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Quasi-identities, Mal'cev conditions and congruence regularity

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Dedicated to the memory of András P. Huhn

This work grew out of our desire to present a uniform approach to the various forms of congruence regularity which have been studied in the literature. We were particularly interested in the result of GRÄTZER [8] that if every algebra in a variety $\mathscr{V}$ contains an element a such that $[a]\alpha=[a]\beta$ implies $\alpha=\beta$ for all congruences α, β on A, then the element a may be chosen from any subalgebra of A. We also wished to study the concept of subregularity introduced by TIMM [15]: an algebra A is *subregular* if for all subalgebras $B\leqq A$ and all congruences α, β on A we have $\alpha=\beta$ whenever $[b]\alpha=[b]\beta$ for all $b\in B$. In particular we wanted a characterization of subregularity via simple identities and quasi-identities similar to those for regularity due to WILLE [16] and CSÁKÁNY [2]. These two topics turned out to be quite closely related (Theorem 2.3).

In the first section the various types of regularity are defined and their local properties are investigated. In particular, we give characterizations in terms of principal congruences similar to those for regularity and weak regularity give in HASHIMOTO [12] and GRÄTZER [8] (Lemma 1.3). We also apply GUMM's Shifting Principle [9] to give local proofs of congruence modularity where possible (Theorem 1.4).

The global relationships between the various forms of regularity are studied in Section 2. The section begins with a general translation principle for converting a Hashimoto-type principal-congruence property into a quasi-identity (Theorem 2.1) which is then applied to yield quasi-identity characterizations for each of the forms of regularity (Theorems 2.2, 2.3, 2.4, 2.5).

The third section contains a general consideration of the relationships between quasi-identities, identities, congruence modularity and n-permutability. Several ways of translating quasi-identities into identities are given (Theorems 3.4, 3.5). We

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also describe a large class of quasi-identities which imply both congruence modularity and n-permutability for some n (Theorem 3.9).

In Section 4 we see that the results of Sections 2 and 3 may be combined to yield identities characterizing each of the forms of regularity and show that, with one exception, each implies congruence modularity and n-permutability for some n.

Our notation and terminology are fairly standard. Note in particular that the lattice of congruences of an algebra A is denoted by $\mathrm{Con}\, A$ with least element 0, the n-generated free algebra in a variety $\mathcal{V}$ is denoted by $F\mathcal{V}(n)$ and by a *constant term* we mean a nullary or constant unary term.

1. Definitions and local relationships. In this section we introduce various degrees of regularity and study these at the local level. A Hashimoto-type principal-congruence characterization is given for each and, where possible, a local proof of congruence modularity is obtained via H. P. Gumm's Shifting Principle.

An algebra A is *regular with respect to* $a_1, \ldots, a_n \in A$ if for all $\alpha, \beta \in \mathrm{Con}\, A$ we have

$$\left(\mathop{\&}_{i=1}^{n} [a_i]\alpha = [a_i]\beta\right) \Rightarrow \alpha = \beta.$$

R: A is *regular* if it is regular with respect to a for each $a \in A$.

R_n: A is *n-regular* if there exist $a_1, \ldots, a_n \in A$ such that A is regular with respect to $a_1, \ldots, a_n$.

SR: A is *subregular* if it is regular with respect to each of its subalgebras, that is, for each $B \leqq A$ and all $\alpha, \beta \in \mathrm{Con}\, A$ we have

$$\left(\mathop{\&}_{b \in B} [b]\alpha = [b]\beta\right) \Rightarrow \alpha = \beta.$$

SR_n: A is *n-subregular* if for all $B \leqq A$ there exist $b_1, \ldots, b_n \in B$ such that A is regular with respect to $b_1, \ldots, b_n$.

Note that 1-regularity is usually referred to as *weak regularity*. Some authors have insisted that the elements $a_1, \ldots, a_n$ in the definition of n-regularity be constant terms: if there are constant terms $o_1, \ldots, o_n$ such that A is regular with respect to $o_1, \ldots, o_n$ then we shall say that A satisfies $\mathsf{R}(o_1, \ldots, o_n)$. We say that a class $\mathcal{V}$ of algebras is *regular* (respectively, *subregular*, etc.) if every algebra in $\mathcal{V}$ is regular (respectively, subregular, etc.).

In TIMM [15] it is pointed out that the algebra $\langle \mathbf{N}; s \rangle$, where s is the successor function, is subregular and it is easily seen that it is not n-regular for any $n \in \mathbf{N}$. The non-zero congruences on $\langle \mathbf{N}; s \rangle$ are all of the form $\Theta(m, k)$ for some $m, k \in \mathbf{N}$ where

$$x \Theta(m,k) y \quad \Leftrightarrow \quad x = y < m \quad \text{or} \quad (x, y \geqq m \;\;\&\;\; x \equiv y \pmod{k}).$$

Since $\Theta(m,k)\leqq\Theta(n,l)\Leftrightarrow n\leqq m$ and $l|k$, we have

$$\mathrm{Con}\,\langle \mathbf{N}; s\rangle \cong 1\oplus[\langle \mathbf{N}; \leqq\rangle^d\times\langle \mathbf{N}; |\rangle^d].$$

Note that $\langle \mathbf{N}; s\rangle$ is congruence-distributive.

The implications in the diagram below are trivial:

$$\begin{array}{ccccccc} & & \mathsf{R}_1 & \longrightarrow & \mathsf{R}_2 & \to\dots & \\ & & \uparrow & & \uparrow & & \\ \mathsf{R} & \to & \mathsf{SR}_1 & \longrightarrow & \mathsf{SR}_2 & \to\dots\to & \mathsf{SR} \\ & & \uparrow & & \uparrow & & \\ & & \mathsf{R}(o_1) & \to & \mathsf{R}(o_1, o_2) & \to\dots & \end{array}$$

In the presence of a one-element subalgebra many of these relations collapse.

1.1. Lemma. (i) *If A has a one-element subalgebra, then on A we have* $\mathsf{SR}\to\mathsf{R}_1$.

(ii) *If there is a constant term o such that $\{o\}\leqq A$, then on A we have* $\mathsf{SR}\to\mathsf{R}(o)$.

The following characterizations using set inclusion rather than equality are often useful. If $\alpha\in\mathrm{Con}\,A$ and $B\leqq A$ then $\alpha{\restriction}B$ denotes the restriction of α to B and $[B]\alpha$ denotes the union over $b\in B$ of the α-blocks $[b]\alpha$.

