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Horn sentences in submodule lattices

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To the memory of András P. Huhn

1. Introduction. Given a ring R with 1, a lattice is said to be representable by R-modules if it is embeddable in the lattice of submodules of some R-module. The class $\mathbf{L}(R)$ of all lattices representable by R-modules is known to be a quasivariety, i.e., to be axiomatizable by universal Horn sentences (cf. HERRMANN and POGUNTKE [9], HUTCHINSON [11] and, for another proof, [3]). The study of these quasivarieties was started in HUTCHINSON [10]. The main problem in this theory is to classify the possible quasivarieties of the form $\mathbf{L}(R)$. This needs to answer the following question:

(1.1) When does the inclusion $\mathbf{L}(R_1) \subseteq \mathbf{L}(R_2)$ hold?

Denoting by R-**Mod** $(\varkappa)$ the category of R-modules with cardinality less than or equal to a given cardinal $\varkappa$, the main result of [10] is the following.

Theorem 1.2 (HUTCHINSON [10]). $\mathbf{L}(R_1) \subseteq \mathbf{L}(R_2)$ *if and only if for each infinite cardinal* $\varkappa$ *there exists an exact embedding functor* R_1-**Mod** $(\varkappa) \to R_2$-**Mod**.

Note that even a stronger result (cf. HUTCHINSON [11B]) is true: $\mathbf{L}(R_1) \subseteq \mathbf{L}(R_2)$ iff there is an exact embedding functor R_1-**Mod** $\to R_2$-**Mod**.

By the help of this theorem, HUTCHINSON [10] proves a number of interesting results concerning (1.1). As the proof and the applications of this theorem require a good command of category theory and a hard technique, it seems reasonable to develop another approach to (1.1). As $\mathbf{L}(R)$ is a quasivariety, the inclusion $\mathbf{L}(R_1) \subseteq \mathbf{L}(R_2)$ holds if and only if every Horn sentence satisfied in $\mathbf{L}(R_2)$ is also satisfied in $\mathbf{L}(R_1)$. Therefore (1.1) can be reduced to the following problem:

(1.3) When does a Horn sentence hold in $\mathbf{L}(R)$?

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Our aim in the present paper is to investigate the connection between properties of rings R and Horn sentences holding in $\mathbf{L}(R)$. We give some answer to (1.3) in Theorem 3.5, which, among others, enables us to give new proofs for some results of HUTCHINSON [10] concerning (1.1). Although our description (Theorems 4.1 and 4.2) of the ring properties that can be characterized by Horn sentences is not complete, it leads to a solution of the following problem of JÓNSSON [13]:

(1.4) Is there a strong Mal'tsev condition for any Horn sentecne χ which characterizes if χ holds in the congruence lattices of algebras of an n-permutable variety?

The connection between ring properties and lattice identities, which are particular Horn sentences, was firstly studied by HERRMANN and HUHN [8]. After András P. Huhn had personally initiated me into their research with C. Herrmann, we with G. Hutchinson settled the case of lattice identities in [12]. The present paper resembles [12] in some extent; e.g., the use of Mal'tsev conditions is the main tool of investigations in both papers. The results of this paper are taken from the author's thesis [4].

2. Preliminaries. By a ring we always mean a ring with 1, and modules are always unitary left modules. Suppose R is a ring, let R-**Mod** denote the class of R-modules. If M is an R-module then Con (M) and Su (M) will stand for the lattice of congruences and that of submodules of R, respectively. For a class $\mathcal{M}$ of modules, let Con $(\mathcal{M})=\{\text{Con}(M)\colon M\in\mathcal{M}\}$ and Su $(\mathcal{M})=\{\text{Su}(M)\colon M\in\mathcal{M}\}$. Then $\mathbf{L}(R)=\mathbf{IS}\,\text{Su}(R\text{-}\mathbf{Mod})$. As $\text{Con}(M)\cong\text{Su}(M)$ for any $M\in R$-**Mod** (cf. BIRKHOFF [1, p. 159]), we have $\mathbf{L}(R)=\mathbf{IS}\,\text{Con}(R\text{-}\mathbf{Mod})$. It is worth pointing out that exactly the same Horn sentences hold in $\mathbf{L}(R)$, Su (R-**Mod**) and Con (R-**Mod**), whence, in many of the forthcoming results, $\mathbf{L}(R)$ can be replaced by any of the other two. The lattice variety generated by $\mathbf{L}(R)$ will be denoted by $\mathbf{HL}(R)$, which consists of all homomorphic images of lattices in $\mathbf{L}(R)$.

For any integers m and n, let $D(m,n)$ denote the sentence (in the first-order language of rings with 1) "$(\exists x)(m\cdot x=n\cdot 1)$" where $k\cdot y$ or ky is an abbreviation for $y+y+\ldots+y$ (k times if $k>0$) or 0 (if $k=0$) or $-|k|\cdot y$ (if $k<0$). $D(m,n)$ is called a *divisibility condition.* Denoting the set of prime numbers by P, a map $S\colon \{0\}\cup P\to\omega+1$ is called a *spectrum* if

(α) $S(0)<\omega$

and

(β) if $S(0)>0$ then $S(p)=\max\{i\colon 0\leqq i \text{ and } p^i \text{ divides } S(0)\}$ holds for all $p\in P$.

For any spectra S_1 and S_2, let $S_1\leqq S_2$ mean that $S_1(0)$ divides $S_2(0)$ and, for all $p\in P$, $S_1(p)\leqq S_2(p)$. Equipped with this (ordering) relation, the set $\mathcal{L}_S$ of all spectra turns into a complete lattice (cf. Theorem 2.1 later). For a ring R, let S_R be the map $\{0\}\cup P\to\omega+1$ defined by $S_R(0)=\operatorname{char} R=\min\{i:\ i\geqq 1 \text{ and } D(0,i) \text{ holds in } R\}$, the characteristic of R (here $\min\emptyset=0$) and, for $p\in P$, $S_R(p)=\min\{i:\ 0\leqq i<\omega \text{ and } D(p^{i+1},p^i) \text{ holds in } R\}$ (here $\min\emptyset=\omega$). Hutchinson [12] has shown that S_R is a spectrum; it will be called the spectrum of R.

Now, for a spectrum S with $S(0)=0$, let $FR(\{x_p:\ p\in P\})$ be the free commutative ring with 1 on the free generating set $\{x_p:\ p\in P\}$, let J_S denote the ideal of this ring generated by $\{p^{S(p)}(px_p-1):\ p\in P \text{ and } S(p)<\omega\}$, and put $R_S=FR(\{x_p:\ p\in P\})/J_S$. For $S(0)=m>0$, we put $R_S=\mathbf{Z}_m$, the factor ring of the ring $\mathbf{Z}$ of integers modulo m.

For an integer n and a prime p, let $\exp(n,p)$ denote $\sup\{i:\ 0\leqq i<\omega \text{ and } p^i \text{ divides } n\}$. Then the main result of [12] is the following

Theorem 2.1 (Hutchinson [12]). (a) $\mathbf{HL}(R)$ *and* S_R *mutually determine each other.*

(b) *The lattice varieties of the form* $\mathbf{HL}(R)$, R *is a ring, form a complete lattice* $\mathcal{L}_R$ *under the inclusion.*

(c) $\mathcal{L}_R$ *is isomorphic to* $\mathcal{L}_S$. *In fact, the map* $\mathcal{L}_R\to\mathcal{L}_S$, $\mathbf{HL}(R)\mapsto S_R$ *is a lattice isomorphism whose inverse is* $\mathcal{L}_S\to\mathcal{L}_R$, $S\mapsto\mathbf{HL}(R_S)$.

(d) $D(0,n)$ *holds in a ring* R *iff* $S_R(0)$ *divides* n *while, for* $m\neq 0$, $D(m,n)$ *holds in* R *iff* $(\forall p\in P)(\exp(m,p)>\exp(n,p)\Rightarrow\exp(n,p)\geqq S_R(p))$.

