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On Stieltjes transform of distributions behaving as regularly varying functions

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1. Introduction. The asymptotics of the Stieltjes transform of distributions T with support in $[0, \infty)$ belonging to a subset, specified below, of the space of (Schwartz) distributions, whose behavior (at zero) is of the type $T \sim x^\nu \log^j x_+$ in the Lojasiewicz sense, or (at ∞) in the Sebastiao E Silva sense, is studied by LAVOINE and MISRA [1], [2]. The aim of this paper is to extend their results to the cases when $T \sim x^\nu L_0(x)_+$, $x \to 0^+$ and $T \sim x^\nu L(x)$, $x \to \infty$. Here L_0 and L belong to the class of slowly varying functions (s.v.f.) introduced by KARAMATA [3] in 1930.

A real valued function $L(x)$ is slowly varying at infinity, if it is positive, measurable on $[a, \infty)$ for some $a > 0$, and such that for each $\lambda > 0$

$$\lim_{x \to \infty} L(\lambda x)/L(x) = 1. \tag{1.1}$$

A function $L_0(x)$ is s.v. at zero if $L_0(1/x)$ is s.v. at infinity. E.g. all positive functions tending to positive constants are s.v. at infinity, products of powers of iterated logarithms, the function $(1/x) \int^x dt/\ln t$, are such, etc.

Slowly varying functions are of frequent occurrence in various branches of analysis (Fourier analysis, number theory, differential equations, Tauberian theorems) and of stochastic processes, whenever more information than the mere fact of convergence is needed. Here we also emphasize the use of s.v.f. in the theory of distributions.

Also, the function $R(x) = x^\nu L(x)$ is called regularly varying at infinity with index ν.

1.1. Following [2] we denote by $J'(r)$, $\operatorname{Re} r > -1$, the space of distributions T with support in $[0, \infty)$, admitting the decomposition

$$T = B + D^k f(x) \tag{1.2}$$

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where k is a non-negative integer, D^k is the distributional differentiation operator of order k, $f(x)$ is a locally integrable function with support in $[a, \infty)$ for some $a \geqq 0$ and such that

$$\int^{\infty} |f(x)x^{-r-k-1}|\,dx < \infty, \tag{1.3}$$

and B is a distribution with support in $[0, a]$. Notice that T is a tempered distribution.

Further, the Stieltjes transform of $T \in J'(r)$ is defined by

$$F(s) = \mathcal{S}_s\{T\} = \langle T_x, (x+s)^{-r-1}\rangle = \tag{1.4}$$

$$= \langle B_x, (x+s)^{-r-1}\rangle + \frac{\Gamma(r+k+1)}{\Gamma(r+1)} \int_a^{\infty} f(x)(x+s)^{-r-k-1}\,dx, \quad s \in C \setminus (-\infty, 0].$$

Here the first term on the right-hand side in the second line of (1.4) exists since $(x+s)^{-r-1}$ coincides on the support of B with some infinitely differentiable function.

Throughout the paper we assume that $s > 0$.

The following result of LAVOINE and MISRA [2] is needed in the sequel:

Lemma 1.1. *Let B be a distribution with support in $[a, b]$ where $0 < a < b < \infty$, then*

$$\mathcal{S}_s\{B\} \to \langle B, x^{-r-1}\rangle, \quad s \to 0^+; \tag{1.5}$$

if $a=0$ then

$$s^{r+1}\mathcal{S}_s\{B\} \to \langle B_x, 1\rangle, \quad s \to \infty. \tag{1.6}$$

1.2. We next give the basic properties of the slowly varying functions ([3], [4]) needed in the paper.

(i) The limit (1.1) holds uniformly in any finite interval $[a, b]$, $a > 0$.

(ii) For any $p > 0$ there holds

$$x^p L(x) \to \infty, \quad x^{-p} L(x) \to 0, \quad x \to \infty.$$

The next property is a Theorem of ALJANČIĆ, BOJANIĆ and TOMIĆ [5], and represents our main tool for the proofs.

(iii) Let $g(x)$ be a Lebesgue integrable function on an interval I. Then

$$\int_I g(x)L(\lambda x)\,dx \sim L(\lambda) \int_I g(x)\,dx, \quad \lambda \to \infty,$$

provided that one of the following conditions is satisfied:

A) $I = [0, b]$, $b < \infty$, and the integrals

$$\int_{0+}^{b} g(x)L(\lambda x)\,dx, \quad \int_{0+}^{b} x^{-p}|g(x)|\,dx \tag{1.7}$$

converge; the latter for some $p > 0$.