1.2. Lemma. (i) *A is regular if and only if*

$$(\forall a\in A)(\forall \alpha, \beta\in\mathrm{Con}\,A)\big[[a]\alpha\subseteqq[a]\beta\Rightarrow\alpha\subseteqq\beta\big].$$

(ii) *A is n-regular if and only if*

$$(\exists a_1, \dots, a_n\in A)(\forall \alpha, \beta\in\mathrm{Con}\,A)\big[(\mathop{\&}_{i=1}^{n}[a_i]\alpha\subseteqq[a_i]\beta)\Rightarrow\alpha\subseteqq\beta\big].$$

(iii) *A is n-subregular if and only if*

$$(\forall B\leqq A)(\exists b_1, \dots, b_n\in B)(\forall \alpha, \beta\in\mathrm{Con}\,A)\big[(\mathop{\&}_{i=1}^{n}[b_i]\alpha\subseteqq[b_i]\beta)\Rightarrow\alpha\subseteqq\beta\big].$$

(iv) *The following are equivalent:*

(a) *A is subregular;*

(b) $(\forall B\leqq A)(\forall\alpha, \beta\in\mathrm{Con}\,A)\big[(\mathop{\&}_{b\in B}[b]\alpha\subseteqq[b]\beta)\Rightarrow\alpha\subseteqq\beta\big]$;

(c) $(\forall B\leqq A)(\forall\alpha, \beta\in\mathrm{Con}\,A)\big[(\alpha{\restriction}B\subseteqq\beta{\restriction}B\,\&\,[B]\alpha=B)\Rightarrow\alpha\subseteqq\beta\big]$.

Proof. These proofs are trivial once we observe that $[a]\alpha\subseteqq[a]\beta$ implies $[a]\alpha=[a](\alpha\wedge\beta)$.

The version of subregularity given in 1.2 (iv) (c) has been useful in the study of injectivity: see, Davey and Kovács [3].

We now give the Hashimoto-type principal-congruence characterizations of the various forms of regularity. The subalgebra generated by $a\in A$ is denoted by $\langle a\rangle$.

1.3. Lemma. (i) *A is regular with respect to $a_1, \ldots, a_n \in A$ if and only if for all $b, c \in A$ there exist $d_{i1}, \ldots, d_{im} \in A$ such that*

$$\Theta(b, c) = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} \Theta(a_i, d_{ij}).$$

(ii) $A \models \mathsf{R} \Leftrightarrow (\forall a \in A)(\forall b, c \in A)(\exists d_1, \ldots, d_m \in A) \quad \Theta(b, c) = \bigvee_{j=1}^{m} \Theta(a, d_j).$

(iii) $A \models \mathsf{R}_n \Leftrightarrow (\exists a_1, \ldots, a_n \in A)(\forall b, c \in A)(\exists d_{i1}, \ldots, d_{im} \in A)$

$$\Theta(b, c) = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} \Theta(a_i, d_{ij}).$$

(iv) $A \models \mathsf{SR} \Leftrightarrow (\forall a \in A)(\forall b, c \in A)(\exists a_1, \ldots, a_n \in \langle a \rangle)(\exists d_1, \ldots, d_n \in A)$

$$\Theta(b, c) = \bigvee_{i=1}^{n} \Theta(a_i, d_i).$$

(v) $A \models \mathsf{SR}_n \Leftrightarrow (\forall a \in A)(\exists a_1, \ldots, a_n \in \langle a \rangle)(\forall b, c \in A)(\exists d_{i1}, \ldots, d_{im} \in A)$

$$\Theta(b, c) = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} \Theta(a_i, d_{ij}).$$

Proof. (ii) is due to Hashimoto [12] and Grätzer [8]. As all proofs are similar, we prove only (iv).

Assume that A is subregular. Let $a, b, c \in A$ and let α be the smallest congruence on A having $[a']\Theta(b, c)$ as a block for all $a' \in \langle a \rangle$. Then for all $a' \in \langle a \rangle$ we have $[a']\alpha = [a']\Theta(b, c)$, whence $\alpha = \Theta(b, c)$ by subregularity. Thus

$$\Theta(b, c) = \bigvee\big(\Theta(a', d) \mid a' \in \langle a \rangle \ \&\ d \in [a']\Theta(b, c)\big).$$

Since $\Theta(b, c)$ is compact, there exist $a_1, \ldots, a_n \in \langle a \rangle$ and $d_i \in [a_i]\Theta(b, c)$ with

$$\Theta(b, c) = \bigvee_{i=1}^{n} \Theta(a_i, d_i).$$

Conversely, suppose that the principal-congruence condition holds. Let $B \leqq A$ and suppose that $\alpha, \beta \in \operatorname{Con} A$ satisfy $[b]\alpha \subseteqq [b]\beta$ for all $b \in B$. Let $a \in B$ and $b, c \in A$ and let $a_i \in \langle a \rangle \subseteqq B$ and $d_{ij} \in A$ be given by the principal-congruence condition. Now suppose that $b \equiv c(\alpha)$. Since $\Theta(a_i, d_i) \subseteqq \Theta(b, c) \subseteqq \alpha$, we have $d_i \in [a_i]\alpha \subseteqq [a_i]\beta$ for all i and hence $a_i \equiv d_i(\beta)$ for all i. Thus $\Theta(b, c) \subseteqq \beta$, whence $b \equiv c(\beta)$. Consequently $\alpha \subseteqq \beta$.

While the proof below of congruence modularity uses Gumm's Shifting Principle, it is closely related to the corresponding proof in Bulman-Fleming, Day and Taylor [1].

1.4. **Theorem.** *If every subalgebra of* A^2 *is subregular, then* $\operatorname{Con} A$ *is modular.*

Proof. By Lemma 3.2 of GUMM [9] it suffices to prove that if $\alpha, \gamma \in \operatorname{Con} A$ and $\Lambda \leqq A^2$ is reflexive and symmetric with $\alpha \cap \Lambda \leqq \gamma \leqq \alpha$, then whenever we have

$$\begin{tikzcd} x \arrow[r, dash, "\Lambda"] \arrow[d, dash] & z \arrow[d, dash, "\alpha"'] \arrow[d, dash, bend left, "\gamma"] \\ y \arrow[r, dash] & u \end{tikzcd}$$

it follows that $x\gamma y$. Let α, γ and Λ be as stated and assume that the relations indicated in the diagram hold.

Consider $\alpha\times\gamma$ and $\gamma\times\gamma$ as congruences on the subregular algebra Λ: note that $\gamma\times\gamma\leqq\alpha\times\gamma$. Denote the diagonal of A^2 by Δ and consider $(a, a)\in\Delta$. Let $(b, c)\in\Lambda$ with $(b, c)\ \alpha\times\gamma\ (a, a)$. Then

$$\begin{aligned} b\,\alpha\,a \quad \& \quad c\,\gamma\,a \quad \& \quad b\,\Lambda\,c &\Rightarrow b\,\alpha\cap\Lambda\,c && \text{as} \quad \gamma\leqq\alpha \\ &\Rightarrow b\,\gamma\,c && \text{as} \quad \alpha\cap\Lambda\leqq\gamma \\ &\Rightarrow b\,\gamma\,a && \text{as} \quad c\,\gamma\,a \\ &\Rightarrow (b, c)\ \gamma\times\gamma\ (a, a). \end{aligned}$$

Thus $[(a, a)]_\Lambda\gamma\times\gamma=[(a, a)]_\Lambda\alpha\times\gamma$.

