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A connection between the unitary dilation and the normal extension of a subnormal contraction

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1. Introduction and theorem. Let H be an infinite dimensional, separable, complex Hilbert space and let T be a bounded (linear) operator on H. If T is a contraction ($\|T\|\leqq 1$) on H, then, by a well-known result of B. Sz.-Nagy [7] there exists a Hilbert space $K\supset H$ and a unitary operator U on K for which

$$(1.1)\qquad T^n = PU^n|H \quad\text{and}\quad T^{*n} = PU^{*n}|H, \quad n=0, 1, 2, \ldots,$$

where P is the orthogonal projection $P\colon K\to H$. If K is the least subspace of K which reduces U and contains H (as will be supposed) then U is the (unique) minimal unitary dilation of T. See Halmos [3], Sz.-Nagy and Foiaş [8].

Next, let T be any subnormal operator, so that there exists a Hilbert space $K'\supset H$ and a normal operator N on K' for which $NH\subset H$ and $T=N|H$. Thus, N is a normal extension of T and, if P' denotes the orthogonal projection $P'\colon K'\to H$,

$$(1.2)\qquad T^n = N^n|H \text{ and } T^{*n} = P'N^{*n}|H, \quad n = 0, 1, 2, \ldots.$$

In case K' is the least subspace of K' which reduces N and contains H (as will be supposed) then N is the (unique) minimal normal extension of T. (See Halmos [3] and, for an extensive treatment of subnormal operators, Conway [2].) The operator T is said to be a pure subnormal operator if it has no normal part.

Henceforth, it will be supposed that T is both a contraction and a pure subnormal operator. Let the associated operators U and N defined above have the corresponding spectral resolutions

$$(1.3)\qquad U = \int_C z\,dG_z \quad\text{on } K \text{ and }\quad N = \int_{D^-} z\,dE_z \quad\text{on } K',$$

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where D is the unit disk $D=\{z: |z|<1\}$ with closure D^- and $C=\{z: |z|=1\}$. The main object of this note is to point out the following explicit connection between the operators U and N:

Theorem. *Let T be a pure subnormal contraction on H with the minimal unitary dilation U on K and the minimal normal extension N on K' of* (1.3). *Then, for any Borel set β on C and for any vector x in H,*

$$\|E(\beta)x\|^2+\int_D h_\beta(z)\,d\|E_z x\|^2=\|G(\beta)x\|^2, \tag{1.4}$$

where

$$h_\beta(z)=(1/2\pi)\int_{\beta_0}\operatorname{Re}(P_z(t))\,dt, \quad \text{with} \quad P_z(t)=(e^{it}+z)/(e^{it}-z) \tag{1.5}$$

and $\beta_0=\{t: 0\leqq t\leqq 2\pi,\ e^{it}\in\beta\}$.

The function $\operatorname{Re}(P_z(t))$ is the Poisson kernel and, as is well known, is positive for all $t\in C$, $z\in D$, while $h_\beta(z)$ is harmonic in D.

The formula (1.4) is contained "between the lines" in [6], pp. 333—334, but apparently has not appeared explicitly in the literature. For completeness, the argument will be given below.

As above, let P and P' denote the orthogonal projections $P: K\to H$ and $P': K'\to H$, so that, by (1.1) and (1.2), for x in H and $n=0, 1, 2, \ldots$, one has $T^n x=PU^n x=N^n x$ and $T^{*n}x=PU^{*n}x=P'N^{*n}x$. Hence, if $f=f(z)$ is analytic in D and continuous in D^- then

$$\int_{D^-} f\,dE_z x=P\int_C f\,dG_z x \quad \text{and} \quad P'\int_{D^-}\bar f\,dE_z x=P\int_C \bar f\,dG_z x.$$

Consequently, if $h=h(z)$ is any real harmonic function in D which is continuous on D^-, then

$$P'\int_{D^-} h\,dE_z x=P\int_C h\,dG_z x, \quad x\in H.$$

Next, let β be any closed subset of C and let φ be a real-valued continuous function on C satisfying $\varphi=1$ on β and $0\leqq\varphi<1$ on $C-\beta$. Then there exists a function $h(z)$, given by the Poisson integral, which is harmonic in D, continuous in D^-, and satisfies $h=\varphi$ on C. Consequently,

$$P'\Big(\int_D h\,dE_z+\int_C \varphi\,dE_z\Big)x=P\int_C \varphi\,dG_z x, \quad x\in H.$$

On forming inner products with x one obtains

$$\int_C \varphi\,d\|E_z x\|^2+\int_D h\,d\|E_z x\|^2=\int_C \varphi\,d\|G_z x\|^2. \tag{1.6}$$

On replacing φ by $\varphi_n=\varphi^n$ and h by the corresponding h_n, $n=1, 2, \ldots$, one sees that $\varphi_1 \geqq \varphi_2 \geqq \ldots$ and the sequence $\{\varphi_n\}$ converges to the characteristic function of β. Similarly, $0 \leqq h_n \leqq 1$, $h_1 \geqq h_2 \geqq \ldots$, and $\{h_n\}$ converges to $h_\beta(z)$ of (1.5). Clearly, one has the relation (1.4), when β is closed, and the extension of (1.4) to arbitrary Borel sets β readily follows.

