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Remarks on the strong summability of numerical and orthogonal series by some methods of Abel type

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In this paper we shall prove that Leindler's theorems on the strong summability of orthogonal series, given in [2], are true for the methods $(A; B_n)$ of Abel type.

1. Let $C^\infty(0, 1\rangle$ be the class of real functions defined in $(0, 1\rangle$ and having derivatives of all orders in $(0, 1\rangle$. Denote by $B_n=\{b_k(r; n)\}_{k=0}^n$ a sequence of functions of the class $C^\infty(0, 1\rangle$ and such that

$$b_0(r; n) \equiv 1; \left(\frac{d^p}{dr^p} b_k(r; n)\right)_{r=1} = \begin{cases} 0 & \text{if } p \neq k, \\ (-1)^p & \text{if } p = k, \end{cases} \tag{1}$$

for $k, p=0, 1, \ldots, n$. As in [3] we write

$$R_k(r; B_n) = \sum_{p=0}^{n} b_p(r; n) \frac{d^p}{dr^p} r^k$$

and $\Delta R_k(r; B_n)=R_k(r; B_n)-R_{k+1}(r; B_n)$ for $k=0, 1, \ldots$ and $r\in(0, 1\rangle$. In [3] the following definition is given: A real numerical series

$$\sum_{k=0}^{\infty} u_k \qquad (S_k = u_0+\ldots+u_k) \tag{2}$$

is summable to s by the method $(A; B_n)$ if the series $\sum_{k=0}^{\infty} r^k u_k$ is convergent in $(0, 1)$ and if the function $L(B_n)$,

$$L(r; B_n) = \sum_{k=0}^{\infty} R_k(r; B_n) u_k \equiv \sum_{k=0}^{\infty} \Delta R_k(r; B_n) S_k \tag{3}$$

$(r\in(0, 1))$ satisfies the condition $\lim_{r\to 1-} L(r; B_n)=s$. The classical Abel method, i.e. $(A; B_0)$ method, will be denoted by (A). We shall write $L(r)$ for $L(r; B_0)$.

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In [3] and [4] there were given fundamental properties of the methods $(A; B_n)$ of summability of numerical and orthogonal series, and some applications of those methods to the Dirichlet problem for some equations of Laplace type. In [3] and [4] it was proved that

Theorem A. *Series* (2) *is summable to s by a method* $(A; B_n)$ *if and only if*

$$\lim_{r\to 1-} (1-r)^p \frac{d^p}{dr^p} L(r) = \begin{cases} s & \text{if } p=0, \\ 0 & \text{if } p=1, \ldots, n. \end{cases}$$

If the sequence B_n is defined as follows:

$$b_0(r;\, n) = 1, \quad b_n(r;\, n) = (r-r^2)^n/n!, \tag{4}$$

$$b_k(r;\, n) = b_k(r;\, n-1) + \frac{r-r^2}{n}\left(b_{k-1}(r;\, n-1) + \frac{d}{dr} b_k(r;\, n-1)\right)$$

for $k=1, \ldots, n-1$ and $n=0, 1, \ldots$, then

$$L(r;\, B_n) = (1-r)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{n} r^k S_k$$

for $n=0, 1, \ldots$ and $r\in\langle 0, 1)$ ([3], [4]).

Moreover in [3] it was proved that

$$\Delta R_k(r;\, B_n) = \sum_{p=0}^{n} W_p(r;\, B_n) \frac{d^p}{dr^p} r^k \tag{5}$$

for $r\in(0, 1\rangle$, $k=0, 1, \ldots$ and $n\geqq 0$, if $W_p(B_n)$ are some functions of the class $C^\infty(0, 1\rangle$ with

$$\left(\frac{d^q}{dr^q} W_p(r;\, B_n)\right)_{r=1} = 0 \quad \text{for} \quad p, q = 0, 1, \ldots, n. \tag{6}$$

From (5)—(6) and from the Taylor formula for $W_p(B_n)$ we obtain

Lemma 1. *Let* $r_0\in(0, 1)$. *Then, for every sequence* B_n, *there exist positive constants* $M_1(r_0)$ *and* $M_2(r_0)$ *depending on* r_0 *such that*

$$M_1(r_0)(1-r)^{n+1}(k+1)^n r^k \leqq |\Delta R_k(r; B_n)| \leqq M_2(r_0)(1-r)^{n+1}(k+1)^n r^k$$

for $k=0, 1, \ldots$ *and* $r\in\langle r_0, 1)$.