By a Horn sentence we mean a universally quantified first order lattice sentence χ of the form

(2.2) $$(p_0\leqq q_0\ \&\ p_1\leqq q_1\ \&\ \dots\ \&\ p_t\leqq q_t)\Rightarrow p\leqq q$$

where $-1\leqq t<\omega$ and $p_0,q_0,p_1,q_1,\dots,p_t,q_t,p,q$ are lattice terms. (In case $t=-1$ the premise is empty and χ is the identity $p\leqq q$.) Let us call χ *regular* if, for any two rings R_1 and R_2, $S_{R_1}=S_{R_2}$ and $\mathbf{L}(R_1)\models\chi$ imply $\mathbf{L}(R_2)\models\chi$. I.e., χ is regular iff the satisfaction of χ in $\mathbf{L}(R)$ depends only on S_R or, equivalently, on $\mathbf{HL}(R)$. By Theorem 2.1 (a), every lattice identity is regular. In Sections 4 and 8, we will deal with ring properties characterizable by regular Horn sentences as we have not succeeded in handling the general case. (This situation resembles [2].) Hutchinson [10] has shown that there are rings R_1 and R_2 such that $S_{R_1}=S_{R_2}$ but $\mathbf{L}(R_1)\neq\mathbf{L}(R_2)$, whence there exist irregular Horn sentences, too. In the forthcoming [6] we will explicitly construct an irregular Horn sentence.

3. Mal'tsev type conditions. Given an integer $n\geqq 2$ and a Horn sentence χ, [3] associates a Mal'tsev condition with χ such that the satisfaction of χ in the congruence lattices of an arbitrary n-permutable variety $\mathcal{U}$ is equivalent to the satisfaction of this Mal'tsev condition in $\mathcal{U}$. Unfortunately, the Mal'tsev conditions in [3] are so complicated that instead of recalling them and adapting them to the special case $\mathscr{U}=R$-**Mod** it is better and shorter to develop them independently. As these conditions will be meaningful only when $\mathscr{U}=R$-**Mod**, they will be referred to as Mal'tsev *type* conditions.

Our Mal'tsev type conditions will be given by certain graphs. First, for any lattice term $p=p(x\colon x\in U)$ we define a graph $G(p)$ associated with p. (Here we adopt the abbreviation $p(x\colon x\in\{x_1, x_2, \dots, x_n\})$ for $p(x_1, \dots, x_n)$. U is assumed to have a fixed order.) The edges of $G(p)$ will be coloured by the variables $x\in U$, and two distinguished vertices, the so-called left and right endpoints, will have special roles. In figures these endpoints will always be placed on the left-hand side and on the right-hand side, respectively. An x-coloured edge connecting the vertices u and v will often be denoted by (u, x, v). Before defining $G(p)$ we introduce two kinds of operations for graphs. We obtain the *parallel connection* of graphs G_1 and G_2 by taking disjoint copies of G_1 and G_2 and identifying their left (right, resp.) endpoints (Figure 3.1).

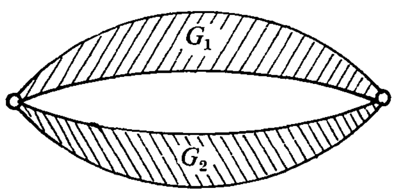

Figure 3.1

Figure 3.2

Similarly, we obtain the *serial connection* of G_1 and G_2 by taking disjoint copies of G_1 and G_2 and identifying the right endpoint of G_1 and the left endpoint of G_2. (The left endpoint of G_1 and the right endpoint of G_2 are the endpoints of the serial connection, cf. Figure 3.2.) Now if p is a variable then $G(p)$ is the following graph

$$\circ\!\xrightarrow{\;p\;}\!\circ\,,$$

which consists of a single edge coloured by p. Let $G(p_1\wedge p_2)$ $(G(p_1\vee p_2)$, resp.) be the parallel connection (serial connection, resp.) of the graphs $G(p_1)$ and $G(p_2)$. This defines $G(p)$ for any lattice term p via induction on the length of p. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Note that $E(G(p))\subseteq V(G(p))\times U\times V(G(p))$ if $p=p(x\colon x\in U)$.

Now let $p=p(x\colon x\in U)$ be a lattice term, let R be a ring, let $M\in R$-**Mod**, and let φ be a map from U into $\mathrm{Su}(M)$. A map $\psi\colon V(G(p))\to M$ will be called a *connecting map* (with respect to φ) if (left endpoint)$\psi=0$ and $b\psi-a\psi\in x\varphi$ holds for every edge $(a, x, b)\in E(G(p))$. For a graph G, let $(\varphi, \psi)\colon G\xrightarrow{c} M$ denote the fact that $\psi\colon V(G)\to M$ is a connecting map with respect to φ. Given a $y\in M$, if there exists a connecting map ψ: $V(G(p))\to M$ such that (right endpoint)$\psi=y$ then y will be said to be *attainable* by $G(p)$ (with respect to φ). Knowing that $X\vee Y= =X+Y=\{x+y\colon x\in X \text{ and } y\in Y\}$ and $X\wedge Y=X\cap Y$ hold for $X, Y\in \mathrm{Su}(M)$, an easy (and therefore omitted) induction on the length of p yields the following

Lemma 3.3. *For any $y\in M$, $y\in p(x\varphi\colon x\in U)$ iff y is attainable by $G(p)$ with respect to φ.*

The following lemma will also be useful.

Lemma 3.4. *Assume that $t(x\colon x\in U)$ is a lattice term, M and K are modules over a ring R, ψ: $M\to K$ is a homomorphism, μ: $U\to \mathrm{Su}(M)$ and φ: $U\to \mathrm{Su}(K)$ are maps, and $x\mu\psi\subseteq x\varphi$ for all $x\in U$. Then $t(x\mu\colon x\in U)\psi\subseteq t(x\varphi\colon x\in U)$.*

Proof. The proof goes via induction on the length of t. If $t\in U$, i.e. t is a variable, then the statement is obvious. If the statement is already true for t_1 and t_2 then for $t=t_1\vee t_2$ we have

$$t(x\mu\colon x\in U)\psi = \big(t_1(x\mu\colon x\in U)+t_2(x\mu\colon x\in U)\big)\psi =$$
$$= t_1(x\mu\colon x\in U)\psi+t_2(x\mu\colon x\in U)\psi \subseteq t_1(x\varphi\colon x\in U)+t_2(x\varphi\colon x\in U) = t(x\varphi\colon x\in U),$$

while in the case $t=t_1\wedge t_2$ we have

$$t(x\mu\colon x\in U)\psi = \big(t_1(x\mu\colon x\in U)\cap t_2(x\mu\colon x\in U)\big)\psi \subseteq$$
$$\subseteq t_1(x\mu\colon x\in U)\psi\cap t_2(x\mu\colon x\in U)\psi \subseteq t_1(x\varphi\colon x\in U)\cap t_2(x\varphi\colon x\in U) = t(x\varphi\colon x\in U).$$

If G is a graph and H is a set then let $H\times G$ denote the graph whose vertex set and edge set are $H\times V(G)$ and $\{((h, a), x, (h, b))\colon (a, x, b)\in E(G)\}$, respectively. Note that $H\times G$ is isomorphic to $\dot{\bigcup}_{h\in H} G$, the disjoint union of $|H|$ copies of G.

Let us fix a ring R and a Horn sentence χ of the form (2.2) where $t\geqq 0$. (The assumption $t\geqq 0$ does not hurt the generality as any lattice identity $p\leqq q$ is equivalent, modulo lattice theory, to the Horn sentence $x\leqq x\Rightarrow p\leqq q$.) Let U be the set of variables occurring in χ. Before formulating Theorem 3.5, we have to define certain modules over R. It seems reasonable to outline our goal roughly before the following tedious definition. In order to obtain a necessary condition for the satisfaction of χ in $\mathbf{L}(R)$ we will start from a "small" module M^0, submodules X^0 for $x\in U$, and an element $f_1\in p(X^0\colon x\in U)$. If $p_0(X^0\colon x\in U)\leqq q_0(X^0\colon x\in U)$ fails then, in order to improve this failure, we will extend X^0, $x\in U$, and M^0 to appropriate X^1 and M^1, respectively. Then, by extending X^1, $x\in U$, and M^1 to X^2 and M^2 if necessary, we will try to remedy the failure of $p_1(X^1\colon x\in U)\leqq q_1(X^1\colon x\in U)$; etc. After ω steps we will obtain $M^\omega=\dot{\bigcup}_{m<\omega} M^m$ and, for $x\in U$, $X^\omega=\bigcup_{m<\omega} X^m$. Now the premise of χ will hold for X^ω, $x\in U$, and the satisfaction of χ in $\mathbf{L}(R)$ will imply $f_1\in q(X^\omega\colon x\in U)=\bigcup_{m<\omega} q(X^m\colon x\in U)$. Lemma 3.3 will be our main tool in doing so.