Hence $B:=[\Delta]_\Lambda\gamma\times\gamma=[\Delta]_\Lambda\alpha\times\gamma$ and $(\gamma\times\gamma)\restriction B=(\alpha\times\gamma)\restriction B$. Consequently $\gamma\times\gamma= =\alpha\times\gamma$ on Λ as Λ is subregular. Since $(x, z), (y, u)\in\Lambda$ with $(x, z)\ \alpha\times\gamma\ (y, u)$ we have $x\,\gamma\,y$, as required. (Note that the symmetry of Λ was not required.)

Similarly it can be proved that if $\mathsf{S}(A^2)\models\mathsf{R}(o_1, \ldots, o_n)$ then $\operatorname{Con} A$ is modular. It follows trivially from Theorem 1.4 that if $\mathsf{S}(A^2)\models\mathsf{SR}_n$ for some n, then $\operatorname{Con} A$ is modular; it seems highly unlikely that a similar conclusion can be made about R_n since the elements with respect to which A is regular cannot be forced into the diagonal.

2. Global relationships. In [2], CSÁKÁNY characterized regularity for varieties via a quasi-identity: a variety $\mathscr{V}$ is regular if and only if there are ternary terms $p_1, \ldots, p_n$ such that

$$\mathscr{V}\models x=y\leftrightarrow \mathop{\&}_{i=1}^{n} p_i(xyz)=z.$$

Much earlier, THURSTON [14] showed that $\mathscr{V}$ is regular if and only if for all $A\in\mathscr{V}$, all $\alpha\in\operatorname{Con} A$ and all $a\in A$ we have that $|[a]\alpha|=1$ implies $\alpha=0$. We now give the corresponding characterizations for our more general forms of regularity. Along the way we shall see that at the global level the various regularities come closer together.

The following translation principle allows us to convert a Hashimoto-type principal-congruence property directly into a quasi-identity.

2.1. Theorem. *Let $\mathcal{V}$ be a variety and let f_i, g_i, r and s be n-ary terms. Then the following are equivalent:*

(i) $\mathcal{V} \models (\mathop{\&}_{i=1}^{m} f_i(\vec{x}) = g_i(\vec{x})) \to r(\vec{x}) = s(\vec{x})$;

(ii) *for all* $A \in \mathcal{V}$ *and all* $\vec{a} \in A^n$

$$\bigvee_{i=1}^{m} \Theta(f_i(\vec{a}), g_i(\vec{a})) \supseteq \Theta(r(\vec{a}), s(\vec{a}));$$

(iii) *the elements* $f_i(\vec{x}), g_i(\vec{x}), r(\vec{x})$ *and* $s(\vec{x})$ *of* $F\mathcal{V}(n)$ *satisfy*

$$\bigvee_{i=1}^{m} \Theta(f_i(\vec{x}), g_i(\vec{x})) \supseteq \Theta(r(\vec{x}), s(\vec{x})).$$

Proof. (i)$\Rightarrow$(ii). Let $A \in \mathcal{V}$ and $\vec{a} \in A^n$ and define α to be $\bigvee_{i=1}^{m} \Theta(f_i(\vec{a}), g_i(\vec{a}))$. In A/α we have $f_i(\vec{b}) = g_i(\vec{b})$ for all i where $b_j := [a_j]\alpha$. Thus, by (i), $r(\vec{b}) = s(\vec{b})$, whence $r(\vec{a})\ \alpha\ s(\vec{a})$ as required.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i). Let $A \in \mathcal{V}$ and $\vec{a} \in A^n$ with $f_i(\vec{a}) = g_i(\vec{a})$ for all i. Let $\varphi\colon F\mathcal{V}(n) \to A$ be a homomorphism with $x_j\varphi = a_j$. Then $\bigvee_{i=1}^{m} \Theta(f_i(\vec{x}), g_i(\vec{x})) \subseteq \ker\varphi$ and so $r(\vec{x}) \ker\varphi\ s(\vec{x})$ by (iii). Thus $r(\vec{a}) = s(\vec{a})$, as required.

In the following result we require the observation that if A is regular with respect to $a_1, \ldots, a_n \in A$ and $\varphi\colon A \to B$ is a surjective homomorphism with $b_i = a_i\varphi$, then B is regular with respect to $b_1, \ldots, b_n$.

2.2. Theorem. *The following are equivalent for any variety $\mathcal{V}$:*

(i) $\mathcal{V} \models \mathsf{SR}_n$;

(ii) $\mathcal{V} \models \mathsf{R}_n$;

(iii) *there exist unary terms* $u_1, \ldots, u_n$ *such that for all* $A \in \mathcal{V}$ *and all* $a \in A$, A *is regular with respect to* $u_1(a), \ldots, u_n(a)$;

(iv) *there exist unary terms* $u_1, \ldots, u_n$ *such that for all* $A \in \mathcal{V}$, *all* $a \in A$ *and all* $\alpha \in \operatorname{Con} A$ *if* $|[u_1(a)]\alpha| = \ldots = |[u_n(a)]\alpha| = 1$, *then* $\alpha = 0$;

(v) *there exist unary terms* $u_1, \ldots, u_n$ *and ternary terms* $p_{11}, \ldots, p_{nm}$ *such that*

$$\mathcal{V} \models (\mathop{\&}_{i=1}^{n} \mathop{\&}_{j=1}^{m} p_{ij}(xyz) = u_i(z)) \leftrightarrow x = y;$$

(vi) *there exist unary terms* $u_1, \dots, u_n$ *and ternary terms* $p_1, \dots, p_m$ *and a selection function* $j \mapsto i_j$ *such that*

$$\mathscr{V} \models \Big(\mathop{\&}_{j=1}^{m} p_j(xyz) = u_{i_j}(z) \Big) \leftrightarrow x = y.$$

Proof. That (i) implies (ii) is trivial. Assume that $\mathscr{V} \models \mathsf{R}_n$. Then there exist $v_1, \dots, v_n \in F\mathscr{V}(\mathrm{N})$ such that $F\mathscr{V}(\mathrm{N})$ is regular with respect to $v_1, \dots, v_n$. Assume that $v_1, \dots, v_n$ depend only upon $x_1, \dots, x_k$; then we can find an onto homomorphism $\psi\colon F\mathscr{V}(\mathrm{N}) \to F\mathscr{V}(\{x, y, z\})$ with $x_i\varphi = z$ for $i = 1, \dots, k$. Thus the image u_i of v_i under φ depends only upon z and $F\mathscr{V}(3)$ is regular with respect to $u_1, \dots, u_n$.

