\end{cases}\quad k=1,2,\dots,n,$$

$$(2.5)\qquad w_{in}(x_{kn})=w_{in}'(x_{kn})=0,\quad w_{in}'''(x_{kn})=\begin{cases}1 & \text{for } i=k\\ 0 & \text{for } i\neq k,\end{cases}\quad k=1,2,\dots,n.$$

The explicit form of fundamental polynomials is given by

(a)

$$w_{in}(x) = \frac{(1-x_{in}^2)^2 T_n^2(x) q_{n-1,i}(x)}{6n^2}, \tag{2.6}$$

where $q_{n-1,i}(x)$ is a polynomial of degree $\leqq n-1$. It is given by

$$q_{n-1,i}(x) = (1-x^2)^{1/2}\left[a_{in}\int_0^x \frac{T_n(t)}{(1-t^2)^{3/2}}\,dt + \int_0^x \frac{l_{in}(t)}{(1-t^2)^{3/2}}\,dt + c_{in}\right] \tag{2.7}$$

where a_{in}, c_{in} are chosen so that $q_{n-1,i}(x)$ is a polynomial of degree $\leqq n-1$.

(b)

$$a_{in} = -\frac{1}{n\pi}\int_{-1}^{1} \frac{t\,l'_{in}(t)}{(1-t^2)^{1/2}}\,dt,$$

$$c_{in} = (1/2)\int_{-1}^{1} \frac{l'_{in}(t)}{(1-t^2)^{1/2}}\,dt - (1/2)\int_0^1 \frac{l_{in}(t)-l_{in}(-t)}{(1-t)^{1/2}(1+t)^{3/2}}\,dt,$$

$$v_{in}(x) = \frac{T_n(x)\,l_{in}^2(x)}{T'_n(x_{in})} + \frac{T_n^2(x)\,s_{n-1,i}(x)}{T'_n(x_{in})} \tag{2.8}$$

where $s_{n-1,i}(x)$ is a polynomial of degree $\leqq n-1$. It is expressed by the formula

(2.9)

$$s_{n-1,i}(x) = (1-x^2)^{1/2}\left[\alpha_{in}\int_0^x \frac{T_n(t)}{(1-t^2)^{3/2}}\,dt + \beta_{in}\int_0^x \frac{l_{in}(t)}{(1-t^2)^{3/2}}\,dt + \int_0^x \frac{F_{in}(t)}{(1-t^2)^{3/2}}\,dt + \gamma_{in}\right]$$

where

$$F_{in}(t) = \frac{(1-t^2)\,l''_{in}(t) - t\,l'_{in}(t) + (n^2+1)\,l_{in}(t)}{2T'_n(x_{in})}, \tag{2.10}$$

$$\alpha_{in} = \frac{-1}{n\pi}\int_{-1}^{1} \frac{tF'_{in}(t) + \beta_{in}\,l'_{in}(t)}{(1-t^2)^{1/2}}\,dt, \tag{2.11}$$

$$\beta_{in} = \frac{1}{2T'_n(x_{in})}\left(\frac{n^2-7}{3} - \frac{5x_{in}^2}{1-x_{in}^2}\right), \tag{2.12}$$

(2.13)

$$2\gamma_{in} = \int_{-1}^{1} \frac{\beta_{in}\,l'_{in}(t) + F'_{in}(t)}{(1-t^2)^{1/2}}\,dt -$$

$$-\int_0^1 \frac{\beta_{in}\big(l_{in}(t)-l_{in}(-t)\big) + F_{in}(t) - F_{in}(-t)}{(1-t)^{1/2}(1+t)^{3/2}}\,dt.$$

(c)

$$u_{in}(x) = \frac{(1-x^2)\, l_{in}^3(x)}{1-x_{in}^2} + \frac{1}{3}\,\frac{(1-x^2)(x-x_{in})}{(1-x_{in}^2)}\, l_{in}^2(x)\, l_{in}'(x) + \tag{2.14}$$

$$+\frac{x_{in}}{3(1-x_{in}^2)}\, v_{in}(x) + \lambda_{in} w_{in}(x)$$

where

$$\lambda_{in} = \frac{x_{in}}{3(1-x_{in}^2)}\left[8n^2 + \frac{13+2x_{in}^2}{1-x_{in}^2}\right].$$