Now the precise definition comes. First we define lattice terms p_i and q_i for $t<i<\omega$: let p_i and q_i be p_j and q_j, respectively, where $j\equiv i \mod (t+1)$ and $0\leqq j\leqq t$. For any integer $m\geqq 0$, we intend to define a graph G^m, a subset F^m of $V(G^m)$, an R-module M^m and submodules X^m of M^m (for all $x\in U$) by induction such that $V(G^m)\subseteq M^m$, M^m is freely generated by F^m and, for all $x\in U$, X^m is the submodule of M^m generated by $\{c-b\colon (b, x, c)\in E(G^m)\}$, in notation $X^m=[c-b\colon (b, x, c)\in E(G^m)]$. (Here we have a map $U\to \mathrm{Su}(M^m)$ which we denote by capitalizing and adding a superscript, e.g. $x\mapsto X^m$ and $y_i\mapsto Y_i^m$ for $x, y_i\in U$.) As G^m and F^m will determine M^m and X^m, $x\in U$, it will suffice to define the former two.

Let $G^0:=G(p)$ and $F^0:=V(G(p))\setminus\{\text{left endpoint}\}$, and, in order to ensure $V(G^0)\subseteq M^0$, identify the left endpoint of G^0 and the zero of M^0.

Assume that G^{m-1}, F^{m-1}, M^{m-1} and X^{m-1}, $x\in U$, have already been defined for some $m\geqq 1$. Now the definition ramifies as we want to define two kinds of our graphs and modules.

(a) Choose a subset S_m of M^{m-1} such that $S_m\subseteq P_m^{m-1}$ where $P_m^{m-1}= =p_m(X^{m-1}\colon x\in U)$.

(b) Choose a subset S_m of M^{m-1} such that $P_m^{m-1}\setminus Q_m^{m-1}\subseteq [S_m]\subseteq P_m^{m-1}$ where P_m^{m-1}, Q_m^{m-1} and $[S_m]$ denote $p_m(X^{m-1}\colon x\in U)$, $q_m(X^{m-1}\colon x\in U)$ and the submodule generated by S_m.

In both cases, we put

$$F^m:= F^{m-1}\cup(\{m\}\times S_m\times(V(G(q_m))\setminus\{\text{left endpoint, right endpoint}\})).$$

We obtain G^m from $G^{m-1}\cup(\{m\}\times S_m\times G(q_m))$ by identifying the zero of M^m and all the (m, s, left endpoint), $s\in S_m$, and by identifying (m, s, right endpoint) and s for every $s\in S_m$. Then $V(G^{m-1})\subseteq V(G^m)\subseteq M^m$ and G^{m-1} is a (weak) subgraph of G^m, i.e., $E(G^{m-1})\subseteq E(G^m)\cap(V(G^{m-1})\times U\times V(G^{m-1}))$. Therefore $X^{m-1}\subseteq X^m$, $x\in U$. Obviously, $F^{m-1}\subseteq F^m$ and $M^{m-1}\subseteq M^m$.

Now we have defined G^m, F^m, M^m and X^m, $x\in U$, for all $m\geqq 0$. Note that, in both cases, these things depend on the choice of $S_1, S_2, S_3, \dots$ because we want to make the following theorem easy to handle. We also note that the choice $S_1=P_1^0$, $S_2=P_2^1$, $S_3=P_3^2$, ... is always possible. Let f_1 denote the right endpoint of $G^0=G(p)$, then we have

Theorem 3.5. (A) *Suppose that* $S_1, S_2, S_3, \dots$ *are chosen according to* (a). *If there exists a non-negative integer* n *such that* $f_1\in q(X^n\colon x\in U)$ *then* χ *holds in* $\mathbf{L}(R)$ *or, equivalently, in* $\mathrm{Su}(R\text{-}\mathbf{Mod})$.

(B) *Suppose that* $S_1, S_2, S_3, \dots$ *are choosen according to* (b). *Then* χ *holds in* $\mathbf{L}(R)$ *if and only if there exists a non-negative integer* n *such that* $f_1\in q(X^n\colon x\in U)$.

Proof. It suffices to prove (A) and the "only if" part of (B).

To prove (A), assume that $f_1\in q(X^n\colon x\in U)$ holds for some n. Let $A\in R\text{-}\mathbf{Mod}$, for $x\in U$ let $X'\in\mathrm{Su}(A)$, let $a_1\in p(X'\colon x\in U)$, and assume that $p_i(X'\colon x\in U)\subseteq$

$\subseteqq q_i(X'\colon x\in U)$ holds for $i\leqq t$ (whence for $i<\omega$ as well). Let φ denote the map $U\to \mathrm{Su}\,(\mathrm{A})$, $x\mapsto X'$. We need to show $a_1\in q(X'\colon x\in U)$. Via induction on m, we intend to define two maps, $\eta^m\colon V(G^m)\to A$ and $\psi^m\colon M^m\to A$ for any $m\geqq 0$ such that

(I_m) $(\varphi,\eta^m)\colon G^m\xrightarrow{c} A$, and $\psi^m\colon M^m\to A$ is a homomorphism extending both η^m and η^0.

By Lemma 3.3, a_1 is attainable by $G(p)=G^0$ with respect to φ. I.e., there is a map $\eta^0\colon V(G^0)\to A$ such that $f_1\eta^0=a_1$ and $(\varphi,\eta^0)\colon G^0\xrightarrow{c} A$. Extend $\eta^0\restriction F^0$ to a homomorphism $\psi^0\colon M^0\to A$. (Here $\restriction$ stands for the restriction.) As M^0 is freely generated by F^0, ψ^0 exists and is uniquely determined. Since $0\eta^0=(\text{left endpoint})\eta^0=0$, ψ^0 extends η^0, too, and (I_0) is satisfied.

Now let $m\geqq 1$ and suppose (I_{m-1}). I.e., $(\varphi,\eta^{m-1})\colon G^{m-1}\xrightarrow{c} A$ and ψ^{m-1} extends both η^{m-1} and η^0. For $x\in U$,

$$X^{m-1}\psi^{m-1}=[c-b\colon (b,x,c)\in E(G^{m-1})]\psi^{m-1}=$$
$$=[c\psi^{m-1}-b\psi^{m-1}\colon (b,x,c)\in E(G^{m-1})]=[c\eta^{m-1}-b\eta^{m-1}\colon (b,x,c)\in E(G^{m-1})]\subseteqq X',$$

whence, by Lemma 3.4,

$$S_m\psi^{m-1}\subseteqq P_m^{m-1}\psi^{m-1}=p_m(X^{m-1}\colon x\in U)\psi^{m-1}\subseteqq p_m(X'\colon x\in U)\subseteqq q_m(X'\colon x\in U).$$

I.e., $S_m\psi^{m-1}\subseteqq q_m(X'\colon x\in U)$. By Lemma 3.3, for every $s\in S_m$, $s_m\psi^{m-1}$ is attainable by $G(q_m)\cong\{m\}\times\{s\}\times G(q_m)$ with respect to φ. I.e., there is a map η_s^m such that $(\varphi,\eta_s^m)\colon \{m\}\times\{s\}\times G(q_m)\xrightarrow{c} A$ and $(m,s,\text{right endpoint})\eta_s^m=s\psi^{m-1}$. Put $\eta^m==\eta^{m-1}\cup\overline{\bigcup_{s\in S_m}}\eta_s^m$. Then η^m is really a map from $V(G^m)$ into A, and it extends η^{m-1}. Further, if $s\in S_m$ then $s\eta^m=(m,s,\text{right endpoint})\eta^m=s\psi^{m-1}$. Now let $\psi^m\colon M^m\to A$ be the unique homomorphism that extends $\eta^m\restriction F^m$. For any $u\in F^{m-1}\subseteqq F^m$, $u\psi^m==u\eta^m=u\eta^{m-1}=u\psi^{m-1}$. Hence $\psi^m\restriction F^{m-1}=\psi^{m-1}\restriction F^{m-1}$ and $[F^{m-1}]=M^{m-1}$ yield that ψ^m extends ψ^{m-1}. For $u\in V(G^m)\setminus F^m$ either $u\in V(G^{m-1})$ and $u\eta^m=u\eta^{m-1}==u\psi^{m-1}=u\psi^m$ or $u\in S_m$ and $u\eta^m=(m,u,\text{right endpoint})\eta^m=u\psi^{m-1}=u\psi^m$. Hence ψ^m is an extension of η^m. As η^{m-1} and η_s^m, $s\in S_m$, are connecting maps, so is η^m. I.e., $(\varphi,\eta^m)\colon G^m\xrightarrow{c} A$, and (I_m) holds.

