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The lattice variety $\mathbf{D} \circ \mathbf{D}$

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Dedicated to the memory of András P. Huhn

Section 1. Introduction. Let $\mathbf{V}$ and $\mathbf{W}$ be varieties of lattices. The product of $\mathbf{V}$ and $\mathbf{W}$, denoted by $\mathbf{V} \circ \mathbf{W}$, consists of all lattices L for which there is a congruence relation Θ such that every congruence class of Θ (as a lattice) is in $\mathbf{V}$ and L/Θ is in $\mathbf{W}$.

In this paper, we investigate in detail the class $\mathbf{D}^2 = \mathbf{D} \circ \mathbf{D}$. This class first appeared in a paper of S. V. Polin [9]. $\mathbf{D}^2$ is a curious class. Usually, one defines a class of algebras and aims at obtaining a structure theorem, while $\mathbf{D}^2$ is defined *via* a structure theorem: members of $\mathbf{D}^2$ are formed from distributive lattices over another distributive lattice.

In Section 2 we exhibit some lattices in $\mathbf{D}^2$. We describe a method to construct lattices freely generated by a poset over $\mathbf{D}^2$; we apply this (Theorem 1, Figure 1) to obtain the free product over $\mathbf{D}^2$ of the one-element and the four-element chain, and (Theorem 2) the free lattice over $\mathbf{D}^2$ generated by the six-element partially ordered set H (see Figures 2 and 3). An example shows (Theorem 3, Figure 4) that $\mathbf{D}^2$ is not locally finite.

In Section 3 we verify the most important property of $\mathbf{D}^2$: it is a variety. This result is a special case of the following result (Theorem 4): Let $\mathbf{V}$ be a lattice variety closed under gluing; then $\mathbf{V} \circ \mathbf{D}$ is a variety. In particular, $\mathbf{D}^2$ is a variety. As a corollary of this theorem, we get that there are continuumly many pairs of varieties whose product is a variety again.

While most known lattice varieties are either modular (contained in $\mathbf{M}$, the variety of modular lattices) or of small height (their height in the lattice of lattice varieties is 4 or less), $\mathbf{D}^2$ is neither. We show that $\mathbf{D}^2$ has large height (Theorem 5): There are continuumly many varieties contained in $\mathbf{D}^2$. Also, $\mathbf{D}^2$ is very far from $\mathbf{L}$ (the variety of all lattices): there are continuumly many varieties containing $\mathbf{D}^2$. Finally, $\mathbf{D}^2$ is almost disjoint from $\mathbf{M}$: $\mathbf{D}^2 \cap \mathbf{M} = \mathbf{D}$.

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The results reported in this paper were discovered in the late seventies. There are some newer results (1982—1985) of the type covered in Section 3. Firstly, there is the result of R. McKenzie (see R. McKenzie and D. Hobby [9]) that $\mathbf{D}\circ\mathbf{V}$ and $\mathbf{M}\circ\mathbf{V}$ are always varieties. The corollary of the main result of Section 3 also follows from McKenzie's result. The result of T. Harrison [6] shows that McKenzie's result is best possible: if $\mathbf{W}$ is a non-modular lattice variety with the property that $\mathbf{W}\circ\mathbf{V}$ is a variety for any given non-modular lattice variety $\mathbf{V}$, then $\mathbf{W}=\mathbf{L}$, the variety of all lattices.

For the basic facts concerning products of lattice varieties, we refer to our paper [4]. For the basic concepts and notation, the reader is referred to [2].

Section 2. Examples. Although it may seem rather restrictive to require a lattice to be in $\mathbf{D}^2$, there are surprisingly many lattices in $\mathbf{D}^2$.