The above representation of $u_{in}(x)$ is new and very useful in obtaining $\int_{-1}^{1} \frac{u_{in}(x)}{(1-x^2)^{1/2}}\, dx.$

3. Preliminaries. Here we shall prove the following lemmas.

Lemma 3.1. *For $k=1, 2, \ldots, n$ we have*

$$\int_{-1}^{1} \frac{T_n(x)\, l_{kn}^2(x)}{T_n'(x_{kn})} (1-x^2)^{-1/2}\, dx = \frac{\pi}{2n^3}\, x_{kn}, \tag{3.1}$$

$$\int_{-1}^{1} \frac{(1-x^2)\, l_{kn}^3(x)}{(1-x_{kn}^2)} (1-x^2)^{-1/2}\, dx = \frac{\pi}{n}\left\{\frac{3}{4} - \frac{1}{4n^2(1-x_{kn}^2)}\right\}, \tag{3.2}$$

$$\int_{-1}^{1} \left(\frac{1-x x_{kn}}{1-x_{kn}^2}\right) l_{kn}^3(x)(1-x^2)^{-1/2}\, dx = \frac{\pi}{n}\left\{\frac{3}{4} + \frac{1}{4n^2(1-x_{kn}^2)}\right\}, \tag{3.3}$$

$$\int_{-1}^{1} \frac{(1-x^2)(x-x_{kn})\, l_{kn}^2(x)\, l_{kn}'(x)}{1-x_{kn}^2} (1-x^2)^{-1/2}\, dx = \frac{\pi}{4n}\left\{-1 + \frac{1}{n^2(1-x_{kn}^2)}\right\}. \tag{3.4}$$

Proof. According to a theorem of Michelli and Rivlin [5] if $g(x)$ is a polynomial of degree $\leqq 4n-1$ then

$$\int_{-1}^{1} g(x)(1-x^2)^{-1/2}\, dx = \tag{3.5}$$

$$= \frac{\pi}{n}\left[\sum_{i=1}^{n} g(x_{in}) + \frac{1}{4n^2}\sum_{i=1}^{n} (1-x_{in}^2)\, g''(x_{in}) - x_{in}\, g'(x_{in})\right]$$

where x_{in}'s are the zeros of $T_n(x)$. First let

$$g(x) = \frac{T_n(x)\, l_{kn}^2(x)}{T_n'(x_{kn})}$$

and note that

$$g(x_{in})=0,\quad g'(x_{in})=\begin{cases}1 & \text{if } i=k\\ 0 & \text{if } i\neq k\end{cases}$$

and

$$g''(x_{in})=\begin{cases}3x_{kn}/(1-x_{kn}^2) & \text{if } i=k\\ 0 & \text{if } i\neq k,\end{cases}$$

on applying (3.5) we obtain (3.1). The proofs of (3.2) and (3.3) are similar, so we omit the details. The proof of (3.4) is on the following lines. Since

$$\int_{-1}^{1}\frac{(1-x^2)(x-x_{kn})l_{kn}^2(x)l'_{kn}(x)}{(1-x_{kn}^2)}(1-x^2)^{-1/2}\,dx=$$

$$=\frac{1}{3}\int_{-1}^{1}\frac{(1-x^2)^{1/2}(x-x_{kn})}{(1-x_{kn}^2)}\frac{d}{dx}l_{kn}^3(x)\,dx=$$

$$=-\frac{1}{3}\int_{-1}^{1}\frac{l_{kn}^3(x)}{(1-x_{kn}^2)}\left((1-x^2)^{1/2}-\frac{x(x-x_{kn})}{(1-x^2)^{1/2}}\right)dx=$$

$$=-\frac{1}{3}\int_{-1}^{1}\frac{l_k^3(x)}{(1-x_{kn}^2)}\left(\frac{2(1-x^2)-(1-xx_{kn})}{(1-x^2)^{1/2}}\right)dx=$$

$$=-\frac{2}{3}\int_{-1}^{1}\frac{(1-x^2)}{(1-x_{kn}^2)}l_{kn}^3(x)(1-x^2)^{-1/2}\,dx+\frac{1}{3}\int_{-1}^{1}\frac{(1-xx_{kn})}{(1-x_{kn}^2)}l_{kn}^3(x)(1-x^2)^{-1/2}\,dx.$$

Now applying (3.2) and (3.3) we obtain (3.4). This proves the lemma.