Now η^m and ψ^m, satisfying (I_m), are defined for all $m\geqq 0$. From (I_n) we conclude that, for $x\in U$,

$$X^n\psi^n=[c-b\colon (b,x,c)\in E(G^n)]\psi^n=[c\psi^n-b\psi^n\colon (b,x,c)\in E(G^n)]=$$
$$=[c\eta^n-b\eta^n\colon (b,x,c)\in E(G^n)]\subseteqq X'.$$

Hence Lemma 3.4 yields $a_1=f_1\eta^0=f_1\psi^n\in q(X^n\colon x\in U)\psi^n\subseteqq q(X'\colon x\in U)$. This proves (A).

To prove the "only if" part of (B), assume that χ holds in $\mathbf{L}(R)$ and F^m, G^m, M^m, X^m $(0\leqq m<\omega,\ x\in U)$ are defined according to (b). Then $F^0\subseteqq F^1\subseteqq F^2\subseteqq\dots$,

$M^0 \subseteq M^1 \subseteq M^2 \subseteq \ldots$ and $G^0 \subseteq G^1 \subseteq G^2 \subseteq \ldots$. Put $F^\omega := \bigcup_{m<\omega} F^m$ and $G^\omega := \bigcup_{m<\omega} G^m$ (i.e., $V(G^\omega) = \bigcup_{m<\omega} V(G^m)$ and $E(G^\omega) = \bigcup_{m<\omega} E(G^m)$). Let M^ω be the R-module freely generated by F^ω and, for $x \in U$, let $X^\omega = [c-b\colon (b, x, c) \in G^\omega]$. It is easy to see that $V(G^\omega) \subseteq M^\omega$, $M^\omega = \bigcup_{m<\omega} M^m$ and, for $x \in U$, $X^\omega = \bigcup_{m<\omega} X^m$. We will show that the submodules X^ω, $x \in U$, satisfy the premise of χ.

Let the map $U \to \mathrm{Su}(M^l)$, $x \mapsto X^l$ be denoted by φ^l, $l \leqq \omega$. Since $X^\omega = \bigcup_{m<\omega} X^m$ and $X^0 \subseteq X^1 \subseteq X^2 \subseteq \ldots$, we obtain that, for any map $\eta\colon V(G(p_j)) \to M^\omega$, $(\varphi^\omega, \eta)\colon G(p_j) \xrightarrow{c} M$ iff there is an m such that $(\varphi^{m-1}, \eta)\colon G(p_j) \xrightarrow{c} M^\omega, j+1 < m < \omega$ and $j \equiv m \mod (t+1)$. Hence, denoting the right endpoint of $G(p_j)$ by r and applying Lemma 3.3, we obtain

$$
\begin{aligned}
(3.6) \qquad p_j(X^\omega\colon x \in U) &= \{b\colon (\exists \eta)(r\eta = b \text{ and } (\varphi^\omega, \eta)\colon G(p_j) \xrightarrow{c} M^\omega)\} = \\
&= \{b\colon (\exists \eta)(\exists m)(j+1 < m < \omega,\ j \equiv m \bmod (t+1),\ r\eta = b \\
&\qquad \text{and } (\varphi^{m-1}, \eta)\colon G(p_j) \xrightarrow{c} M^\omega)\} = \\
&= \bigcup(\{b\colon (\exists \eta)(r\eta = b \text{ and } (\varphi^{m-1}, \eta)\colon G(p_j) \xrightarrow{c} M^\omega)\}\colon \\
&\qquad j+1 < m < \omega \text{ and } j \equiv m \bmod (t+1)) = \\
&= \bigcup(p_j(X^{m-1}\colon x \in U)\colon j+1 < m < \omega \text{ and } j \equiv m \bmod (t+1)).
\end{aligned}
$$

Since $(\varphi^m, \text{identical map})\colon \{m\} \times \{s\} \times G(q_m) \xrightarrow{c} M^\omega$, Lemma 3.3 yields $s \in q_m(X^m\colon x \in U)$ for any $s \in S_m$, $m < \omega$. Therefore $[S_m] \subseteq q_m(X^m\colon x \in U)$. For $j+1 < m < \omega$ and $j \equiv m \mod (t+1)$ we obtain

$$
\begin{aligned}
p_j(X^{m-1}\colon x \in U) &= p_m(X^{m-1}\colon x \in U) = P_m^{m-1} = \\
&= (P_m^{m-1} \setminus Q_m^{m-1}) \cup (P_m^{m-1} \cap Q_m^{m-1}) \subseteq [S_m] \cup Q_m^{m-1} \subseteq \\
&\subseteq q_m(X^m\colon x \in U) \cup q_m(X^{m-1}\colon x \in U) \subseteq \\
&\subseteq q_m(X^\omega\colon x \in U) \cup q_m(X^\omega\colon x \in U) = q_m(X^\omega\colon x \in U) = q_j(X^\omega\colon x \in U).
\end{aligned}
$$

This inclusion and (3.6) yield $p_j(X^\omega\colon x \in U) \subseteq q_j(X^\omega\colon x \in U)$, whence the premise of χ holds for X^ω, $x \in U$. As $\mathrm{Su}(M^\omega) \in \mathbf{L}(R)$, $p(X^\omega\colon x \in U) \subseteq q(X^\omega\colon x \in U)$. Lemma 3.3 yields $f_1 \in p(X^\omega\colon x \in U)$ as $(\varphi^\omega, \text{identical map})\colon G(p) \xrightarrow{c} M$. An argument analogous to (3.6) shows that $q(X^\omega\colon x \in U) = \bigcup_{m<\omega} q(X^m\colon x \in U)$. Hence we have $f_1 \in p(X^\omega\colon x \in U) \subseteq q(X^\omega\colon x \in U) = \bigcup_{m<\omega} q(X^m\colon x \in U)$. Therefore there is an n such that $f_1 \in q(X^n\colon x \in U)$, which completes the proof.

4. Regular Horn sentences. Let U denote the set $\{x, y, z, t, e\}$ of variables, and define the following lattice terms over U (the meet and join will be denoted by $\cdot$ and

$+$, respectively):

$$p := (x+y)(z+t), \quad w_{-1} := (x+z)(y+t), \quad w_0 := x,$$
$$s_{i+1} := (w_i+t)(y+z) \quad \text{and} \quad w_{i+1} := (s_{i+1}+p)(x+z) \quad \text{for} \quad i \geqq 0.$$

By induction, this defines s_i and w_j for all $i \geqq 1$, $j \geqq -1$. Now let m, n and k be non-negative integers, put

$$p_0 := \big((e+w_{n-1})w_{-1}+x\big)z, \quad q_0 := e,$$
$$q := \big((w_k+y)(z+t)+w_{mn}\big)(x+y+e)+x+z,$$

and let $\chi(m, n, k)$ denote the Horn sentence

$$p_0 \leqq q_0 \Rightarrow p \leqq q.$$

Theorem 4.1. *For any ring R and non-negative integers m, n, k, the Horn sentence $\chi(m, n, k)$ holds in $\mathbf{L}(R)$ if and only if there exists a non-negative integer i such that the divisibility condition $D(mn^{i+1}, kn^i)$ holds in R.*

Note that, in virtue of Theorems 2.1 (d) and 4.1, $\chi(m, n, k)$ is regular. To avoid the feeling that $(\exists i)\big(D(mn^{i+1}, kn^i)\big)$ in the above theorem is just a haphazard ring property we state the following result, which is almost the converse of Theorem 4.1. While we have collected all we need to prove Theorem 4.1, the following theorem will be proved only in Section 8.