Suppose that $A \in \mathscr{V}$, $a \in A$ and $\alpha, \beta \in \mathrm{Con}\, A$ with $[u_i(a)]\alpha \subseteq [u_i(a)]\beta$ for all i. Let $s, t \in A$ with $s\ \alpha\ t$ and define $\varphi\colon F\mathscr{V}(\{x, y, z\}) \to A$ by $x\varphi = s$, $y\varphi = t$, $z\varphi = a$. Then $x\ \bar{\alpha}\ y$ where $\bar{\alpha}$ denotes the inverse image of α under φ. Now $v\ \bar{\alpha}\ u_i(z)$ implies $v\varphi\ \alpha\ u_i(a)$; hence $v\varphi\ \beta\ u_i(a)$ and so $v\ \bar{\beta}\ u_i(z)$. Thus $[u_i(z)]\bar{\alpha} \subseteq [u_i(z)]\bar{\beta}$ for all i and consequently $\bar{\alpha} \subseteq \bar{\beta}$ since $F\mathscr{V}(\{x, y, z\})$ is regular with respect to $u_1(z), \dots, u_n(z)$. Hence

$$s\ \alpha\ t \Rightarrow x\ \bar{\alpha}\ y \Rightarrow x\ \bar{\beta}\ y \Rightarrow s\ \beta\ t$$

and thus $\alpha \subseteq \beta$. Hence (ii) implies (iii).

That (v) follows from (iii) is a direct consequence of the principal-congruence characterization of regularity with respect to $a_1, \dots, a_n$ given in Lemma 1.3 (i) and the translation to quasi-identities given in Theorem 2.1: take $A = F\mathscr{V}(\{x, y, z\})$, $a_i = u_i(z)$, $b = x$, $c = y$ and $d_{ij} = p_{ij}(xyz)$. The equivalence of (v) and (vi) is clear.

The combination of Theorem 2.1 and Lemma 1.3 (v) shows that (v) implies (i). It remains to prove that (iv) implies (iii).

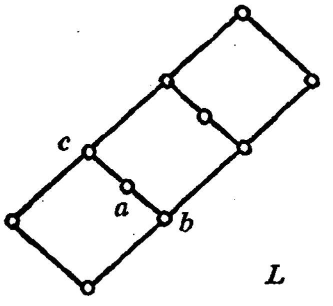
Suppose that $[u_i(a)]\alpha \subseteq [u_i(a)]\beta$ for $i = 1, \dots, n$; then $[u_i(a)]\alpha = [u_i(a)]\alpha \wedge \beta$. Consider the congruence $\alpha/(\alpha \wedge \beta)$ on $A/(\alpha \wedge \beta)$. The block of $[u_i(a)]\alpha \wedge \beta$ in $\alpha/(\alpha \wedge \beta)$ is a singleton for all i and hence, by (iv), we have $\alpha/(\alpha \wedge \beta) = 0$ in $\mathrm{Con}\, A/(\alpha \wedge \beta)$. Thus $\alpha = \alpha \wedge \beta$ and so $\alpha \subseteq \beta$, as required.

The choice between (v) and (vi) is a matter of taste: in (v) the emphasis is on the unary terms while the emphasis in (vi) is on the ternary terms. The equivalence of SR_1 and R_1 was observed by GRÄTZER [8]. It is tempting to replace (iv) by

(iv)′ $\quad (\forall A \in \mathscr{V})(\exists a_1, \dots, a_n \in A)(\forall \alpha \in \mathrm{Con}\, A)(|[a_1]\alpha| = \dots = |[a_n]\alpha| = 1) \Rightarrow \alpha = 0.$

An algebra A with this property might be called *regular at* 0 *with respect to* $a_1, \dots, a_n$. But this property is not preserved by homomorphisms and so the proof method used above is not applicable. The lattice L drawn below is regular at 0 with respect to a as it is subdirectly irreducible and both ab and ac are critical edges (that is,

$\Theta(a, b)=\Theta(a, c)$ is the monolith of L). Since $L/\Theta(a, b)$ is a four-element chain it is not regular at 0 with respect to any one of its elements.



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The proof of our next result is now easy and is omitted.

2.3. Theorem. *The following are equivalent for any variety* $\mathcal{V}$:

(i) $\mathcal{V} \models \mathsf{SR}$;

(ii) $(\exists n \in \mathrm{N}) \mathcal{V} \models \mathsf{SR}_n$; (ii)' $(\forall A \in \mathcal{V})(\exists n \in \mathrm{N}) A \models \mathsf{SR}_n$;

(iii) $(\exists n \in \mathrm{N}) \mathcal{V} \models \mathsf{R}_n$; (iii)' $(\forall A \in \mathcal{V})(\exists n \in \mathrm{N}) A \models \mathsf{R}_n$;

(iv) *for all* $A \in \mathcal{V}$, *all* $B \leqq A$ *and all* $\alpha \in \mathrm{Con}\, A$ *if* $|[b]\alpha| = 1$ *for all* $b \in B$, *then* $\alpha = 0$;

(v) *there exist* $n \in \mathrm{N}$, *unary terms* $u_1, \ldots, u_n$ *and ternary terms* $p_1, \ldots, p_n$ *such that*

$$\mathcal{V} \models \big(\mathop{\&}_{i=1}^{n} p_i(xyz) = u_i(z)\big) \leftrightarrow x = y.$$

Call a variety $\mathcal{V}$ *locally regular with respect to unary terms* $u_1, \ldots, u_n$ (and write $\mathcal{V} \models \mathsf{LR}(u_1, \ldots, u_n)$) if

$$(\forall A \in \mathcal{V})(\forall a \in A)\big[\big(\mathop{\&}_{i=1}^{n} [u_i(a)]\alpha = [u_i(a)]\beta\big) \Rightarrow [a]\alpha = [a]\beta\big].$$

This concept was introduced, under a different name, in the important but unpublished paper Hagemann [10] where a characterization via identities was obtained; no quasi-identity characterization was given. It is clear from Theorem 2.2 that, at the varietal level, we have $\mathsf{R}_n \to \mathsf{LR}_n$. The proof of the following result should by now be an easy exercise.

2.4. Theorem. *The following are equivalent for any variety* $\mathcal{V}$ *and unary terms* $u_1, \ldots, u_n$:

(i) $\mathcal{V} \models \mathsf{LR}(u_1, \ldots, u_n)$;

(ii) *for all* $A \in \mathcal{V}$, *all* $a \in A$ *and all* $\alpha, \beta \in \mathrm{Con}\, A$

$$\mathop{\&}_{i=1}^{n} \big([u_i(a)]\alpha \subseteqq [u_i(a)]\beta\big) \Rightarrow [a]\alpha \subseteqq [a]\beta;$$

(iii) *for all* $A\in\mathscr{V}$ *and all* $a, b\in A$ *there exist* $d_{i1}, \ldots, d_{im}\in A$ *such that*

$$\Theta(a, b) = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} \Theta(u_i(a), d_{ij});$$

(iv) *for all* $A\in\mathscr{V}$, *all* $a\in A$ *and all* $\alpha\in\mathrm{Con}\, A$ *if* $|[u_1(a)]\alpha|=\ldots=|[u_n(a)]\alpha|=1$, *then* $|[a]\alpha|=1$;

(v) *there exist binary terms* $p_{11}, \ldots, p_{nm}$ *such that*

$$\mathscr{V}\models\Big(\mathop{\&}_{i=1}^{n} \mathop{\&}_{j=1}^{m} p_{ij}(xy) = u_i(x)\Big) \leftrightarrow x = y;$$

(vi) *there exist binary terms* $p_1, \ldots, p_k$ *and a selection function* $j\mapsto i_j$ *such that*

$$\mathscr{V}\models\Big(\mathop{\&}_{j=1}^{k} p_j(xy) = u_{i_j}(x)\Big) \leftrightarrow x = y.$$

Note that $\mathsf{R}(o_1, \ldots, o_n)$ for constant terms $o_1, \ldots, o_n$ implies LR_n and if the terms in the definition of LR_n can be chosen to be constants then we obtain the reverse implication. Thus Theorem 2.4 yields the quasi-identity characterization of $\mathsf{R}(o_1, \ldots, o_n)$.