Lemma 3.2. *For* $k=1,2,\ldots,n$ *we have*

(3.6) $$\int_{-1}^{1}l'_{kn}(t)(1-t^2)^{-1/2}\,dt=\frac{\pi}{n}\left[T'_n(x_{kn})-\frac{x_{kn}}{1-x_{kn}^2}\right],$$

and

(3.7) $$\int_{-1}^{1}tl''_{kn}(t)(1-t^2)^{-1/2}\,dt=$$
$$=\frac{\pi}{n}\frac{x_{kn}}{1-x_{kn}^2}+\frac{\pi}{2}nT'_n(x_{kn})-\frac{(2+x_{kn}^2)}{(1-x_{kn}^2)}\int_{-1}^{1}l'_{kn}(t)(1-t^2)^{-1/2}\,dt.$$

Proof. It is well known that

(3.8) $$l_{kn}(x)=\frac{1}{n}+\frac{2}{n}\sum_{r=1}^{n-1}T_r(x_{kn})T_r(x)$$

and

(3.9) $$T'_{2r}(x) = 4r \sum_{i=1}^{r} T_{2i-1}(x), \quad T'_{2r-1}(x) = (2r-1)\left[1+2 \sum_{i=1}^{r} T_{2i}(x)\right].$$

Therefore

(3.10) $$\int_{-1}^{1} l'_{kn}(t)(1-t^2)^{-1/2}\,dt = \frac{2\pi}{n} \sum_{r=1}^{n/2} (2r-1)T_{2r-1}(x_{kn}).$$

But a simple computation shows that

(3.11) $$2\sum_{r=1}^{[n/2]} (2r-1)T_{2r-1}(x_{kn}) = T'_n(x_{kn}) - x_{kn}/(1-x_{kn}^2).$$

From (3.10), (3.11) we obtain (3.6). For the proof of (3.7) we first note from (3.8), (3.9)

$$\int_{-1}^{1} tl''_{kn}(t)(1-t^2)^{-1/2}\,dt = \frac{2}{n}\sum_{r=1}^{n/2} T_{2r-1}(x_{kn}) \int_{-1}^{1} tT''_{2r-1}(t)(1-t^2)^{-1/2}\,dt +$$

$$+\frac{2}{n}\sum_{r=1}^{n/2} T_{2r}(x_{kn}) \int_{-1}^{1} tT''_{2r}(t)(1-t^2)^{-1/2}\,dt.$$

But

$$\int_{-1}^{1} tT''_{2r}(t)(1-t^2)^{-1/2}\,dt = 0,$$

and

$$2\int_{-1}^{1} tT''_{2r-1}(t)(1-t^2)^{-1/2}\,dt = ((2r-1)^2-1) \int_{-1}^{1} T'_{2r-1}(t)(1-t^2)^{-1/2}\,dt =$$

$$= ((2r-1)^3-(2r-1))\pi.$$

Therefore

(3.12) $$\int_{-1}^{1} tl''_{kn}(t)(1-t^2)^{-1/2}\,dt = \frac{\pi}{n}\sum_{r=1}^{n/2} T_{2r-1}(x_{kn})((2r-1)^3-(2r-1)).$$

But a simple computation shows that

(3.13) $$\sum_{r=1}^{n/2} (2r-1)^3 T_{2r-1}(x_{kn}) = \frac{x_{kn}}{1-x_{kn}^2} + \frac{n^2T'_n(x_{kn})}{2} - \frac{3n(1+x_{kn}^2)}{(1-x_{kn}^2)} \int_{-1}^{1} \frac{l'_{kn}(t)}{(1-t^2)^{1/2}}\,dt.$$

From (3.10), (3.17), (3.13) we obtain (3.7). This proves Lemma 3.2.