Theorem 4.2. *Let χ be a regular Horn sentence. Assume that there is a ring R^* of characteristic 0, i.e., $S_{R^*}(0)=0$, such that χ holds in $\mathbf{L}(R^*)$. Then there are positive integers m_χ, n_χ and k_χ such that, for any ring R, χ holds in $\mathbf{L}(R)$ if and only if $D(m_\chi n_\chi^{i+1}, k_\chi n_\chi^i)$ holds in R for some integer $i \geqq 0$.*

Proof of Theorem 4.1. We will apply Theorem 3.5 (B) with the choice $S_j = p_j(H^{j-1}\colon h \in U)$. The graph $G^0 = G(p)$ is given in Figure 4.3, whence $X = X^0 = [f_2]$, $Y = Y^0 = [f_1 - f_2]$, $Z = Z^0 = [f_3]$ and $T = T^0 = [f_1 - f_3]$. Since $G(q_0) = G(q_j)$, $0 \leqq j < \omega$, has no "inner vertex", i.e., $|V(G(q_j))| = 2$, we have $F^j = F^0 = \{f_1, f_2, f_3\}$ and $M^j = M^0$, $0 \leqq j < \omega$. Let F and M denote F^0 and M^0, respectively. As the only edge of $G(q_j) = G(q_0)$ is coloured by e, all the edges in $E(G^j) \setminus E(G^0)$ are coloured

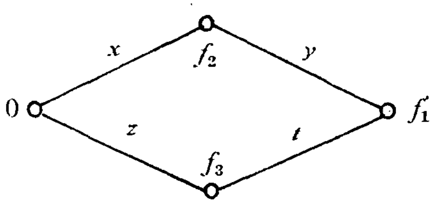

Figure 4.3

by e, and we have $X^j=X^0$, $Y^j=Y^0$, $Z^j=Z^0$, $T^j=T^0$, $E^0=\{0\}$ and $E^j=[E^{j-1}\cup p_0(H^{j-1}\colon h\in U)]=E^{j-1}+p_0(H^{j-1}\colon h\in U)$. We claim that

$$
\begin{aligned}
&P=[f_1],\quad W_j=[f_2+jf_3] \quad\text{for}\quad j\geqq -1,\\
(4.4)\qquad &S_i=[f_1-f_2-if_3] \quad\text{for}\quad i\geqq 1,\quad\text{and}\\
&E^j=P_0^{j-1}=\{rf_3\colon r\in R \text{ and } n^jr=0\}
\end{aligned}
$$

where V stands for $v(H^0\colon h\in U)$ if $v\in\{p, w_j, s_i\}$. These formulas can be obtained by an elementary calculation, only a part of which will be presented. As any element a of M can uniquely be written of the form $a=r_1f_1+r_2f_2+r_3f_3$ where $r_1, r_2, r_3\in R$, we can compute as follows.

$$P=(X+Y)\cap(Z+T)=$$

$$=\{a\in M\colon (\exists r_1, r_2, r_3, r_4\in R)(a=r_1f_2+r_2(f_1-f_2)=r_3f_3+r_4(f_1-f_3))\}=$$

$$=\{a\in M\colon (\exists r_1, r_2, r_3, r_4\in R)(a=r_2f_1+(r_1-r_2)f_2=r_4f_1+(r_3-r_4)f_3)\}=$$

$$=\{a\in M\colon (\exists r_1, r_2, r_3, r_4\in R)(a=r_2f_1+(r_1-r_2)f_2=r_4f_1+(r_3-r_4)f_3 \quad\text{and}$$

$$r_2=r_4, r_1-r_2=0, r_3-r_4=0)\}=\{a\in M\colon (\exists r_2\in R)(a=r_2f_1)\}=\{r_2f_1\colon r_2\in R\}=[f_1].$$

The rest of (4.4) follows similarly via induction. Another elementary computation of the same nature yields

$$q(H^j\colon h\in U)=\{af_1+bf_2+cf_3\colon a, b, c\in R \text{ and } (\exists r\in R)(mn^{j+1}r=kn^ja)\}.$$

Therefore $f_1=1f_1+0f_2+0f_3\in q(H^j\colon h\in U)$ iff $D(mn^{j+1}, kn^j)$ holds in R, and a reference to Theorem 3.5 (B) completes the proof.

5. Systems of ring equations. Let u and v be natural numbers and, for $i<v$, let $f_i(y_j\colon j<u)$ be a ring term (i.e., a term in the language of *unitary* rings). Then

$$f_i(y_j\colon j<u)=0,\quad 0\leqq i<v,$$

is called a system of ring equations. This system is said to be solvable in a ring R iff there exist elements r_j, $j<u$, in R such that $f_i(r_j\colon j<u)=0$ for $i<v$.

Lemma 5.1. *For any Horn sentence χ there is a set $\{E_n\colon n<\omega\}$ of systems of ring equations such that*

(i) *for any ring R, χ holds in $\mathbf{L}(R)$ iff there exists an $n<\omega$ such that E_n is solvable in R;*

(ii) *$E_0, E_1, E_2, \ldots$ is a weakening sequence in the sense that, for any $n<\omega$ and any ring R, if E_n is solvable in R then so is E_{n+1}.*

The proof will only be outlined as it is relatively easy but would need a lot of technical preliminaries. First, consider a Mal'tsev condition $(\exists n<\omega)(U_n)$, where $U_0, U_1, U_2, \ldots$ is a weakening sequence of strong Mal'tsev conditions, such that, for any congruence permutable variety $\mathcal{V}$, χ holds in Con $(\mathcal{V})$ iff $(\exists n<\omega)$ (U_n holds in $\mathcal{V}$). The existence of this Mal'tsev condition was proved by JÓNSSON [13]; a Mal'tsev condition of this kind is explicitly given in [3]. We can easily associate a system E_n of ring equations with each U_n such that, for any ring R, U_n holds in R-**Mod** (which is a congruence permutable variety) iff E_n is solvable in R (cf., e.g., [2, Claim 5.1] or [12, proof of Theorem 2] where analogous or particular cases are handled).

Corollary 5.2. *Let R be the direct product of two rings, R_1 and R_2. Then $\mathbf{L}(R)=\mathbf{L}(R_1)\vee\mathbf{L}(R_2)$ in the lattice of quasivarieties of lattices.*

Proof. We only need to show that an arbitrary Horn sentence χ holds in $\mathbf{L}(R)$ iff it holds in both $\mathbf{L}(R_1)$ and $\mathbf{L}(R_2)$. It is easy to see that a system of ring equations is solvable in R iff it is solvable both in R_1 and R_2. Now if $\mathbf{L}(R_1)\models\chi$ and $\mathbf{L}(R_2)\models\chi$ then, by Lemma 5.1, there are m and k such that the appropriate E_m and E_k are solvable in R_1 and R_2, respectively. Put $n=\max\{m, k\}$. Then E_n is solvable in R_1 and R_2, whence it is solvable in R and $\mathbf{L}(R)\models\chi$. Conversely, if $\mathbf{L}(R)\models\chi$ is assumed then $\mathbf{L}(R_1)\models\chi$ and $\mathbf{L}(R_2)\models\chi$ follows similarly and even more easily.

6. Two results of G. Hutchinson. In this section we will deduce two results of HUTCHINSON [10] from the results of Sections 3 and 5.

Corollary 6.1 (HUTCHINSON [10, Proposition 2]). *Assume that R_1 and R_2 are rings and there is a homomorphism of R_1 into R_2 (preserving 1, of course). Then $\mathbf{L}(R_2)\subseteq\mathbf{L}(R_1)$.*

Proof. Let $\varphi\colon R_1\to R_2$ be a ring homomorphism. It suffices to show that any Horn sentence holding in $\mathbf{L}(R_1)$ holds in $\mathbf{L}(R_2)$, too. But this is evident by Lemma 5.1 as φ maps any solution of E_n in R_1 to a solution of E_n in R_2.

