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Behavior of the extended Cauchy representation of distributions

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1. Introduction. There is an interesting correspondence between the spaces $\mathcal{O}'_\alpha=\mathcal{O}'_\alpha(\mathbf{R})$ of distributions and the class of functions that are analytic in the complex plane $\mathbf{C}$ with a boundary on $\mathbf{R}$, except for a set of points lying in $\mathbf{C}-\mathbf{R}$, and vanish at the point of infinity. In the present paper we make a study of this correspondence. As we shall see it depends essentially on the support of distributions, order relation of included functions, and their distributional boundary value in either half plane

$$\Delta^+=\{z\in\mathbf{C}:\ \mathrm{Im}(z)>0\},\quad \Delta^-=\{z\in\mathbf{C}:\ \mathrm{Im}(z)<0\},\quad z=x+iy.$$

The problem under study is motivated by some facts from the theory of integrals of the Cauchy type. Namely, to every function u: $\mathbf{R}\to\mathbf{C}$ Hölder continuous with compact support equal K there corresponds the function

$$\hat{u}(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{u(t)}{t-z}\,dt,\quad z\bar{\in}K.$$

If $v(z)\equiv\hat{u}(z)$, then: (1) v is an analytic function in $\mathbf{C}-K$; (2) $v(z)$ has the boundary values $v^+(x)$ and $v^-(x)$ in $\mathbf{C}$; (3) $v(z)=O(1/|z|)$ as $|z|\to\infty$. Conversely, given a function v which satisfies the conditions (1)—(3), then it is the Cauchy integral of some u with $\mathrm{supp}\,u=K$.

If T is a distribution, then the notation T_t is used to indicate that the testing functions on which T is defined have t as their variable. The pairing between a testing function space and its dual is denoted by $\langle T,\varphi\rangle$. The space of $C^\infty=C^\infty(\mathbf{R})$ functions having compact support is denoted by $\mathcal{D}=\mathcal{D}(\mathbf{R})$; its dual $\mathcal{D}'=\mathcal{D}'(\mathbf{R})$ is the space of Schwartz distributions on $\mathbf{R}$. As regards the general properties of the spaces $\mathcal{O}_\alpha$ and $\mathcal{O}'_\alpha$ we refer to [1].

2. Definitions. In order to describe the correspondence in question in a condensed form we introduce some classes of functions.

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Let K be a closed subset of $\mathbf{R}$ and let $\{a_1, a_2, \dots, a_n\}$ $(n \in N)$ be a finite set of distinct complex points lying in $\Delta^+ U \Delta^-$. A function f is said to belong to the class $\{M\}$ if

(m.1) f is analytic in the domain $\mathbf{C}-(K \cup \{a_1, a_2, \dots, a_n\})$, a_k being a pole of order α_k $(k=1, 2, \dots, n)$;

(m.2) $f(z)$ converges (weakly) in either half plane to a $\mathcal{D}'$-boundary value;

$$\text{(m.3)} \qquad f(z) = O(1/|z|) \quad \text{as} \quad |z| \to \infty.$$

We use the notation $\{M_c\}$ for the class of functions in $\{M\}$ that satisfy (m.1) when K is a compact set. Also, a function f is said to belong to the class $\{M_0\}$ if it satisfies the conditions (m.1), (m.2) and the condition

$$\text{(m.4)} \qquad f(\infty) \equiv \lim_{z \to \infty} f(z) = 0.$$

Thus we have the inclusions $\{M_c\} \subset \{M\} \subset \{M_0\}$. Further, the class of functions that satisfy the conditions (m.1)—(m.3) when the set of poles is empty is denoted by $\{A\}$. The class $\{A_c\}$ is the subclass of $\{A\}$ relative to compact set K in (m.1). If here the condition (m.3) is replaced by (m.4) we have the class $\{A_0\}$.

Remark. The arbitrary sets involved in (m.1) are not necessarily the same for all functions in a class defined above.

Now denote by $R(z)$ a meromorphic function, vanishing at the point $z=\infty$, with prescribed poles $a_1, a_2, \dots, a_n$ (in $\Delta^+ \cup \Delta^-$) and their principal parts. Let $T \in \mathcal{O}'_\alpha$ $(\alpha \geqq -1)$. The function $\hat{F}$ from $\mathbf{C}-(\text{supp}\, T \cup \{a_1, a_2, \dots, a_n\})$ to $\mathbf{C}$ defined by

$$\text{(1)} \qquad \hat{F}(z) = (1/2\pi i)\langle T_t, 1/(t-z)\rangle + R(z)$$

will be referred to as *the extended Cauchy representation of* T.