Let us start with small varieties. Obviously, N_5 (the five-element nonmodular lattice) is in $\mathbf{D}^2$, hence $\mathbf{N}_5$ (the variety generated by N_5) is contained in $\mathbf{D}^2$. $\mathbf{N}_5$ has 16 covers (B. Jónsson and I. Rival [7]; 15 of them are generated by the lattices of Figures 3—11 in Section V.2 of [2] and their duals; the 16th is $\mathbf{N}_5\vee\mathbf{M}_3$). All but two are contained in $\mathbf{D}^2$. The exceptions are the (self-dual) variety generated by Figure 11 and $\mathbf{N}_5\vee\mathbf{M}_3$. M_3 does not belong to $\mathbf{D}^2$ because it is simple and it does not belong to $\mathbf{D}$. In fact, the only simple lattice in $\mathbf{D}^2$ is the two-element chain. Which modular lattices belong to $\mathbf{D}^2$? A modular lattice is non-distributive iff it contains M_3; hence, a modular lattice belongs to $\mathbf{D}^2$ iff it is distributive.

It is easy to check whether a lattice belongs to $\mathbf{D}^2$. For a lattice variety $\mathbf{V}$, and a lattice L, let $\Theta(L,\mathbf{V})$ be the smallest congruence relation on L such that $L/\Theta(L,\mathbf{V})$ is in $\mathbf{V}$. Now, L belongs to $\mathbf{D}^2$ iff all $\Theta(L,\mathbf{D})$ congruence classes are distributive. The "if" parts is obvious. Conversely, if L belongs to $\mathbf{D}^2$ by virtue of the congruence relation Θ, then $\Theta\geqq\Theta(L,\mathbf{D})$; since all Θ classes are distributive, so are the $\Theta(L,\mathbf{D})$ classes. $\Theta(L,\mathbf{D})$ can be described as follows: it is the join of all principal congruence relations $\Theta(a,b)$, where $a<b$ is a "violation of the distributive identity"; that is, there are $x,y,z\in L$ such that $(x\wedge y)\vee z=a$ and $(x\vee z)\wedge(y\vee z)=b$.

Using this, we can find the largest homomorphic image of a lattice that belongs to $\mathbf{D}^2$. Indeed, for a lattice L, first form $\Theta(L,\mathbf{D})$. Then form the join Φ of all $\Theta(a,b)$, where $a<b$ is a violation of the distributive law in some $\Theta(L,\mathbf{D})$ congruence class. Obviously, $\Phi\leqq\Theta(L,\mathbf{D})$; in L/Φ, $\Theta(L,\mathbf{D})/\Phi$ has distributive congruence classes, and $\Theta(L,\mathbf{D})/\Phi=\Theta(L/\Phi,\mathbf{D})$. Hence Φ is the same as $\Theta(L,\mathbf{D}^2)$. (We can describe similarly the congruence $\Theta(L,\mathbf{V}\circ\mathbf{W})$ for arbitrary lattice varieties $\mathbf{V}$ and $\mathbf{W}$.)

We apply this observation to determine some lattices freely generated by partially ordered sets over $\mathbf{D}^2$. Let C_n denote the n-element chain, and $A*B$ the free product of A and B. $L=C_2*C_2$ is in $\mathbf{D}^2$; so the $\mathbf{D}^2$-free product of C_2 and C_2 is L (see Figure 6 of Section VI.1 of [2]). However, the free product of C_1 and C_4

(see Figure 7 of Section VI.1 of [2]) does not lie in $\mathbf{D}^2$. Applying the construction of $\Phi = \Theta(L, \mathbf{D}^2)$ to this lattice $L = C_1 * C_4$ we obtain the lattice of Figure 1.