Proposition 6.2 (HUTCHINSON [10]). *Let R_1 and R_2 be rings with the same spectrum $S=S_{R_1}=S_{R_2}$, and assume that either R_1 and R_2 are torsion free or $S(0)$, the characteristic of R_1 and R_2, is a square free (i.e., divisible by p^2 for no prime p) positive number. Then $\mathbf{L}(R_1)=\mathbf{L}(R_2)$.*

Proof. First we prove the statement under the following stronger assumption: either $S(0)$ is a prime or R_1 and R_2 are torsion free. It is sufficient to show that $\mathbf{L}(R_1)$ and $\mathbf{L}(R_2)$ satisfy exactly the same Horn sentences. Therefore it suffices to show that an appropriate construction needed by Theorem 3.5 (B) does not depend (in a sense to be defined later) on the choice of $R\in\{R_1, R_2\}$.

Let $F=\{f_1, f_2, \ldots, f_t\}$ be a set, let $R\in\{R_1, R_2\}$, and let M be the free R-module generated by F. A submodule C of M will be called *normal*, if it is of the form $[\sum(c_{ij}f_j\colon 1\leqq j\leqq t)\colon i<n_C]$ with a suitable $n_C<\omega$ and integers c_{ij}. The form $[\sum(c_{ij}f_j\colon 1\leqq j\leqq t)\colon i<n_C]$ will be called a *normal form* of C. Note that if only F is fixed then distinct submodules (necessarily over distinct rings) may have identical normal forms. We need

Claim 6.3. Assume that $C, D\in \mathrm{Su}(M)$ are given by the respective normal forms $[\sum(c_{ij}f_j\colon 1\leqq j\leqq t)\colon i<n_C]$ and $[\sum(d_{ij}f_j\colon 1\leqq j\leqq t)\colon i<n_D]$. Then there are normal forms of $C+D$ and $C\cap D$ that depend only on the normal forms $[\sum(c_{ij}f_j\colon 1\leqq j\leqq t)\colon i<n_C]$ and $[\sum(d_{ij}\colon 1\leqq j\leqq t)\colon i<n_D]$ but do not depend on $R\in\{R_1, R_2\}$.

Proof of Claim 6.3. The statement is trivial for $C+D$ as $[\sum(e_{ij}f_j\colon 1\leqq j\leqq t)\colon i<n_C+n_D]$, where $e_{ij}=c_{ij}$ for $i<n_C$ and $e_{ij}=d_{i-n_C,j}$ for $n_C\leqq i<n_C+n_D$, is a normal form of $C+D$. Dealing wiht $C\cap D$, put $n=n_C+n_D$, and let $\mathbf{y}$ and $\mathbf{r}$ stand for n-dimensional column vectors. Then the system of linear equations

$$\sum(c_{ij}y_i\colon\ i<n_C)-\sum(d_{ij}y_{n_C+i}\colon\ i<n_D)=0 \quad (1\leqq j\leqq t)$$

can be written of the form $B\mathbf{y}=\mathbf{0}$ for a suitable integer matrix B. It is easy to see that, denoting the entries of $\mathbf{r}$ by r_i,

$$C\cap D=\Big\{\sum\big(r_i\sum(c_{ij}f_j\colon\ 1\leqq j\leqq t)\colon\ i<n_C\big)\colon\ \mathbf{r}\in R^n \text{ and } B\mathbf{r}=\mathbf{0}\Big\}.$$

A classical matrix diagonalization method of Frobenius yields that there are integer square matrices A and C of appropriate sizes such that A and C are invertible, their inverses are integer matrices and ABC is a diagonal matrix, i.e., the jth entry of the ith row is 0 whenever $i\neq j$ (cf. Frobenius [7]; this result is quoted with a proof in [12, p. 284 and Appendix]). Denoting $C^{-1}\mathbf{r}$ by $\mathbf{r}'$ and observing that, by the existence of A^{-1}, $B\mathbf{r}=\mathbf{0}$ is equivalent to $AB\mathbf{r}=\mathbf{0}$, we have

$$\{\mathbf{r}\in R^n\colon\ B\mathbf{r}=\mathbf{0}\}=\{\mathbf{r}\in R^n\colon\ (ABC)(C^{-1}\mathbf{r})=\mathbf{0}\}=$$

$$=\Big\{\mathbf{r}\in R^n\colon\ (\exists\mathbf{r}'\in R^n)\big((ABC)\mathbf{r}'=\mathbf{0} \text{ and } \mathbf{r}=C\mathbf{r}'\big)\Big\}.$$

As ABC is diagonal, $(ABC)\mathbf{r}'=\mathbf{0}$ is equivalent to $g_0r_0'=0, g_1r_1'=0, \ldots, g_{n-1}r_{n-1}'=0$ where the integers $g_0, g_1, \ldots, g_{n-1}$ are the diagonal entries of ABC and $r_0', \ldots, r_{n-1}'$ are the entries of $\mathbf{r}'\in R^n$. For each i, the equation $g_ir_i'=0$ either makes no restriction on r_i' or implies $r_i'=0$. Really, if R is torsion free then $g_i\neq 0$ implies $r_i'=0$; if $S(0)=S_R(0)=p$ is a prime and p does not divide g_i then there is a g' such that $g'g_i\equiv 1 \bmod (p)$ and $g_ir_i'=0$ implies $r_i'=1r_i'=g'g_ir_i'=g'0=0$ while $g_ir_i'=0$ holds for all $r_i'\in R$ when p divides g_i. Put $I=\{i\colon\ i<n$ and $g_ir_i'=0$ holds for any

$r'_i \in R\}$, and let h_{il}, $0 \leqq i, l < n$, be the entries of the matrix C. Then $r_i = \sum(h_{il} r'_l \colon l < n)$ and we have

$$C \cap D =$$
$$= \{\sum(r_i \sum(c_{ij} f_j \colon 1 \leqq j \leqq t) \colon i < n_C) \colon (\exists \mathbf{r}' \in R^n)(\mathbf{r} = C\mathbf{r}' \text{ and } r'_l = 0 \text{ for all } l \notin I)\} =$$
$$= \{\sum(\sum(h_{il} r'_l \colon l < n) \cdot \sum(c_{ij} f_j \colon 1 \leqq j \leqq t) \colon i < n_C) \colon \mathbf{r}' \in R^n \text{ and } r'_l = 0 \text{ for } l \notin I\} =$$
$$= \{\sum(r'_l \sum(\sum(h_{il} c_{ij} \colon i < n_C) f_j \colon 1 \leqq j \leqq t) \colon l < n) \colon \mathbf{r}' \in R^n \text{ and } r'_l = 0 \text{ for } l \notin I\} =$$
$$= [\sum(\sum(h_{il} c_{ij} \colon i < n_C) f_j \colon 1 \leqq j \leqq t) \colon l \in I],$$

proving Claim 6.3.