2. We shall say that series (2) is strongly $(A; B_n)$-summable to s with exponent $q>0$ if the function $H(B_n, q)$,

$$H(r;\, B_n, q) = \sum_{k=0}^{\infty} |\Delta R_k(r;\, B_n)|\, |S_k - s|^q \tag{7}$$

$(r\in(0, 1))$ satisfies the condition

$$\lim_{r\to 1-} H(r;\, B_n, q) = 0. \tag{8}$$

By (7), (8) and Lemma 1, we obtain

Lemma 2. *Series* (2) *is strongly* $(A;\, B_n)$*-summable to* s *with exponent* $q>0$ *if and only if*

$$\lim_{r\to 1-}(1-r)^{n+1}\sum_{k=0}^{\infty}(k+1)^n r^k |S_k-s|^q=0. \tag{9}$$

Lemma 2 implies

Corollary 1. *For a fixed* n*, the methods* $(A;B_n)$*, for the strong summability of numerical series* (2) *are equivalent, i. e. if* B_n *and* B_n^* *are two sequences having properties* (1)*, then* $\lim_{r\to 1-} H(r;\, B_n,\, q)=0$ *if and only if* $\lim_{r\to 1-} H(r;\, B_n^*,\, q)=0$.

Corollary 2. *If series* (2) *is strongly* $(A;\, B_n)$*-summable to* s *with exponent* $q_1>0$*, then it is strongly* $(A;\, B_n)$*-summable to* s *with every exponent* $0<q<q_1$.

The next statement is obvious:

Lemma 3. *If series* (2) *is strongly* $(C,\, 1)$*-summable to* s *with exponent* $q>0$*, i.e.*

$$\lim_{n\to\infty}(1/(n+1))\sum_{k=0}^{n}|S_k-s|^q=0,$$

then

$$\lim_{n\to\infty}(1/(n+1)^p)\sum_{k=0}^{n}(k+1)^{p-1}|S_k-s|^q=0$$

for every $p>1$.

Applying Lemma 3, we shall prove

Lemma 4. *If series* (2) *is strongly* $(C,\, 1)$*-summable to* s *with exponent* $q>0$*, then condition* (9) *is satisfied for* $n=0,\, 1,\, \ldots$.

Proof. If (2) is strongly $(C,\, 1)$-summable with exponent $q>0$, then $|S_k-s|^q=o(k)$. Hence the series $\sum_{k=0}^{\infty}(k+1)^n r^k|S_k-s|^q$ is convergent in $\langle 0,\, 1)$ and for $n=0,\, 1,\, \ldots$. Applying the Abel transformation, we get

$$(1-r)^{n+1}\sum_{k=0}^{\infty}(k+1)^n r^k|S_k-s|^q=(1-r)^{n+2}\sum_{k=0}^{\infty}(k+1)^{n+1}r^k l_k$$

for $r\in\langle 0,\, 1)$ and $n=0,\, 1,\, \ldots$, where

$$l_k=(1/(k+1)^{n+1})\sum_{p=0}^{k}(p+1)^n|S_p-s|^q.$$

By Lemma 3 and by Toeplitz's theorem ([1], p. 14), we obtain (9) for $n=0,\, 1,\, \ldots$. Thus the proof is complete.

Lemma 5. *Suppose that for series* (2) *condition* (9) *holds with* $n=p+1$ $(p\in N)$ *and with some exponent* $q>0$. *Then* (9) *also holds with* $n=p$ *and with exponent* q.

Proof. Let

$$U_n(r;\, q) = (1-r)^{n+1} \sum_{k=0}^{\infty} (k+1)^n r^k |S_k - s|^q$$

for $r\in\langle 0, 1)$. Hence

$$rU_p(r;\, q) = (1-r)^{p+1} \int_0^r (1-t)^{-p-2} U_{p+1}(t;\, q)\, dt$$

for $r\in\langle 0, 1)$. But,

$$\int_0^r (1-t)^{-p-2} U_{p+1}(t;\, q)\, dt = o\big((1-r)^{-p-1}\big) \quad (\text{as } r\to 1-)$$

if $\lim_{r\to 1-} U_{p+1}(r;\, q)=0$. This proves our statement.