4. Proof of Theorem 1. First we will show that

$$\int_{-1}^{1} w_{kn}(t)(1-t^2)^{-1/2}\,dt = \frac{(1-x_{kn}^2)^2}{12n^2}\int_{-1}^{1} l_{kn}'(t)(1-t^2)^{-1/2}\,dt = \tag{4.1}$$

$$= \frac{\pi}{12n^3}(1-x_{kn}^2)^2\left(T_n'(x_{kn})-\frac{x_{kn}}{1-x_{kn}^2}\right).$$

Since

$$T_n^2(x) = (1+T_{2n}(x))/2, \tag{4.2}$$

it follows from (2.6) and orthogonal properties of Tchebycheff polynomials

$$\int_{-1}^{1} w_{kn}(t)(1-t^2)^{-1/2}\,dt =$$

$$= ((1-x_{kn}^2)/12n^2)\int_{-1}^{1}(1+T_{2n}(t))q_{n-1,k}(t)(1-t^2)^{-1/2}\,dt = \tag{4.3}$$

$$= ((1-x_{kn}^2)^2/12n^2)\int_{-1}^{1} q_{n-1,k}(t)(1-t^2)^{-1/2}\,dt.$$

On differentiating (2.7) twice it follows that

$$(1-x^2)q_{n-1,k}''(x)-xq_{n-1,k}'(x)+q_{n-1,k}(x) = a_{kn}T_n'(x)+l_{kn}'(x). \tag{4.4}$$

Next, we note that

$$\int_{-1}^{1}((1-x^2)q_{n-1,k}''(x)-xq_{n-1}'(x))(1-x^2)^{-1/2}\,dx = \tag{4.5}$$

$$= \int_{-1}^{1}(d((1-x^2)^{1/2}q_{n-1}'(x))/dx)\,dx = 0.$$

Therefore on using (4.4) and (4.5) we obtain

$$\int_{-1}^{1} q_{n-1,k}(x)(1-x^2)^{-1/2}\,dx = \tag{4.6}$$

$$= a_{kn}\int_{-1}^{1} T_n'(x)(1-x^2)^{-1/2}\,dx+\int_{-1}^{1} l_{kn}'(x)(1-x^2)^{-1/2}\,dx.$$

We also note that n is an even positive integer. Therefore $T_n'(x)$ is an odd polynomial of x, and it follows that

$$\int_{-1}^{1} T_n'(x)(1-x^2)^{-1/2}\,dx = 0. \tag{4.7}$$

On using (4.6), (4.7) and (4.3) we obtain (4.1). We also obtain from (3.6) the second part of (4.1). Next we will prove that

$$(4.8)\quad \int_{-1}^{1} v_{in}(x)(1-x^2)^{-1/2}\,dx = -\frac{\pi}{4n}(1-x_{in}^2)\,T'(x_{in})+\left(\frac{3+2(2n^2+1)(1-x_{in}^2)}{12n^2}\right)\int_{-1}^{1}\frac{l_{in}'(t)}{(1-t^2)^{1/2}}\,dt.$$

On using (2.8), (3.1) and (4.2) we obtain

$$(4.9)\quad \int_{-1}^{1} v_{in}(x)(1-x^2)^{-1/2}\,dx = \frac{\pi}{2n^3}x_{in}+\frac{1}{2T_n'(x_{in})}\int_{-1}^{1}\frac{(1+T_{2n}(t))s_{n-1}(t)}{(1-t^2)^{1/2}}\,dt = \frac{\pi}{2n^3}x_{in}+\frac{1}{2T_n'(x_{in})}\int_{-1}^{1}\frac{s_{n-1}(t)}{(1-t^2)^{1/2}}\,dt.$$

Next, differentiating twice we obtain from (2.9)

$$(4.10)\quad (1-x^2)s_{n-1}''(x)-xs_{n-1}'(x)+s_{n-1}(x) = \alpha_{in}T_n'(x)+\beta_{in}l_{in}'(x)+F_{in}'(x).$$