Now, returning to the proof of Proposition 6.2, we intend to show that it is possible to choose subsets S_m in Theorem 3.5 (B) so that $F^m = \{f_1, f_2, \dots, f_{t_m}\}$ be the same for $R = R_1$ and $R = R_2$ and, for $x \in U$, X^m be given in a normal form independent of $R \in \{R_1, R_2\}$. This is clearly true for $m = 0$; to start our induction step let us assume that this is true for $m-1$, $m \geqq 1$. Then, by Claim 6.3, $P_m^{m-1} = p_m(X^{m-1} \colon x \in U)$ also has a normal form $[\sum(c_{ij}^{(m)} f_j \colon 1 \leqq j \leqq t_{m-1}) \colon i < n^{(m)}]$ which does not depend on $R \in \{R_1, R_2\}$. Put $S_m := \{s_i \colon i < n^{(m)}\}$ where $s_i = \sum(c_{ij}^{(m)} f_j \colon 1 \leqq j \leqq t_{m-1})$. Then F^m does clearly not depend on $R \in \{R_1, R_2\}$ and, by Claim 6.3,

$$X^m = X^{m-1} + \sum([v-u] \colon (u, x, v) \in E(G^m) \setminus E(G^{m-1})) =$$
$$= X^{m-1} + \sum(\sum([v-u] \colon (u, x, v) \in E(\{m\} \times \{s_i\} \times G(q_m))) \colon i < n^{(m)})$$

can be given by a normal form not depending on $R \in \{R_1, R_2\}$. Now a final use of Claim 6.3 yields that, for all m, $q(X^m \colon x \in U)$ can be given by a normal form, say, $[\sum(d_{ij}^{(m)} f_j \colon i \leqq j \leqq t_m) \colon i < k^{(m)}]$ which does not depend on $R \in \{R_1, R_2\}$. Let $\mathbf{y} = (y_0, y_1, \dots, y_{k^{(m)}-1})$, and observe that $f \in q(X^m \colon x \in U)$ iff the following system E_m of *productless* ring equations

$$\sum(d_{i1}^{(m)} y_i \colon i < k^{(m)}) - 1 = 0,$$
$$\sum(d_{ij}^{(m)} y_i \colon i < k^{(m)}) = 0 \quad \text{for} \quad 1 < j \leqq t_m,$$

which does not depend on $R \in \{R_1, R_2\}$, is solvable in R. Based on the afore-mentioned result of Frobenius, it has been shown in [12] (cf. Theorem 2.1 (d) and [12, Theorem 3]) that the solvability of any system of productless ring equations in an arbitrary ring depends only on the spectrum of this ring. But now $S_{R_1} = S = S_{R_2}$, whence E_m is solvable in R_1 iff it is solvable in R_2. Hence Theorem 3.5 (B) proves the proposition under the stronger assumption we considered.

The case $S(0) = 1$ being trivial, consider the case $S(0) = p_0 p_1 \dots p_n$ where $p_0, p_1, \dots, p_n$ are distinct primes. It is known that, for $R \in \{R_1, R_2\}$, R is isomorphic to a direct product $\prod_{i \leqq n} R^{(i)}$ where $S_{R^{(i)}}(0) = p_i$, $i \leqq n$ (cf., e.g., McCoy [14, Theorem 28]). By Corollary 5.2, $\mathbf{L}(R)=\bigvee_{i\leqq n}\mathbf{L}(R^{(i)})$, whence Proposition 6.2 follows from its special case we have already proved.

7. Two sufficient conditions for regularity. Consider a Horn sentence χ of the form (2.2).

Proposition 7.1. *If all q_i, $0\leqq i\leqq t$, are join-free then χ is regular.*

Proof. We will use Theorem 3.5 (B) with $S_m:=P_m^{m-1}$. As $G(q_m)$ has no inner vertex, $F^m=F^{m-1}=\ldots=F^0$, if $x\in U$ occurs in q_m then $X^m=X^{m-1}+P_m^{m-1}$, and $X^m=X^{m-1}$ if $x\in U$ does not occur in q_m. Hence there are lattice terms q_m' such that $q(X^m\colon x\in U)=q_m'(X^0\colon x\in U)$, $0\leqq m$, and these q_m' do not depend on the ring in question. By Theorem 3.5 (B), $\mathbf{L}(R)\models\chi$ is equivalent to $(\exists m)\big(f_1\in q_m'(X^0\colon x\in U)\big)$.

Now let us fix a $y\in U$ and consider the Horn sentence χ_k: $y\leqq y\Rightarrow p\leqq q_k'$, $k\geqq 0$. If we apply Theorem 3.5 (B) to χ_k with $S_m=\emptyset$, $1\leqq m$, then $X^m=X^{m-1}=\ldots=X^0$ for every $x\in U$. Hence $f_1\in q_k'(X^0\colon x\in U)$ is equivalent to $\mathbf{L}(R)\models\chi_k$. But χ_k, being modulo lattice theory equivalent to the lattice identity $p\leqq q_k'$, is regular by Theorem 2.1 (a). We have seen that $\mathbf{L}(R)\models\chi$ is equivalent to $(\exists m)\big(\mathbf{L}(R)\models\chi_m\big)$, whence the regularity of χ_m completes the proof.

Note that Proposition 7.1 applies for $\chi(m,n,k)$ occurring in Theorem 4.1. We say that χ satisfies the Whitman condition (W) if the finitely presented lattice $FL(U;p_0\leqq q_0,p_1\leqq q_1,\ldots,p_t\leqq q_t)$ satisfies (W) (cf. [5]).

Proposition 7.2. *If χ satisfies (W) then χ is regular.*

Proof. By [5, Corollary 5.3] there are lattice identites $\varkappa_m$, $m<\omega$, such that, for any n-permutable variety $\mathscr{V}$, $\mathrm{Con}(\mathscr{V})\models\chi$ iff $(\exists m)(\mathrm{Con}(\mathscr{V})\models\varkappa_m)$. In particular, $\mathbf{L}(R)\models\chi$ iff $\mathrm{Con}(R\text{-}\mathbf{Mod})\models\chi$ iff $(\exists m)(\mathrm{Con}(R\text{-}\mathbf{Mod})\models\varkappa_m)$ iff $(\exists m)\big(\mathbf{L}(R)\models\varkappa_m\big)$, whence the regularity of the lattice identites $\varkappa_m$ (cf. Theorem 2.1 (a)) completes the proof.

8. Proof of Theorem 4.2. With the notations of Section 2, let us recall

Claim 8.1. (Hutchinson [12, Proposition 4 and the proof of Theorem 5] or, more explicitly, [2, Proposition 6.2]). If $S_1,S_2\in\mathscr{L}_S$ and $S_1\leqq S_2$ then R_{S_1} is a homomorphic image of R_{S_2}.

Given a spectrum $S\in\mathscr{L}_S$, let $\{p\colon p\in P$ and $S(p)<\omega\}$ be denoted by $T(S)$. Let S be called *cofinite* iff $T(S)$ is finite. Note that $S(0)=0$ for any cofinite $S\in\mathscr{L}_S$. If S is an arbitrary spectrum and H is a finite subset of P then the spectrum $S[H]$ defined by $S[H](0)=0$, $S[H](p)=S(p)$ for $p\in H$ and $S[H](p)=\omega$ for $p\in P\setminus H$ is cofinite, and we have $S\leqq S[H]$ and $T(S[H])\subseteq H$.

Now let us fix a regular Horn sentence χ which holds in $\mathbf{L}(R^*)$ for some ring R^* with $S_{R^*}(0)=0$. Put $S^*=S_{R^*}$. Since χ is regular, it holds in $\mathbf{L}(R_{S^*})$ by Theorem 2.1 (c). Let S^0 denote the zero spectrum, i.e., $S^0(x)=0$ for all $x\in\{0\}\cup P$, and put $R^0=R_{S^0}$. (Note that S^0 is not the smallest element of $\mathscr{L}$.) Then $S^0\leqq S^*$, whence, by Corollary 6.1 and Claim 8.1, χ holds in $L(R^0)$.

Now consider the system of ring equations $E_0, E_1, E_2, \ldots$ associated with χ by Lemma 5.1.

Claim 8.2. Let $S\in\mathscr{L}_S$ with $S(0)=0$ and let n be a non-negative integer. If E_n is solvable in R_S then there is a finite subset H of P such that E_n is solvable in $R_{S[H]}$.

Proof. Let E_n consist of the ring equations $f_i(y_j\colon j<u)=0$, $i<v$, and assume that $f_i(a_j+J_S\colon j<u)=0+J_S$, i.e., $f_i(a_j\colon j<u)\in J_S$, $i<v$, for certain elements $a_i\in FR(\{x_p\colon p\in P\})$, $i<v$ (cf. Section 2). As we have only finitely many $f_i(a_j\colon j<u)$, there is a finite subset A of $\{p^{S(p)}(px_p-1)\colon p\in P \text{ and } S(p)<\omega\}$ such that all the $f_i(a_j\colon j<u)$, $i<v$, belong to the ideal generated by A. Put $H=\{p\colon p\in P,\ S(p)<\omega$ and $p^{S(p)}(px_p-1)\in A\}$. Then H is finite, and $A\subseteqq J_{S[H]}$ yields $f_i(a_j\colon j<u)\in J_{S[H]}$ for all $i<v$. Hence the system of $a_j+J_{S[H]}$, $j<u$, is a solution of E_n in $R_{S[H]}$.