It is known that $\|G_z x\|^2$ is, for each x in H, $x \neq 0$, equivalent to arc length Lebesgue measure on C; see Sz.-Nagy and Foiaş [8], p. 84. The absolute continuity of E_z on C follows from (1.4). (That is, $E(\beta)=0$ whenever β is a Borel set on C having arc length measure zero.) Other proofs of this last result are given in Conway and Olin [2], p. 35, Olin [4] and Putnam [5] (see also [6]).

2. U as the sweep of N. It may be noted that (1.4) of the Theorem or, equivalently, (1.6), in which φ is now any continuous function on C and $h=h(z)=\hat{\varphi}(z)$ is its harmonic extension to D^- (via the Poisson integral), can be interpreted in terms of the sweep of a measure. (For the concept of "sweep" see Conway [2], p. 334.) Thus, one has

$$\int_{D^-} \hat{\varphi}\, d\mu = \int_C \varphi\, d\hat{\mu},$$

where $d\mu = d\|E_z x\|^2$ and $d\hat{\mu} = d\|G_z x\|^2$, so that $\hat{\mu}$ on C is the sweep of μ on D^-.

3. Two corollaries.

Corollary 1. *Under the hypotheses of the Theorem, suppose that, in addition,*

$$x = E(C)x \quad \text{for some} \quad x \in H, \quad x \neq 0. \tag{3.1}$$

Then

$$E_z \text{ on } C \text{ is equivalent to arc length measure on } C, \tag{3.2}$$

that is, $E(\beta)=0$ for a Borel set β of C if and only if β has arc length measure zero.

Proof. In view of the remarks at the end of section 1, it is sufficient to show that β has arc length measure zero whenever $E(\beta)=0$. It follows from (3.1) and (1.4) that $E(|z|<1)\ x=0$, so that $\|E_z x\|^2 = \|G_z x\|^2$, z on C. However, as noted above, $\|G_z x\|^2$ is equivalent to arc length measure on C and the proof is complete.

For use below, let $\alpha_t = \{e^{is}: 0 \leqq s \leqq t\}$, $0 \leqq t \leqq 2\pi$, and put $E_t = E(\alpha_t)$ and $G_t = G(\alpha_t)$. If $\Delta = [a, b] \subset [0, 2\pi]$, let $\Delta E = E_b - E_a$ and $\Delta G = G_b - G_a$.

Corollary 2. *Suppose that for some $x \in H$, $x \neq 0$,*

$$\inf_{\Delta} \{\|\Delta G x\|^2 / |\Delta|\} = 0, \tag{3.3}$$

where Δ is any subinterval of $[0, 2\pi]$ *and $|\Delta| > 0$ is its length. Then* (3.1), *hence also* (3.2), *holds.*

Proof. By (1.4) and (1.5),

$$(3.4)\qquad \|\Delta Ex\|^2/|\Delta| + (1/2\pi)\int_D (1/|\Delta|)\Big(\int_\Delta \mathrm{Re}\,(P_z(t))\,dt\Big)\,d\|E_z x\|^2 = \|\Delta Gx\|^2/|\Delta|.$$

By (3.3), there exists a sequence of intervals Δ_n, $|\Delta_n|>0$, $n=1, 2, \ldots$, where $\Delta_n= =[a_n, b_n]$ and $a_n, b_n\to c$ for some c in $[0, 2\pi]$. It follows from (3.4) and Fatou's lemma that

$$\int_D \mathrm{Re}\,(P_z(c))\,d\|E_z x\|^2 = 0.$$

Consequently, $E(D)x=0$, that is, (3.1), and hence (3.2).

4. Remarks. It follows from Corollary 2 that if E_z on C is not equivalent to arc length measure on C (so that there exists a Borel set β on C of positive arc length measure for which $E(\beta)=0$), then, for every x in H, $x\neq 0$, there exists a constant k_x for which $\|\Delta Gx\|^2/|\Delta|\geqq k_x>0$ for all subintervals Δ of $[0, 2\pi]$. This result can be regarded as a refinement of the relation

$$(4.1)\qquad \log\,(d\|G_t x\|^2/dt)\in L(0, 2\pi),$$

which is valid whether or not E_z on C is equivalent to arc length measure on C. In fact, (4.1) holds for arbitrary completely nonunitary (not necessarily subnormal) contractions T; see Sz.-Nagy and Foiaş [8], p. 84.

Since for each $x\in H$, both $\|E_t x\|^2$ and $\|G_t x\|^2$ are absolutely continuous on $[0, 2\pi]$, it is seen from (3.4) with $\Delta=[t, t+\Delta t]$ that

$$(4.2)\qquad d\|E_t x\|^2/dt + (1/2\pi)\int_D \mathrm{Re}\,(P_z(t))\,d\|E_z x\|^2 = d\|G_t x\|^2/dt$$

holds a.e. on $0\leqq t\leqq 2\pi$. In fact, the relation corresponding to (4.2) but with the equality replaced by "$\leqq$" follows from Fatou's lemma. That, in fact, equality holds (a.e.) follows from an integration and the fact that $x=E(D^-)x=G(C)x$.

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