2.5. Corollary. *The following are equivalent for any variety* $\mathscr{V}$ *and constant terms* $o_1, \ldots, o_n$:

(i) $\mathscr{V}\models\mathsf{R}(o_1, \ldots, o_n)$;

(ii) *there exist binary terms* $p_{11}, \ldots, p_{nm}$ *such that*

$$\mathscr{V}\models\Big(\mathop{\&}_{i=1}^{n} \mathop{\&}_{j=1}^{m} p_{ij}(xy) = o_i\Big) \leftrightarrow x = y;$$

(iii) *there exist binary terms* $p_1, \ldots, p_k$ *and a selection function* $j\mapsto i_j$ *such that*

$$\mathscr{V}\models\Big(\mathop{\&}_{j=1}^{k} p_j(xy) = o_{i_j}\Big) \leftrightarrow x = y.$$

3. Quasi-identities, congruence modularity and permutability. In this section we give the general translation from quasi-identities to identities and investigate the relationship between quasi-identities, congruence modularity and n-permutability.

Lemmas 3.1 and 3.2 are simply restatements of Mal'cev's description of principal congruences. If $Z\subseteq A^2$, then $\Theta(Z)$ denotes the smallest congruence containing Z.

3.1. Lemma. *Let* $Z\subseteq A^2$ *and let* $(c, d)\in A^2$. *Then* $(c, d)\in\Theta(Z)$ *if and only if for some* $k, l\in\mathbf{N}$ *there exist* $(l+2)$*-ary terms* $w_1, \ldots, w_k$, *there exists* $\bar{e}\in A^l$ *and* *there are pairs* (a_i, b_i) *such that*

$$c = w_1(a_1, b_1, \vec{e})$$

$$w_1(b_1, a_1, \vec{e}) = w_2(a_1, b_1, \vec{e})$$

$$\vdots$$

$$w_k(b_k, a_k, \vec{e}) = d,$$

and $(a_i, b_i) \in Z$ *for all* i.

3.2. Lemma. *Let* $Z \subseteqq A^2$ *and let* $(c, d) \in A^2$. *Then* $(c, d) \in \Theta(Z)$ *if and only if for some* $k, l \in \mathbf{N}$ *there exist* $(l+1)$*-ary terms* $w_1, \ldots, w_k$, *there exists* $\vec{e} \in A^l$ *and there are pairs* (a_i, b_i) *such that*

$$c = w_1(a_1, \vec{e})$$

$$w_1(b_1, \vec{e}) = w_2(a_2, \vec{e})$$

$$\vdots$$

$$w_k(b_k, \vec{e}) = d,$$

and $(a_i, b_i) \in Z$ *or* $(b_i, a_i) \in Z$ *for all* i.

Recall that A is called *k-permutable* if for all $\alpha, \beta \in \operatorname{Con} A$ we have $\alpha \vee \beta = \alpha \circ \beta \circ \alpha \ldots$ (with k factors).

Clearly the last line of Lemma 3.2 is needed to guarantee symmetry. Hagemann [10] showed that if $\mathcal{V}$ is a k-permutable variety then for all $A \in \mathcal{V}$, if R is a reflexive subalgebra of A^2, then $R \circ \ldots \circ R$ (with $k-1$ factors) is a congruence. Using this we can simplify Lemma 3.2. The result was rediscovered by Lakser [13] and Duda [5].

3.3. Lemma. *Assume that* A *belongs to a* $(k+1)$*-permutable variety. Let* $Z \subseteqq A^2$ *and let* $(c, d) \in A^2$. *Then* $(c, d) \in \Theta(Z)$ *if and only if for some* $l \in \mathbf{N}$ *there exist* $(l+1)$*-ary terms* $w_1, \ldots, w_k$, *there exists* $\vec{e} \in A^l$ *and there are pairs* (a_i, b_i) *such that*

$$c = w_1(a_1, \vec{e})$$

$$w_1(b_1, \vec{e}) = w_2(a_2, \vec{e})$$

$$\vdots$$

$$w_k(b_k, \vec{e}) = d,$$

and $(a_i, b_i) \in Z$ *for all* i.

The translation from quasi-identities to identities is obtained by combining one of these lemmas with the principal-congruence translation given in the previous section.

3.4. Theorem. *Let $\mathscr{V}$ be a variety and let f_i, g_i, r and s be n-ary terms. Then the following are equivalent:*

(i) $\mathscr{V} \models \left(\mathop{\&}\limits_{i=1}^{m} f_i(\vec{x}) = g_i(\vec{x})\right) \to r(\vec{x}) = s(\vec{x})$;

(ii) *for some* $k \in \mathbf{N}$ *there exist* $(n+2)$*-ary terms* $t_1, \ldots, t_k$ *and pairs* $(u_j, v_j) \in \{(f_i, g_i) | i = 1, \ldots, m\}$ *such that* $\mathscr{V}$ *satisfies the identities*

$$\begin{aligned} r(\vec{x}) &= t_1(u_1(\vec{x}), v_1(\vec{x}), \vec{x}) \\ t_1(v_1(\vec{x}), u_1(\vec{x}), \vec{x}) &= t_2(u_2(\vec{x}), v_2(\vec{x}), \vec{x}) \\ &\vdots \\ t_k(v_1(\vec{x}), u_2(\vec{x}), \vec{x}) &= s(\vec{x}); \end{aligned}$$

(iii) *for some* $k \in \mathbf{N}$ *there exist* $(n+1)$*-ary terms* $t_1, \ldots, t_k$ *and pairs* $(u_j, v_j) \in \{(f_i, g_i), (g_i, f_i) \mid i = 1, \ldots, m\}$ *such that* $\mathscr{V}$ *satisfies*

$$\begin{aligned} r(\vec{x}) &= t_1(u_1(\vec{x}), \vec{x}) \\ t_1(v_1(\vec{x}), \vec{x}) &= t_2(u_2(\vec{x}), \vec{x}) \\ &\vdots \\ t_k(v_k(\vec{x}), \vec{x}) &= s(\vec{x}). \end{aligned}$$