Let us observe that every function f in $\{A_0\}$ ($\{M_0\}$) is sectionally analytic in $\mathbf{C}$ with a boundary on $\mathbf{R}$ (except for the poles), that is, it can be decomposed into two independent functions $f^+(z)$ and $f^-(z)$ such that $f(z)=f^+(z)$ for $z \in \Delta^+$, $f(z)=$ $=f^-(z)$ for $z \in \Delta^-$ (the half planes being punctured at the points of poles).

3. Main result. We need the following

Lemma [5]. *If $f^+(z)$ is a function analytic in Δ^+ with $f^+(z)=O(1/|z|)$ as $|z| \to \infty$ in Δ^+, and if $f^+(x+i\varepsilon)$ converges to $\mathcal{D}'$-boundary value f_x^+ as $\varepsilon \to +0$, then*: 1) *f_x^+ belongs to $\mathcal{O}'_\alpha$ for all $\alpha<0$*; 2) *$f^+(x+i\varepsilon)$ converges to $\mathcal{O}'_\alpha$-boundary value f_x^+ as $\varepsilon \to$ $\to +0 (\alpha<0)$*; 3) *f_x^+ generates the Cauchy representation*

$$\text{(2)} \qquad (1/2\pi i)\langle f_t^+, 1/(t-z)\rangle = \begin{cases} f^+(z) & \textit{for} \quad z \in \Delta^+, \\ 0 & \textit{for} \quad z \in \Delta^-. \end{cases}$$

For a function $f^-(z)$ analytic in Δ^- and satisfying here the conditions similar to ones of $f^+(z)$, we have

(3) $$-(1/2\pi i)\langle f_t^-, 1/(t-z)\rangle = \begin{cases} f^-(z) & \text{for } z\in\Delta^-, \\ 0 & \text{for } z\in\Delta^+. \end{cases}$$

The distributional version of the previous discussion concerning the integral of the Cauchy type leads to the following

Theorem. *Let* $T\in\mathcal{O}'_\alpha$ $(\alpha\geqq -1)$ *with* supp $T=K$ *and let* $\{a_1, a_2, \dots, a_n\}$ $(n\in N)$ *be a set of distinct complex points located in* $\Delta^+\cup\Delta^-$. *If* $f(z)\equiv\hat{F}(z)$, *then* $f\in\{M_0\}$. *Conversely, given an* $f\in\{M\}$, *then it is the extended Cauchy representation of some* $T\in\mathcal{O}'_\alpha$ *for all* $\alpha\in[-1, 0)$ *with* supp $T=K$.

Proof. Consider the direct part of the theorem. To prove the statement (m.1) it suffices to note that the Cauchy representation $\hat{T}(z)$ of T is an analytic function in the domain $\mathbf{C}-K$ ([1, p. 56]). The statement (m.2) follows directly from [4, Theorem 2]:

$$f_x^+ = \hat{F}_x^+ = T_x/2-(1/2\pi i)(T_x * \mathrm{vp}\, 1/x)+R(x),$$

$$f_x^- = \hat{F}_x^- = -T_x/2-(1/2\pi i)(T_x * \mathrm{vp}\, 1/x)+R(x).$$

Observe that the rational function $R(x)$ is a regular distribution (in $\mathcal{O}'_\alpha$ for all $\alpha<0$). As regards the statement (m.4) it is a simple consequence of the hypothesis $R(\infty)=0$ and the fact that every sequence of functions $\varphi_n(t)=1/(t-z_n)$ converges to zero in $\mathcal{O}'_\alpha$ $(\alpha\geqq -1)$ as $z_n\to\infty(n\to\infty)$.