On using (4.5), (4.7) we obtain

$$(4.11)\quad \int_{-1}^{1}\frac{s_{n-1}(x)}{(1-x)^{1/2}}\,dx = \beta_{in}\int_{-1}^{1}\frac{l_{in}'(x)}{(1-x^2)^{1/2}}\,dx+\int_{-1}^{1}\frac{F_{in}'(x)}{(1-x^2)^{1/2}}\,dx,$$

where β_{in} is given by (2.12) and $F_{in}(x)$ by (2.10). From (2.10) we obtain

$$\int_{-1}^{1}\frac{F_{in}'(x)}{(1-x^2)^{1/2}}\,dx = \int_{-1}^{1}\frac{(1-t^2)l_{in}'''(t)-3tl_{in}''(t)+n^2l_{in}'(t)}{2T_n'(x_{in})(1-t^2)^{1/2}}\,dt =$$

$$= \int_{-1}^{1}\frac{(1-t^2)l_{in}'''(t)-tl_{in}''(t)}{2T_n'(x_{in})(1-t^2)^{1/2}}\,dt+\int_{-1}^{1}\frac{-2tl_{in}''(t)+n^2l_{in}'(t)}{2T_n'(x_{in})(1-t^2)^{1/2}}\,dt =$$

$$= \int_{-1}^{1}\frac{(d/dt)((1-t^2)^{1/2}l_{in}''(t))}{2T_n'(x_{in})}\,dt+\int_{-1}^{1}\frac{-2tl_{in}''(t)+n^2l_{in}'(t)}{2T_n'(x_{in})(1-t^2)^{1/2}}\,dt =$$

$$= \int_{-1}^{1}\frac{-2tl_{in}''(t)+n^2l_{in}'(t)}{2T_n'(x_{in})(1-t^2)^{1/2}}\,dt.$$

Therefore, on using (4.9), (4.11), (2.12) together with the above statement it follows that

$$(4.12)\quad \int_{-1}^{1} v_{in}(x)\frac{1}{(1-x^2)^{1/2}}\,dx = \frac{\pi}{2n^3}x_{in}-\frac{(1-x_{in}^2)}{2n^2}\int_{-1}^{1}\frac{xl_{in}''(x)}{(1-x^2)^{1/2}}\,dx+ +(1-x_{in}^2)\left(\frac{1}{4}+\frac{1}{4n^2}\left(\frac{n^2-7}{3}-\frac{5x_{in}^2}{1-x_{in}^2}\right)\right)\int_{-1}^{1}\frac{l_{in}'(t)}{(1-t^2)^{1/2}}\,dt.$$

From (3.7) we have

$$\int_{-1}^{1} \frac{t l_{in}''(t)}{(1-t^2)^{1/2}}\,dt = \frac{\pi}{n}\frac{x_{in}}{1-x_{in}^2} + \frac{\pi}{2} nT_n'(x_{in}) - \frac{2+x_{in}^2}{1-x_{in}^2}\int_{-1}^{1} \frac{l_{in}'(t)}{(1-t^2)^{1/2}}\,dt. \tag{4.13}$$

Now, on using (4.12) and (4.13) we obtain (4.8). Lastly, from (2.14), (3.2), (3.4), (4.1), (4.8) and (3.7) after simplyfying we obtain

$$\int_{-1}^{1} u_{in}(x)\frac{1}{(1-x^2)^{1/2}}\,dx =$$

$$= \frac{\pi}{n}\left\{1 + \frac{1}{3(1-x_{in}^2)}\left(\frac{1}{n^2}-1\right) - \frac{1}{2n^2(1-x_{in}^2)^2} + \frac{(1-x_{in}^2)T_n''(x_{in})}{4} + \frac{T_n''(x_{in})}{2n^2}\right\}. \tag{4.14}$$

From (4.1), (4.8) and (4.14) one can prove Theorem 1.

The proofs of Theorem 2 and Theorem 3 follow easily from Theorem 1, so we omit the details.

References

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