Since $E_0, E_1, E_2, \ldots$ is a weakening sequence, the first n_0 of its members can be omitted without the loss of generality, for any $n_0<\omega$. Therefore, by Lemma 5.1, we may assume that E_0 is solvable in R^0. Hence, by Claim 8.2, we can fix a cofinite spectrum S' such that E_0, and therefore every E_n, is solvable in $R_{S'}$. (Indeed, let $S'=S^0[H]$ for an appropriate $H\subseteqq P$.)

For $j<\omega$, let $U_j:=\{S_R\colon R$ is a ring and E_j is solvable in $R\}$. Then $U_j\subseteqq\mathscr{L}_S$. For each $S\in U_j$ choose a ring $B_{j,S}$ such that E_j is solvable in $B_{j,S}$ and the spectrum of $B_{j,S}$ is S. Put $A_j:=\prod(B_{j,S}\colon S\in U_j)$, the direct product of $B_{j,S}$, $S\in U_j$, and let $S_j:=\bigvee(S\colon S\in U_j)$.

Claim 8.3. The spectrum of A_j is S_j, S_j is cofinite, E_j is solvable in A_j, and, for any ring R, if E_j is solvable in R then $S_R\leqq S_j$.

Proof. E_j is clearly solvable in A_j as it is solvable in all the direct factors of A_j. Similarly, a divisibility condition $D(m,n)$, which is a particular ring equation, holds in A_j iff $D(m,n)$ holds in every $B_{j,S}$, $S\in U_j$. As $S'\in U_j$ and $S'(0)=0$, $S_j(0)=0$ and the characteristic of $A_j=B_{j,S'}\times\prod(B_{j,S}\colon S\in U_j\setminus\{S'\})$ is also 0. Further, by Theorem 2.1 (d), we have

$$\min\{i\colon A_j\models D(p^{i+1},p^i)\}=\min\bigcap(\{i\colon B_{j,S}\models D(p^{i+1},p^i)\}\colon S\in U_j)=$$

$$=\min\bigcap(\{i\colon i=\exp(p^i,p)\geqq S(p)\}\colon S\in U_j)=\sup\{S(p)\colon S\in U_j\}=S_j(p)$$

for any $p\in P$. Consequently, S_j is the spectrum of A_j. From $S'\leqq S_j$ we obtain $T(S')\supseteqq T(S_j)$, whence S_j is cofinite. Finally, if E_j is solvable in a ring R then $S_R\in U_j$ yields that $S_R\leqq\bigvee(S\colon S\in U_j)=S_j$.

Now let $I:=\{S\in\mathcal{L}_S\colon \chi$ holds in $\mathbf{L}(R_S)\}$, and let $(S_j]$ denote $\{S\in\mathcal{L}_S\colon S\leqq S_j\}$, the principal ideal of $\mathcal{L}_S$ generated by S_j.

Claim 8.4. $I=\bigcup((S_j]\colon j<\omega)$.

Proof. If $S\in I$ then, by Lemma 5.1, there is a $j<\omega$ such that E_j is solvable in R_S. Hence $S=S_{R_S}\in(S_j]$ by Claim 8.3. Conversely, assume that $S\in(S_j]$ for some $j<\omega$. By Lemma 5.1 and Claim 8.3, χ holds in $\mathbf{L}(A_j)$. By Theorem 2.1 (c) and Claim 8.3, A_j and R_{S_j} have the same spectrum S_j. The regularity of χ yields that χ holds in $\mathbf{L}(R_{S_j})$, too. Now $S\in I$ follows from $S\leqq S_j$, Claim 8.1 and Corollary 6.1.

Now we obtain $S_0\leqq S_1\leqq S_2\leqq\dots$ from the fact that $E_0, E_1, E_2, \dots$ is a weakening sequence. Hence $T(S_0)\supseteqq T(S_1)\supseteqq T(S_2)\supseteqq\dots$. Since $T(S_0)$ is finite, so is $H:=\bigcap(T(S_j)\colon j<\omega)$. Put $S:=\bigvee(S_j\colon j<\omega)$, then $T(S)\subseteqq H$. Define m_χ, n_χ and k_χ as follows:

$$m_\chi:=\prod(p^{S(p)+1}\colon p\in T(S)),\quad n_\chi:=\prod(p\colon p\in H\setminus T(S))$$

and

$$k_\chi:=\prod(p^{S(p)}\colon p\in T(S)).$$

Then m_χ, n_χ and k_χ are positive integers.

Assume that $D(m_\chi n_\chi^{i+1}, k_\chi n_\chi^i)$ holds in a ring R for some $i<\omega$. Then, by Theorem 2.1 (d), we have $S(p)=\exp(k_\chi n_\chi^i, p)\geqq S_R(p)$ for $p\in T(S)$ and $S(p)=\omega>i=\exp(k_\chi n_\chi^i, p)\geqq S_R(p)$ for $p\in H\setminus T(S)$. For $p\in H$, $S(p)$ is the limit of the increasing sequence $S_0(p), S_1(p), S_2(p), \dots$, whence the finiteness of H yields the existence of a $j<\omega$ such that $T(S_j)=H$ and $S_R(p)\leqq S_j(p)$ for all $p\in H$. Then $S_R\leqq S_j$, and Claim 8.4 yields that $S_R\in I$. I.e., χ holds in $\mathbf{L}(R_{S_R})$. Since R_{S_R} and R have the same spectrum and χ is regular, χ holds in $\mathbf{L}(R)$.

Conversely, assume that R is a ring and χ holds in $\mathbf{L}(R)$. As R_{S_R} and R have the same spectrum and χ is regular, $S_R\in I$. By Claim 8.4, there is a $j<\omega$ such that $T(S_j)=H$ and $S_R\leqq S_j\leqq S$. Put $i=\max\{S_j(p)\colon p\in H\setminus T(S)\}$, then i is a non-negative integer. (Here $\max\emptyset=0$.) For $p\in T(S)$, $\exp(k_\chi n_\chi^i, p)=S(p)\geqq S_R(p)$ while, for $p\in H\setminus T(S)$, $\exp(k_\chi n_\chi^i, p)=i\geqq S_j(p)\geqq S_R(p)$. Hence, by Theorem 2.1 (d), $D(m_\chi n_\chi^{i+1}, k_\chi n_\chi^i)$ holds in R. This completes the proof of Theorem 4.2.

9. On a problem of Jónsson. In this section we will give a negative answer to (1.4), the afore-mentioned problem of Jónsson [13]. Let $n\geqq 2$ be an integer, and consider $\chi(0, n, 1)$ from Theorem 4.1. Then we have

Proposition 9.1. *There is no strong Mal'tsev condition U such that, for any congruence permutable variety $\mathcal{V}$, $\chi(0, n, 1)$ holds in* Con $(\mathcal{V})$ *iff U holds in $\mathcal{V}$.*

Proof. Assume the contrary, and let E be a system of ring equations such that, for any ring R, U holds in R-**Mod** iff E is solvable in R (cf. the proof of Lemma 5.1). Since $D(0, n^i)$ holds in $\mathbf{Z}_{n^i}$, $\chi(0, n, 1)$ holds in $\mathbf{L}(\mathbf{Z}_{n^i})$ by Theorem 4.1, and we infer that E is solvable in $\mathbf{Z}_{n^i}$, $i<\omega$. Therefore E is solvable in the direct product $R=\prod(\mathbf{Z}_{n^i}\colon i<\omega)$, whence $\chi(0, n, 1)$ holds in $\mathbf{L}(R)$. It follows from Theorem 4.1 that there is a $j<\omega$ such that $D(0, n^j)$ holds in R. Consequently, $D(0, n^j)$ holds in every direct factor $\mathbf{Z}_{n^i}$ of R. In particular, $D(0, n^j)$ holds in $\mathbf{Z}_{n^{j+1}}$, which is a contradiction.

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