Conversely, suppose given an $f\in\{M\}$. Then in view of Lemma the assertions (m.2) and (m.3) together imply $f_x^+\in\mathcal{O}'_\alpha$, $f_x^-\in\mathcal{O}'_\alpha$ for all $\alpha<0$. Now define $T_x= =f_x^+-f_x^-$. Since $(f_x^+-f_x^-)\in\mathcal{O}'_\alpha$ for all $\alpha<0$ and the Cauchy kernel belongs to $\mathcal{O}'_\alpha$ for all $\alpha\geqq -1$, we can associate to T the Cauchy representation

$$\hat{T}(z) = (1/2\pi i)\langle T_t, 1/(t-z)\rangle = (1/2\pi i)\langle (f_t^+ - f_t^-), 1/(t-z)\rangle$$

for all $\alpha\in[-1, 0)$. Clearly, $\hat{T}$ is analytic in $\mathbf{C}-$supp T and vanishes at the point $z=\infty$; moreover, it is easy to show that in this situation $\hat{T}(z)=O(1/|z|)$ as $|z|\to\infty$. To prove that f is the extended Cauchy representation of T first we shall show that the function H from $\mathbf{C}-(\text{supp } T\cup\{a_1, a_2, \dots, a_n\})$ to $\mathbf{C}$ defined by

(4) $$H(z) = f(z)-\hat{T}(z)$$

is meromorphic in $\mathbf{C}$. In fact, after a simple computation we have

$$\langle (H_x^+ - H_x^-), \varphi\rangle = \langle (f_x^+ - f_x^-), \varphi\rangle = \langle (\hat{T}_x^+ - \hat{T}_x^-), \varphi\rangle$$

for all $\varphi\in\mathcal{D}$. Since $T_x=\hat{T}_x^+-\hat{T}_x^-$ it follows

(5) $$\langle H_x^+, \varphi\rangle = \langle H_x^-, \varphi\rangle$$

for all $\varphi\in\mathcal{D}$. Further, let $\Delta=\{z\in\mathbf{C}:\ -d<\operatorname{Im}(z)<d\}$ be the strip of the half height

$$d = \operatorname{Min}\{|\operatorname{Im}(a_1)|, |\operatorname{Im}(a_2)|, \ldots, |\operatorname{Im}(a_n)|\}.$$

Let (a, b) be an arbitrary finite open interval in $\mathbf{R}$, E^+ and E^- two open rectangles contained in Δ which have (a, b) as a common edge. Evidently, H is an analytic function in Δ except the boundary on $\mathbf{R}$ consisting of the set $\operatorname{supp} T\cup K$. Applying the distributional analytic continuation principle ([6], [3, p. 244]) the equality (5) implies that the function H is analytic in $E^+\cup(a, b)\cup E^-$, and consequently, in all of Δ. Thus H is analytic everywhere in $\mathbf{C}$ except for the poles a_k of f, and as a meromorphic function which vanishes at the point of infinity it may be written uniquely in the form

$$H(z) = \sum_{k=1}^{n} \sum_{p=1}^{\alpha_k} B_{k,p}/(z-a_k)^p, \tag{6}$$

where the coefficients $B_{k,p}$ must be determined (by means of f). Since the function $\hat{T}$ is analytic in $\mathbf{C}-\operatorname{supp} T$, using Theorem on the partial fraction expansion of rational functions ([2]) from (4) we get

$$B_{k,\alpha_k-m} = (1/m!) \lim_{z\to a_k} d^m[(z-a_k)^{\alpha_k} f(z)]/dz^m \tag{7}$$

$(m=0, 1, 2, \ldots, \alpha_k-1)$. Returning to equality (4) with (6) and (7) it follows the representation

$$f(z) = (1/2\pi i)\langle T_t, 1/(t-z)\rangle + \sum_{k=1}^{n} \sum_{p=1}^{\alpha_k} B_{k,p}/(z-a_k)^p. \tag{8}$$

So we have established that the given $f\in\{M\}$ is the extended Cauchy representation of $T_x=(f_x^+-f_x^-)\in\mathcal{O}_\alpha'$ for all $\alpha\in[-1, 0)$. Next we have to prove that $\operatorname{supp} T=K$. First let K be a closed proper subset of $\mathbf{R}$. Since the function f is analytic on the open set $\mathbf{R}-K$, it follows that

$$\langle f_x^+, \varphi\rangle = \lim_{\varepsilon\to+0} \langle f^+(x+i\varepsilon), \varphi\rangle = \lim_{\varepsilon\to+0} \langle f^-(x-i\varepsilon), \varphi\rangle = \langle f_x^-, \varphi\rangle$$