Proof. Assume that (i) holds. Then by Theorem 2.1 we have $r(\vec{x}) \equiv s(\vec{x})(\Theta(Z))$ on $F\mathscr{V}(n)$ where $Z = \{(f_i, g_i) \mid i = 1, \ldots, m\}$. Thus by Lemma 3.1, for some $k, l \in \mathbf{N}$ there exist $(l+2)$-ary terms $w_1, \ldots, w_k$ and n-ary terms $h_1, \ldots, h_l$ and pairs $(u_j, v_j) \in Z$ such that (in $F\mathscr{V}(n)$)

$$\begin{aligned} r(\vec{x}) &= w_1(u_1(\vec{x}), v_1(\vec{x}), h_1(\vec{x}), \ldots, h_l(\vec{x})) \\ w_1(v_1(\vec{x}), u_1(\vec{x}), h_1(\vec{x}), \ldots, h_l(\vec{x})) &= w_2(u_2(\vec{x}), v_2(\vec{x}), h_1(\vec{x}), \ldots, h_l(\vec{x})) \\ &\vdots \\ w_k(v_k(\vec{x}), u_k(\vec{x}), h_1(\vec{x}), \ldots, h_l(\vec{x})) &= s(\vec{x}). \end{aligned}$$

Thus (ii) holds: define $t_j(y, z, \vec{x}) = w_j(y, z, h_1(\vec{x}), \ldots, h_l(\vec{x}))$. That (ii) implies (i) is trivial. In the same way, Lemma 3.2 yields the equivalence of (i) and (iii).

In just the same way, Theorem 2.1 and Lemma 3.3 combine to yield a simpler Mal'cev condition in the $(k+1)$-permutable case.

3.5. Theorem. *Let $\mathscr{V}$ be a $(k+1)$-permutable variety and let f_i, g_i, r and s be n-ary terms. Then the following are equivalent:*

(i) $\mathscr{V} \models \left(\mathop{\&}\limits_{i=1}^{m} f_i(\vec{x}) = g_i(\vec{x})\right) \to r(\vec{x}) = s(\vec{x})$;

(ii) *there exist* $(n+1)$*-ary terms* $t_1, \ldots, t_k$ *and pairs* $(u_j, v_j) \in \{(f_i, g_i) \mid i = 1, \ldots, m\}$

such that $\mathscr{V}$ *satisfies the identities*

$$\begin{aligned} r(\vec{x}) &= t_1(u_1(\vec{x}), \vec{x}) \\ t_1(v_1(\vec{x}), \vec{x}) &= t_2(u_2(\vec{x}), \vec{x}) \\ &\vdots \\ t_k(v_k(\vec{x}), \vec{x}) &= s(\vec{x}). \end{aligned}$$

If a variety $\mathscr{V}$ is k-permutable we shall write $\mathscr{V} \models \mathsf{P}_k$ and if $\mathscr{V}$ is k-permutable for some $k \in \mathbf{N}$ then we write $\mathscr{V} \models \mathsf{P}_*$. Whenever every algebra in $\mathscr{V}$ has a modular congruence lattice we write $\mathscr{V} \models \mathsf{CM}$. We require the identities for k-permutability (Hagemann and Mitschke [11]) and for congruence modularity Day [4].

3.6. Lemma. *Let* $\mathscr{V}$ *be a variety.*

(a) *Let* $k \geqq 2$. *Then* $\mathscr{V} \models \mathsf{P}_k$ *if and only if there are 3-ary terms* $p_1, \ldots, p_{k-1}$ *such that* $\mathscr{V}$ *satisfies*

$$\begin{aligned} x &= p_1(xzz), \\ p_i(xxz) &= p_{i+1}(xzz) \quad \textit{for all } i, \\ p_{k-1}(xxz) &= z. \end{aligned}$$

(b) $\mathscr{V} \models \mathsf{CM}$ *if and only if for some* $n \geqq 2$ *there exist 4-ary terms* $m_0, \ldots, m_n$ *such that* $\mathscr{V}$ *satisfies*

$$\begin{aligned} m_0(xyzw) &= x, \\ m_i(xyyx) &= x && \textit{for all } i, \\ m_i(xxww) &= m_{i+1}(xxww) && \textit{for even } i, \\ m_i(xyyw) &= m_{i+1}(xyyw) && \textit{for odd } i, \\ m_n(xyzw) &= w. \end{aligned}$$

(c) $\mathscr{V} \models \mathsf{CM}$ *if and only if for some* $n \geqq 2$ *there exist 4-ary terms* $m_0', \ldots, m_n'$ *such that* $\mathscr{V}$ *satisfies*

$$\begin{aligned} m_0'(xyzw) &= x, \\ m_i'(xyyx) &= x && \textit{for all } i, \\ m_i'(xyyw) &= m_{i+1}'(xyyw) && \textit{for even } i, \\ m_i'(xxww) &= m_{i+1}'(xxww) && \textit{for odd } i, \\ m_n'(xyzw) &= w. \end{aligned}$$

When the condition given in (b) above holds we shall write $\mathscr{V} \models \mathsf{CM}_n$. Similarly for the condition in (c) we write $\mathscr{V} \models \mathsf{CM}_n'$. For $n=2$ the m_i and m_i' are interdefinable but do not seem to be for $n \geqq 3$. Clearly $\mathsf{CM}_n \Rightarrow \mathsf{CM}_{n+1}' \Rightarrow \mathsf{CM}_{n+2}$

and hence $\mathsf{V}\,\mathsf{CM}_n = \mathsf{V}\,\mathsf{CM}'_n = \mathsf{CM}$. We shall refer to the terms m_i and the terms m'_i as the *Day terms.*

3.7. Lemma. *Let $\mathscr{V}$ be a variety. If $\mathscr{V} \models \mathsf{CM}_n$ and the Day terms satisfy $m_i(xxxz) = m_i(xzzz)$ for all (even or odd) i, then $\mathscr{V} \models \mathsf{P}_n$. Similarly if $\mathscr{V} \models \mathsf{CM}'_n$ and the Day terms satisfy $m'_i(xxxz) = m'_i(xzzz)$ for all (even or odd) i, then $\mathscr{V} \models \mathsf{P}_n$.*

Proof. Assume that $\mathscr{V} \models \mathsf{CM}_n$ with $m_i(xxxz) = m_i(xzzz)$ for all i. Define 3-ary terms $p_1, \ldots, p_{n-1}$ by

$$p_i(xyz) = \begin{cases} m_i(xxyz) & \text{for odd } i, \\ m_i(xyzz) & \text{for even } i. \end{cases}$$

Then by Lemma 3.6 (b) and our extra assumption, we have

$$p_1(xzz) = m_1(xxzz) = m_0(xxzz) = x,$$

$$(i \text{ odd}) \quad p_i(xxz) = m_i(xxxz) = m_i(xzzz) = m_{i+1}(xzzz) = p_{i+1}(xzz),$$

$$(i \text{ even}) \quad p_i(xxz) = m_i(xxzz) = m_{i+1}(xxzz) = p_{i+1}(xzz),$$

$$p_{n-1}(xxz) = p_n(xzz) = m_n(\ldots z) = z.$$

Thus, by Lemma 3.6 (a), we have $\mathscr{V} \models \mathsf{P}_n$. The proof for $\mathsf{CM}'_n \to \mathsf{P}_n$ is similar.