Lemmas 2 and 5 prove

Theorem 1. *For a fixed integer* $n\geqq 0$, *the following conditions are equivalent for every numerical series* (2), *for every* $q>0$ *and every sequence* B_n: (*here* $H(r;\, q)\equiv \equiv H(r;\, B_0, q)$).

(a) $\lim_{r\to 1-} H(r;\, B_n, q) = 0,$

(b) $\lim_{r\to 1-} (1-r)^{n+1} \sum_{k=0}^{\infty} (k+1)^n r^k |S_k - s|^q = 0,$

(c) $\lim_{r\to 1-} (1-r)^p \dfrac{d^p}{dr^p} H(r;\, q) = 0$ *for* $m = 0, 1, \ldots, n.$

3. Let $\{\varphi_k(x)\}_{k=0}^{\infty}$ be a real and orthonormal system on the interval $\langle 0, 1\rangle$. We shall consider the strong summability of orthogonal series

$$(10) \qquad \sum_{k=0}^{\infty} c_k \varphi_k(x) \quad \text{with} \quad \sum_{k=0}^{\infty} c_k^2 < \infty$$

by the methods $(A;\, B_n)$. The strong summability of (10) by the methods $(A;\, B_n)$, defined by the sequence (4), was examined by L. Leindler [2].

Let $S_k(x)=\sum_{p=0}^{k} c_p \varphi_p(x)$ and let f be the function given by the Riesz—Fischer theorem, having expansion (10). By $L(r, x;\, B_n)$ and $H(r, x;\, B_n, q)$ with $s=f(x)$ $\big(r\in\langle 0, 1),\ x\in\langle 0, 1\rangle\big)$ we denote the functions as in (3) and (7) but for series (10). As usual ([1]) we say that two methods of summability are equivalent in L^2 if the summability of series (10) in a set E of positive measure by one of those methods implies the summability of (10) to the same sum almost everywhere in E by the other method.

In [3] we proved the following

Theorem B. *The methods* $(A; B_n)$, $n=0, 1, \ldots,$ *and the Cesàro* $(C, 1)$ *method of summability for orthogonal series* (10) *are equivalent in* L^2.

In [2] L. Leindler proved

Theorem C. *If orthogonal series* (10) *is Abel-summable to* $f(x)$ *in* $\langle 0, 1\rangle$ *almost everywhere, then*

$$\lim_{r\to 1-} (1-r)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{n} r^k |S_k(x)-f(x)|^q = 0$$

for any $n\in N$ *and* $q>0$ *in* $\langle 0, 1\rangle$ *almost everywhere.*

Applying Theorem C and Corollary 1, or arguing as in [2], we obtain

Theorem 2. *If orthogonal series* (10) *is* (A)*-summable to* $f(x)$ *in* $\langle 0,1\rangle$ *almost everywhere, then it is strongly* $(A; B_n)$*-summable to* $f(x)$ *in* $\langle 0, 1\rangle$ *almost everywhere with every exponent* $q>0$ *and every sequence* B_n $(n=0, 1, \ldots)$.

Applying Lemma 1 and arguing as in the proof of Theorem 5 given in [2], we can prove

Theorem 3. *Suppose that* α *and* q *are two positive numbers and* B_n $(n\geqq 0)$ *is a sequence having properties* (1). *If the coefficients of series* (10) *satisfy the condition*

$$\sum_{k=1}^{\infty} c_k^2 k^{2\alpha} < \infty,$$

then

$$H(r, x; B_n, q) = o_x((1-r)^\alpha)$$

if $q\alpha<1$; *and*

$$H(r, x; B_n, q) = \begin{cases} o_x((1-r)^\alpha) & \text{if } n+1 > q\alpha, \\ o_x((1-r)^\alpha |\log (1-r)|^{1/q}) & \text{if } n+1 = q\alpha, \\ O_x((1-r)^{(n+1)/q}) & \text{if } n+1 < q\alpha \end{cases}$$

in the case $q\alpha\geqq 1$ *but* $0<q\leqq 2$, *almost everywhere in* $\langle 0, 1\rangle$ *as* $r\to 1-$.

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