B) $I=[a, \infty)$, $a>0$, and the integral

$$\int_a^{\infty} x^p |g(x)| \, dx \tag{1.8}$$

converges for some $p>0$.

C) $I=[0, \infty)$ and (1.7) and (1.8) both hold.

1.3. Among the various definitions of the behaviour (at zero and at infinity) of generalized functions, we use the following two:

Definition 1.1 (cf. LOJASIEWICZ [6]). The distribution $T\in\mathcal{D}'_+$ behaves at zero as $x^\nu L_0(x)_+$, i.e.

$$T \sim x^\nu L_0(x)_+, \quad x \to 0^+, \quad \operatorname{Re}\nu > -1,$$

if there exist $a>0$ and a distribution R with support in $[0, a]$ such that $T=x^\nu L_0(x)_+ + R$ for $x\in[0, a]$ and

$$t^{-\nu-1} L_0^{-1}(t)_+ \langle R_x, \varphi(x/t)\rangle \to 0, \quad t \to 0^+ \tag{1.9}$$

for each function φ infinitely differentiable on some neighborhood of $[0, \infty)$. The subscript "+" means that the functions bearing it are equal to zero for $x \leqq 0$.

Definition 1.2 (cf. SEBASTIAO E SILVA [7]). The distribution $T\in\mathcal{D}'_+$ behaves at infinity as $x^\nu L(x)$ i.e.

$$T \sim x^\nu L(x), \quad x \to \infty$$

if for some $a>1$ and $x\in[a, \infty)$, there holds $T_x=D^k f(x)$ and

$$L^{-1}(x) x^{-k-\nu} f(x) \to \frac{1}{(\nu+1)_k}, \quad x \to \infty$$

(where $(\nu+1)_k=(\nu+1)(\nu+2)\dots(\nu+k)$, $k>1$ and $(\nu+1)_0=1$).

2. Results. We prove the following two theorems giving the behavior of the Stieltjes transform $\mathcal{S}_s$ of the distribution $T\in J'(r)$ at zero and at infinity respectively. Notice that $\mathcal{S}_s\{T\}=F(s)$ is a (holomorphic) function and the asymptotics has to be understood accordingly.

Theorem 2.1. *Let* $\operatorname{Re} r > -1$, $\operatorname{Re}\nu > -1$, $\operatorname{Re}(r-\nu)>0$, *further let* $L_0(x)$ *be slowly varying at zero and* $T\in J'(r)$. *If*

$$T \sim x^\nu L_0(x)_+, \quad x \to 0^+ \tag{2.1}$$

then

$$\mathcal{S}_s\{T\} \sim B(\nu+1, r-\nu) s^{\nu-r} L_0(s), \quad s \to 0^+.$$

Theorem 2.2. *Let* $\operatorname{Re} r > -1$, $\operatorname{Re} \nu > -1$, $\operatorname{Re}(r-\nu) > 0$, *further let* $L(x)$ *be slowly varying at* ∞ *and* $T \in J'(r)$. *If*

$$T \sim x^{\nu} L(x), \quad x \to \infty \tag{2.2}$$

then

$$\mathcal{S}_s\{T\} \sim B(\nu+1, r-\nu) s^{\nu-r} L(s), \quad s \to \infty.$$

By taking in Theorem 2.1 $L_0(x)=1$ and $L_0(x)=\ln^j x$, $j \in N$, one obtains respectively Theorems 3.2. I in [1] and 2.1 in [2] for $\operatorname{Re} \nu > -1$ of Lavoine and Misra. Similarly, by taking in Theorem 2.2, $k=0$, $L(x)=1$, $L(x)=\ln^j x$ one obtains their Theorem 5.1. III in [1] and Theorem 3.1 in [2].

3. Proofs. Proof of Theorem 2.1. By virtue of (1.2) and (2.1) one has

$$T = x^{\nu} L_0(x)_+ + R + B + D^k f(x); \tag{3.1}$$

here the supports of R, B, $f(x)$ are in $[0, a]$, $[a, b]$, $[b, \infty)$ respectively, and R satisfies (1.9); $L_0(x)$ may be chosen conveniently in $[a, \infty)$. We apply the Stieltjes transform to the distributions at both sides of (3.1), multiply by $(s^{\nu-r} L_0(s))^{-1}$, and then estimate each summand of the right-hand side.