3.8. Lemma. *On any variety we have:*
(i) $\mathsf{CM}_2 \leftrightarrow \mathsf{CM}'_2 \leftrightarrow \mathsf{P}_2$;
(ii) $\mathsf{CM}'_3 \leftrightarrow \mathsf{P}_3$.

Proof. (i) Let m_1 be the nontrivial term for CM_2. Then $m'_1(xyzw) := m_1(wzyx)$ is the corresponding term for CM'_2. Thus $\mathsf{CM}_2 \to \mathsf{CM}'_2$ and similarly $\mathsf{CM}'_2 \to \mathsf{CM}_2$. The term for P_2 is given by $p_1(xyz) := m_1(xxyz)$. Thus $\mathsf{CM}_2 \to \mathsf{P}_2$ and the converse holds by the previous lemma since $m_1(xxxz) = m_2(xxxz) = z = m_2(xzzz) = m_1(xzzz)$.

(ii) It is easily seen that the Day terms for CM'_3 satisfy $m'_i(xxxz) = m'_i(xzzz)$ and hence $\mathsf{CM}'_3 \to \mathsf{P}_3$ by the previous lemma. If p_1 and p_2 are the terms for P_3 then terms m'_1 and m'_2 for CM'_2 may be defined by $m'_1(xyzw) := p_1(xyz)$ and $m'_2(xyzw) := := p_2(yzw)$; the identities for CM'_2 are easily checked. Thus $\mathsf{P}_3 \to \mathsf{CM}'_3$.

Hagemann [10] observed that for varieties we have $\mathsf{R} \to \mathsf{CM}$ and $\mathsf{R} \to \mathsf{P}_*$. Since regularity is given by a quasi-identity, it is natural to ask which quasi-identities yield CM and P_*.

3.9. Theorem. *Let $\mathscr{V}$ be a variety, let f_i, g_i be $(n+2)$-ary terms $(n \geqq 0)$ and let h_i be unary terms such that $\mathscr{V}$ satisfies*

$$\Big(\mathop{\&}_{i=1}^{m} f_i(xy\bar{z}) = g_i(xy\bar{z})\Big) \to x = y$$

and

$$f_i(xx\underline{z}) = g_i(xx\underline{z}) = h_i(z),$$

where $\underline{z}=(z...z)$. *Then* $\mathscr{V}\models \mathsf{CM}\,\&\,\mathsf{P}_*$.

Proof. Assume that $\mathscr{V}$ satisfies the quasi-identity and identities above, and let $t_1, \ldots, t_k$ be the $(n+4)$-ary terms given by Theorem 3.4. Thus there are pairs $(u_j, v_j)\in\{(f_i, g_i) \mid i=1, \ldots, m\}$ such that $\mathscr{V}$ satisfies

$$\begin{aligned} x &= t_1\big(u_1(xy\vec{z}),\, v_1(xy\vec{z}),\, xy\vec{z}\big)\\ t_1\big(v_1(xy\vec{z}),\, u_1(xy\vec{z}),\, xy\vec{z}\big) &= t_2\big(u_2(xy\vec{z}),\, v_2(xy\vec{z}),\, xy\vec{z}\big)\\ &\ \,\vdots\\ t_k\big(v_k(xy\vec{z}),\, u_k(xy\vec{z}),\, xy\vec{z}\big) &= y, \end{aligned}$$

and there exist unary terms $w_j\in\{h_1, \ldots, h_m\}$ such that

$$u_j(xx\underline{z}) = v_j(xx\underline{z}) = w_j(z).$$

We shall prove that $\mathscr{V}\models \mathsf{CM}_{2k+1}\,\&\,\mathsf{P}_{2k+1}$. Define the Day terms as follows:

$$\begin{aligned} m_0(xyzw) &= x,\\ m_{2j-1}(xyzw) &= t_j\big(u_j(yz\underline{w}),\, v_j(yz\underline{w}),\, xw\underline{w}\big),\\ m_{2j}(xyzw) &= t_j\big(v_j(yz\underline{w}),\, u_j(yz\underline{w}),\, xw\underline{w}\big),\\ m_{2k+1}(xyzw) &= w. \end{aligned}$$

Rather than introduce w_j into the calculations we shall repeatedly use the fact that $u_j(xx\underline{z})$ and $v_j(xx\underline{z})$ are equal and independent of x. For $0<j<k$ we have

$$\begin{aligned} m_{2j-1}(xyyx) &= t_j\big(u_j(yy\underline{x}),\, v_j(yy\underline{x}),\, xx\underline{x}\big) =\\ &= t_j\big(v_j(yy\underline{x}),\, u_j(yy\underline{x}),\, xx\underline{x}\big) = m_{2j}(xyyx) =\\ = t_j\big(v_j(xx\underline{x}),\, u_j(xx\underline{x}),\, xx\underline{x}\big) &= t_{j+1}\big(u_{j+1}(xx\underline{x}),\, v_{j+1}(xx\underline{x},\, xx\underline{x}\big) =\\ &= t_{j+1}\big(u_{j+1}(yy\underline{x}),\, v_{j+1}(yy\underline{x}),\, xx\underline{x}\big) = m_{2j+1}(xyyx). \end{aligned}$$

A similar calculation shows that $m_1(xyyx)=x$ and it follows by induction that $m_i(xyyx)=x$ for all i. Now

$$m_0(xxww) = x = t_1\big(u_1(xw\underline{w}),\, v_1(xw\underline{w}),\, xw\underline{w}\big) = m_1(xxww)$$

and similarly

$$m_{2k}(xxww) = t_k\big(v_k(xw\underline{w}),\, u_k(xw\underline{w}),\, xw\underline{w}\big) = w = m_{2k+1}(xxww),$$

and for $0<j<k$ we find

$$\begin{aligned} m_{2j}(xxww) &= t_j\big(v_j(xw\underline{w}),\, u_j(xw\underline{w}),\, xw\underline{w}\big) =\\ &= t_{j+1}\big(u_{j+1}(xw\underline{w}),\, v_{j+1}(xw\underline{w}),\, xw\underline{w}\big) = m_{2j+1}(xxww). \end{aligned}$$

Hence $m_i(xxww)=m_{i+1}(xxww)$ for i even. Finally, for $1\leq j\leq k$ we have

$$m_{2j-1}(xyyw) = t_j(u_j(yy\underline{w}), v_j(yy\underline{w}), xw\underline{w}) =$$
$$= t_j(v_j(yy\underline{w}), u_j(yy\underline{w}), xw\underline{w}) = m_{2j}(xyyw),$$

and thus $m_i(xyyw)=m_{i+1}(xyyw)$ for i odd. Consequently $\mathscr{V}\models \mathsf{CM}_{2k+1}$ by Lemma 3.6 (b).