for all φ with support disjoint from K $(\varphi\in\mathcal{D}(\mathbf{R}-K))$. Thus $\langle(f_x^+-f_x^-), \varphi\rangle = \langle T_x, \varphi\rangle=0$ for all such φ. Hence we may conclude that $\operatorname{supp} T=K$. The assumption that $\operatorname{supp} T\subset K$ properly leads to the conclusion that there exists an open interval $(a, b)\subset(K-\operatorname{supp} T)$ on which $(f_x^+-f_x^-)$ is zero of $\mathcal{D}'((a, b))$. Therefore (by the analytic continuation principle) f would be analytic on $(\mathbf{R}-K)\cup(a, b)$ contrary to the hypothesis. For the same reason $\operatorname{supp} T=\mathbf{R}$ in the case $K=\mathbf{R}$. Finally suppose that there exists a distribution $S\in\mathcal{O}_\alpha'$ $(\alpha\geqq-1)$ distinct from T and such that $f(z)=\hat{S}(z)+H(z)$ for $z\in\mathbf{C}-(K\cup\{a_1, a_2, \ldots, a_n\})$. According to [4, Theorem 2] we have $f_x^+-f_x^-=\hat{S}_x^+-\hat{S}_x^-=S_x$. Hence $T_x=S_x$ on $\mathcal{D}$ and this implies $T_x=S_x$ on $\mathcal{O}_\alpha$ (since $\mathcal{D}$ is dense in $\mathcal{O}_\alpha$ for all $\alpha\in\mathbf{R}$). But this contradicts the hypothesis on S. Thus, the distribution T is unique. The proof is complete.

In particular, if all poles a_k of the function f are simple $(k=1, 2, \dots, n)$, then in the representation (8) instead of double sum we have

$$\sum_{k=1}^{n} \operatorname{res}[f(z), a_k]/(z-a_k).$$

4. Consequences. First assume that the set of poles of the function $\hat{F}$ is empty. In this case $\hat{F}$ is reduced to the Cauchy representation $\hat{T}$ of T. Thus we have at once

Corollary 1. *Let* $T\in\mathcal{O}'_\alpha$ $(\alpha\geqq-1)$ *with* $\operatorname{supp} T=K$. *If* $f(z)\equiv\hat{T}(z)$, *then* $f\in\{A_0\}$. *Conversely, given an* $f\in\{A\}$, *then it is the Cauchy representation of some* $T\in\mathcal{O}'_\alpha$ *for all* $\alpha\in[-1, 0)$ *with* $\operatorname{supp} T=K$.

Nevertheless we can prove the second part of this Corollary directly, that is, without intervention of the meromorphy. In fact, since the distributions f_x^+ and f_x^- belong to $\mathcal{O}'_\alpha$ $(\alpha<0)$ we may define $T_x=f_x^+-f_x^-$. As T_x is a linear continuous functional on $\mathcal{O}_\alpha$ generating the Cauchy integral $\hat{T}(z)$ we have for all $\alpha\in[-1, 0)$

$$\hat{T}(z) = (1/2\pi i)\langle f_t^+, 1/(t-z)\rangle-(1/2\pi i)\langle f_t^-, 1/(t-z)\rangle.$$

Using the formulas (2) and (3) we get at once the required result

$$\hat{T}(z) = \begin{cases} f^+(z) & \text{for } z\in\Delta^+, \\ f^-(z) & \text{for } z\in\Delta^-, \end{cases}$$

that is, $f(z)=\hat{T}(z)$. So we have proved by Lemma that the given $f\in\{A\}$ is the Cauchy representation of some $T\in\mathcal{O}'_\alpha$.

Denote in Schwartz's notation by $\mathcal{E}'=\mathcal{E}'(\mathbf{R})$ the space of distributions on $\mathbf{R}$ with compact support (recall that $\mathcal{E}'\subset\mathcal{O}'_\alpha$ for all $\alpha\in\mathbf{R}$, but an $T\in\mathcal{O}'_\alpha$ with compact support belongs to $\mathcal{E}'$). From Theorem we derive

Corollary 2. *Let* $T\in\mathcal{E}'$ *with* $\operatorname{supp} T=K$. *If* $f(z)\equiv\hat{F}(z)$, *then* $f\in\{M_c\}$. *Conversely, given an* $f\in\{M_c\}$, *then it is the extended Cauchy representation of some* $T\in\mathcal{E}'$ *with* $\operatorname{supp} T=K$.

We have to comment only the assertion (m.3). The function $\hat{F}$ around the point $z=0$ has the Laurent expansion of the form

$$\hat{F}(z) = c_0+c_1/z+c_2/z^2+\dots$$

which converges uniformly and absolutely outside the smallest disk containing K and all poles a_k $(k=1, 2, \dots, n)$. The fact that $\hat{F}(z)$ vanishes as $z\to\infty$ implies that $c_0=0$, and the required result follows at once.