To treat the first term, put in the occurring integral $s=1/\lambda$, $x=s/u$, and to the obtained integral

$$\lambda^{r-\nu} \int_0^{\infty} u^{r-\nu-1}(1+u)^{-r-1} L(\lambda u)\, du \quad \text{with} \quad L(x) = L_0(1/x)$$

apply the result 1.2 (iii) C). Thus for the first term one obtains

$$(s^{\nu-r} L_0(s))^{-1} \int_0^{\infty} \frac{x^{\nu} L_0(x)\, dx}{(x+s)^{r+1}} \to B(\nu+1, r-\nu), \quad s \to 0^+.$$

To complete the proof one has to show that the remaining three terms tend to zero with s. The second term is such due to (1.9) and to the property 1.2 (ii) (with $L_0(s) = L(1/s)$). The third term is such due to (1.5) of Lemma 1.1, and to the property 1.2 (ii) as above. The fourth term is

$$(s^{\nu-r} L_0(s))^{-1} \mathcal{S}_s\{D^k f(x)\} = (s^{\nu-r} L_0(s))^{-1} \frac{\Gamma(k+r+1)}{\Gamma(r+1)} \int_b^{\infty} f(x)(x+s)^{-r-k-1}\, dx.$$

The occurring integral is bounded by an absolute constant and $(s^{\nu-r} L_0(s))^{-1} \to 0$, $s \to 0^+$, as before, which completes the proof.

Proof of Theorem 2.2. Because of (2.2) the function $f(x)$ in (1.2) is of the form

$$f(x) = c x^{k+\nu} L(x)(1+\omega(x)),$$

where $\omega(x)$ is a locally integrable function with support in $[a, \infty)$ and such that $\omega(x)\to 0$, $x\to\infty$, and $c^{-1}=(\nu+1)_k$. Hence

$$T = cD^k\{x^{k+\nu}L(x)_+\}+B^1+D^k\{x^{k+\nu}L(x)\,\omega(x)\} \tag{3.2}$$

where B^1 is with support in $[0, a]$; $L(x)$ may be chosen conveniently in $[0, a]$. Now we proceed as in the proof of Theorem 2.1, i.e. apply the Stieltjes transform to both sides of (3.2), multiply by $(s^{\nu-r}L(s))^{-1}$ and estimate each term on the right-hand side separately.

We apply to the integral occurring in the first term the result 1.2 (iii) C), yielding

$$\mathcal{S}_s\{D^k x^{k+\nu}L(x)\}\sim B(\nu+1, r-\nu)s^{\nu-r}L(s), \quad s\to\infty.$$

Now we see that the second term tends to zero for $s\to\infty$; this time we have to use 1.2 (ii) and (1.6) of Lemma 1.1. The third term I_3, say, is estimated as follows

$$|I_3| \leqq (L(s))^{-1}\int_0^{a/\sqrt{s_0}} s^{\mathrm{Re}\,\nu+k}L(sy)\,dy+(L(s))^{-1}M(s)\int_{a/\sqrt{s_0}}^{\infty} y^{\mathrm{Re}(\nu-r)-1}L(sy)\,dy$$

where $M(s)=\sup|\omega(sy)|$ for $y\geqq as_0^{-1/2}$. Hence $|I_3|\leqq\varepsilon_1+\varepsilon_2$ for $s\geqq s_0$, since $|\omega(sy)|$ is bounded, $M(s)\to 0$, $s\to\infty$, and by applying to the occurring integrals (1.2), (iii) A) and (1.2), (iii), B) respectively.

Remark. By altering slightly the method of proof used above we can obtain similar results when the distributions behave as some functions more general than the regularly varying ones. Thus the following result holds:

Theorem 3.1. *Let* $\mathrm{Re}\,r>-1$, $T\in J'(r)$, *and let* $h(x)$ *be such that for some* $p>0$, $\int_a^\infty |h(x)x^{p+\nu}|dx$ *converges* $(a>0)$. *If*

$$T\sim x^\nu h(x), \quad x\to\infty$$

then $\mathcal{S}_s\{T\}\sim ms^{-r-1}$, $s\to\infty$, *where m is a constant that can be calculated.*

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