By Lemma 3.7, to show that $\mathscr{V}\models \mathsf{P}_{2k+1}$ it suffices to show that the Day terms defined above satisfy $m_i(xxxz)=m_i(xzzz)$ for odd i (and hence for all i). But for $1\leq j\leq k$ we find

$$m_{2j-1}(xxxz) = t_j(u_j(xx\underline{z}), v_j(xx\underline{z}), xz\underline{z}) =$$
$$= t_j(u_j(zz\underline{z}), v_j(zz\underline{z}), xz\underline{z}) = m_{2j-1}(xzzz),$$

as required.

These considerations lead us to ask for compact collections of identities characterizing $\mathsf{CM}\,\&\,\mathsf{P}_*$ and $\mathsf{CM}\,\&\,\mathsf{P}_k$. Note that $\mathsf{CM}\,\&\,\mathsf{P}_k$ is equivalent to $\mathsf{CM}_k\,\&\,\mathsf{P}_k$. Our Lemma 3.7 gives a useful set of identities which imply $\mathsf{CM}_k\,\&\,\mathsf{P}_k$ while Lemma 3.8 shows that there is noting to do for $k=2, 3$.

4. Applications to congruence regularity. It is a simple exercise to apply the results of Section 3 to the various forms of regularity (and we leave all of the details to the reader). For example, we obtain at once that, at the varietal level,

$$(\mathsf{R}(o_1, \ldots, o_n) \text{ or } \mathsf{R}_n \text{ or } \mathsf{SR}) \to \mathsf{CM}\,\&\,\mathsf{P}_*.$$

Since every variety satisfies $\mathsf{LR}(x)$, Theorem 2.4 shows that in Theorem 3.9 we cannot drop the additional assumption that $f_i(xx\underline{z})$ and $g_i(xx\underline{z})$ are independent of x.

Combining Theorems 2.2 and 2.3 with Theorem 3.5 gives the identities which characterize R_n and SR.

4.1. Theorem. *Let $\mathscr{V}$ be a variety. Then $\mathscr{V}\models \mathsf{R}_n$ if and only if there exist unary terms $u_1, \ldots, u_n$, and for some $k\in\mathbf{N}$ there are 4-ary terms $t_1, \ldots, t_k$ and 3-ary terms $p_1, \ldots, p_k$ and there is a selection function $j\mapsto i_j$ such that $\mathscr{V}$ satisfies*

$$x = t_1(p_1(xyz), xyz)$$
$$t_1(u_{i_1}(z), xyz) = t_2(p_2(xyz), xyz)$$
$$\vdots$$
$$t_k(u_{i_k}(z), xyz) = y,$$

and $p_j(xxz)=u_{i_j}(z)$ for all j.

4.2. Theorem. *Let $\mathscr{V}$ be a variety. Then $\mathscr{V}\models \mathsf{SR}$ if and only if for some $n\in\mathbf{N}$ there exist unary terms $u_1, \ldots, u_n$, 4-ary terms $t_1, \ldots, t_n$ and 3-ary terms*

$p_1, \ldots, p_n$ *such that* $\mathscr{V}$ *satisfies*

$$\begin{aligned} x &= t_1(p_1(xyz), xyz) \\ t_1(u_1(z), xyz) &= t_2(p_2(xyz), xyz) \\ &\vdots \\ t_n(u_n(z), xyz) &= y, \end{aligned}$$

and $p_j(xxz)=u_j(z)$ *for all* j.

This characterization of subregularity and the quasi-identity characterization from Theorem 2.3 were obtained independently by DUDA [6, 7].

If we combine Theorems 2.4 and 3.4 to give identities for $\mathsf{LR}(u_1, \ldots, u_n)$ we do not immediately obtain the identities given by HAGEMANN [10]. Theorem 3.4 and the following lemma, whose proof we leave to the reader, provide the translation from our identities to his.

4.3. Lemma. *Let* $n \geqq 2$ *and* $l \geqq 0$. *The following are equivalent for a variety* $\mathscr{V}$:

(i) *there exist* $(n+l)$*-ary terms* $p_1, \ldots, p_s$ *and* $q_1, \ldots, q_s$ *such that* $\mathscr{V}$ *satisfies*

$$\Big(\mathop{\&}_{i=1}^{s} p_i(x_1 \ldots x_n y_1 \ldots y_l) = q_i(x_1 \ldots x_n y_1 \ldots y_l)\Big) \leftrightarrow x_1 = \ldots = x_n;$$

(ii) *there exist* $(n+l)$*-ary terms* $u_1, \ldots, u_t$ *and* $(l+1)$*-ary terms* $v_1, \ldots, v_t$ *such that* $\mathscr{V}$ *satisfies*

$$\Big(\mathop{\&}_{j=1}^{t} u_j(x_1 \ldots x_n y_1 \ldots y_l) = v_j(x_1 y_1 \ldots y_l)\Big) \leftrightarrow x_1 = \ldots = x_n.$$

Moreover the translation between (i) *and* (ii) *can be achieved in such a way that on* $\mathscr{V}$ *we have*

$$\begin{aligned} &\{p_i(x \ldots x y_1 \ldots y_l) = q_i(x \ldots x y_1 \ldots y_l) \mid i = 1, \ldots, s\} = \\ &= \{u_j(x \ldots x y_1 \ldots y_l) = v_j(x y_1 \ldots y_l) \mid j = 1, \ldots, t\}. \end{aligned}$$

4.4. Theorem. *Let* $\mathscr{V}$ *be a variety. Then the following are equivalent:*

(i) $\mathscr{V} \models \mathsf{LR}(u_1, \ldots, u_n)$;

(ii) *for some* $k \in \mathbf{N}$ *there exist* 4*-ary terms* $t_1, \ldots, t_k$ *and binary terms* $p_1, \ldots, p_k$ *and a selection function* $j \mapsto i_j$ *such that* $\mathscr{V}$ *satisfies*

$$\begin{aligned} x &= t_1(p_1(xy), u_{i_1}(x), xy) \\ t_1(u_{i_1}(x), p_1(xy), xy) &= t_2(p_2(xy), u_{i_2}(x), xy) \\ &\vdots \\ t_k(u_{i_k}(x), p_k(xy), xy) &= y, \end{aligned}$$

and $p_j(xx)=u_{i_j}(x)$ *for all* j;

(iii) *for some* $k \in \mathbf{N}$ *there exist* 3*-ary terms* $t_1, \ldots, t_k$ *and binary terms* $p_1, \ldots, p_k$

and $q_1, \dots, q_k$ *and a selection function* $j \mapsto i_j$ *such that* $\mathcal{V}$ *satisfies*

$$x = t_1(p_1(xy), xy)$$
$$t_1(q_1(xy), xy) = t_2(p_2(xy), xy)$$
$$\vdots$$
$$t_k(q_k(xy), xy) = y,$$

and $p_j(xx) = q_j(xx) = u_{i_j}(x)$ *for all* j.

Condition (iii) of this theorem is precisely the characterization of $\mathrm{LR}(u_1, \dots, u_n)$ given in HAGEMANN [10].

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