Consequently, to every pair $(T, \{a_1, a_2, \dots, a_n\})$ with $T\in\mathcal{E}'$, $n\in N$, there corresponds an $f\in\{M_c\}$ and to every $f\in\{M_c\}$ there corresponds a pair $(T, \{a_1, a_2, \dots, a_n\})$ with $T\in\mathcal{E}'$, $n\in N$.

Corollary 3. *Let* $T\in\mathscr{E}'$ *with* $\operatorname{supp} T=K$. *If* $f(z)\equiv\hat{T}(z)$, *then* $f\in\{A_c\}$. *Conversely, given an* $f\in\{A_c\}$, *then it is the Cauchy representation of some* $T\in\mathscr{E}'$ *with* $\operatorname{supp} T=K$.

Thus one can place distributions in $\mathscr{E}'$ into a one-to-one correspondence with functions in $\{A_c\}$.

It may happen that $f\equiv\hat{F}\in\{M\}$ (any given $f\in\{M\}$ is the extended Cauchy representation of some $T\in\mathcal{O}'_\alpha$ $(\alpha\geqq-1)$ with $\operatorname{supp} T=K$). For example, the function f defined by

$$f(z)\equiv\hat{F}(z)=(1/2\pi i)\langle \operatorname{vp} 1/t,\, 1/(t-z)\rangle+1/(z-i)$$

belongs to $\{M\}$ with $K=R$. This follows from

$$f^+(z)=1/2z+1/(z-i),\quad z\in\Delta^+-\{i\},$$

$$f^-(z)=-1/2z+1/(z-1),\quad z\in\Delta^-$$

with $f_x^+=1/2(x+i0)+1/(x-i)$, $f_x^-=1/2(x-i0)+1/(x-i)$. Conversely, given $f^+(z)$ and $f^-(z)$ we reconstruct $f(z)$ starting with $f_x^+-f_x^-=\operatorname{vp} 1/x$.

In addition, by means of the second part of Theorem we solve the following boundary value

Problem 1. *Let T be a given distribution in* $\mathcal{O}'_\alpha$ $(\alpha\geqq-1)$. *Find a function* $f\in\{M\}$ *whose* $\mathscr{D}'$*-boundary values* f_x^+ *and* f_x^- *satisfy the condition* $f_x^+-f_x^-=T_x$ *on* $\mathbf{R}$.

The general solution is given by (8), where $B_{k,p}$ are arbitrary real or complex coefficients.

It is of interest to sketch the following results: if in Corollaries 2—3 we replace (via condition (m.2)) the convergence in the $\mathscr{D}'$ topology by one in $\mathcal{O}'_\alpha$ for a given $\alpha\in[-1,0)$, we get new corollaries 2.1—3.1 respectively.

Fact 1. Corollaries 2—3 are equivalent to Corollaries 2.1—3.1.

To prove this first observe that $f\in\{M\}$ remains in $\{M\}$ if we substitute the convergence in $\mathscr{D}'$ for one in $\mathcal{O}'_\alpha$ $(\alpha\in\mathbf{R})$. Next, we use the representation (8) or Lemma.

Also, if in Problem 1 we replace $T\in\mathcal{O}'_\alpha$ $(\alpha\geqq-1)$ by $T\in\mathcal{O}'_\alpha$ $(-1\leqq\alpha<0)$ and the convergence in $\mathscr{D}'$ by one in $\mathcal{O}'_\alpha$, we come to

Problem 1.1. *Let T be a given distribution in* $\mathcal{O}'_\alpha$ $(-1\leqq\alpha<0)$. *Find a function* $f\in\{M\}$ *whose* $\mathcal{O}'_\alpha$*-boundary values* f_x^+ *and* f_x^- *satisfy the condition* $f_x^+-f_x^-=T_x$ *on* $\mathbf{R}$.

Fact 2. Problem 1 with $T\in\mathcal{O}'_\alpha$ $(-1\leqq\alpha<0)$ is equivalent to Problem 1.1.

Similarly, substituting under previous conditions the class $\{M\}$ for $\{M_c\}$ and $\{A_c\}$ we come to an equivalent Problem 1.2 and Problem 1.3, respectively.

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