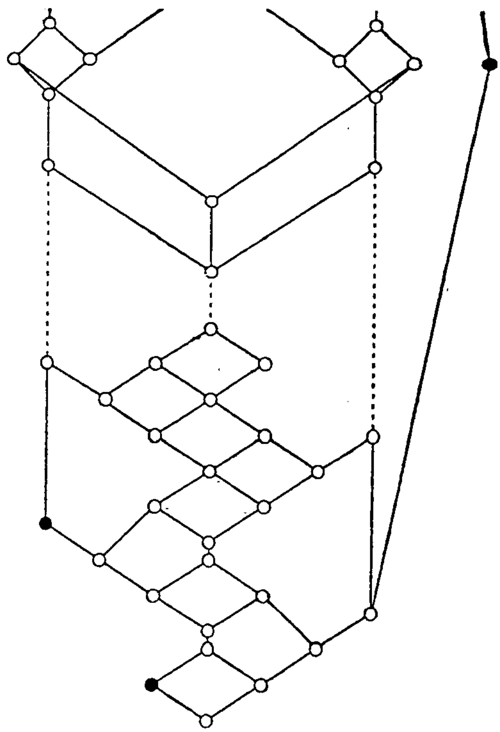

Figure 1

Theorem 1. *The lattice of Figure 1 is the* $\mathbf{D}^2$*-free product of* C_1 *and* C_4.

The free lattice L over the partially ordered set H (see Figure 2) plays an important role in [11] (see also [5]). Applying the method described above to this lattice L, we obtain the lattice freely generated by H over $\mathbf{D}^2$; see Figure 3.

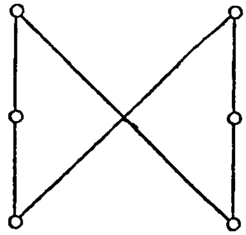

Figure 2

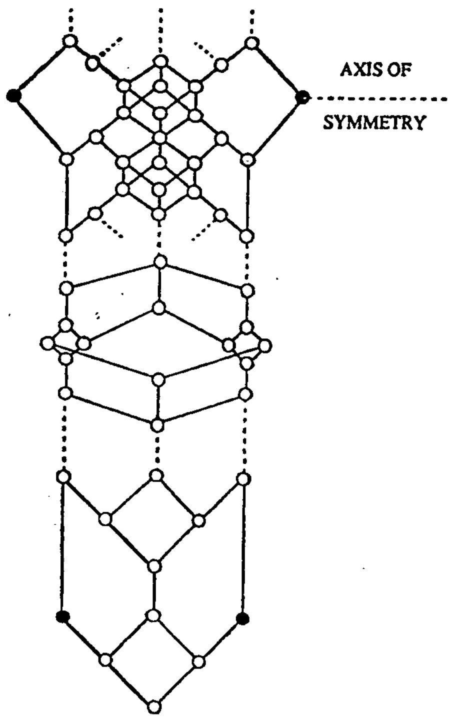

Figure 3

Theorem 2. *The lattice of Figure* 3 *is the* $\mathbf{D}^2$*-free lattice over* H.

Our final example of a lattice in $\mathbf{D}^2$ is Figure 4. Since this is a 3-generated infinite lattice, we conclude that:

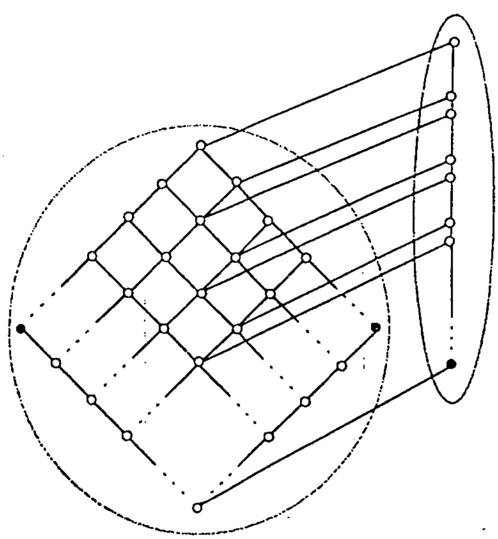

Figure 4

Theorem 3. $\mathbf{D}^2$ *is not locally finite.*

Section 3. $\mathbf{D}^2$ is a variety. We proved in [4] that if $\mathbf{V} \circ \mathbf{D}$ is a variety, then $\mathbf{V} \circ \mathbf{D}$ is closed under gluing. In this section we prove the following theorem:

Theorem 4. *Let* $\mathbf{V}$ *be a lattice variety. If* $\mathbf{V}$ *is closed under gluing, then* $\mathbf{V} \circ \mathbf{D}$ *is a variety.*

Proof. Let $\mathbf{V}$ be a lattice variety closed under gluing, let $L \in \mathbf{V} \circ \mathbf{D}$, and let Φ be a congruence relation of L. Given such an L, there is a smallest congruence relation Θ establishing that $L \in \mathbf{V} \circ \mathbf{D}$. To show that $\mathbf{V} \circ \mathbf{D}$ is a variety it is sufficient to show that for all choices of L and Φ, $L/\Phi \in \mathbf{V} \circ \mathbf{D}$. Since $\mathbf{V} \circ \mathbf{D}$ is a quasi-variety, L/Φ belongs to it iff all finitely generated sublattices of L/Φ belong. Hence we can assume that L is finitely generated. Therefore, L/Φ is a finite distributive lattice. In [4] we have observed that it is sufficient to prove that $L/\Phi \in \mathbf{V} \circ \mathbf{D}$ for Φ satisfying $\Theta \wedge \Phi = \omega$.

Since L/Θ is finite, Φ can be written as the join of n congruences of the form $\Theta(a, b)$ (called *minimal*), where $[a]\Theta$ covers $[b]\Theta$ in L/Θ. We prove that $L/\Phi \in \mathbf{V} \circ \mathbf{D}$ by induction on n. Let $\Phi = \Phi' \vee \Theta(a, b)$. Since L/Φ is isomorphic to $(L/\Phi')/(\Phi/\Phi')$, and Φ/Φ' is minimal in L/Φ', we can assume without loss of generality that $\Phi = \Theta(a, b)$ for such a pair a, b.

We claim that $L/\Phi \in \mathbf{V} \circ \mathbf{D}$ is established by the congruence relation $(\Theta \vee \Phi)/\Phi$. By the Second Isomorphism Theorem (see [2]) $(L/\Phi)/(\Theta \vee \Phi/\Phi)$ is a homomorphic image of L/Θ, hence this lattice is distributive. The behavior of a $\Theta(u, v)$, u covers v, in a distributive lattice is well known (see [2], Chapter II); in particular, every congruence class is a singleton or a covering pair. Each congruence class of L/Φ modulo $\Theta \vee \Phi/\Phi$ lies in $\mathbf{V}$ because it is either isomorphic to a congruence class of L modulo Θ or it is isomorphic to a lattice described in the following lemma.

Lemma 1. *Let* K *be a lattice, and let* $\mathbf{V}$ *be a lattice variety closed under gluing. Let* Θ *be a congruence relation on* K *with two congruence classes which as lattices are in* $\mathbf{V}$. *Let* Φ *be a congruence relation on* K *satisfying* $\Theta \wedge \Phi = \omega$. *Then* $K/\Phi \in \mathbf{V}$.

Proof. Let A and B be the congruence classes of K, with A the zero of L/Θ. Let D be the set of those elements of A that are congruent to some element of B modulo Φ. Let I be the set of those elements of B that are congruent to some element of A. We claim that D is a dual ideal of A, and I is an ideal of B.

Let $a_1, a_2 \in D$. There are $b_1, b_2 \in B$ satisfying $a_1 \equiv b_1(\Phi)$ and $a_2 \equiv b_2(\Phi)$. Then

$$a_1 \wedge a_2 \equiv b_1 \wedge b_2(\Phi).$$

Since $b_1 \wedge b_2 \in B$, we conclude that $a_1 \wedge a_2 \in D$. Also, if $a \in D$ and $x \in A$, then there is a $b \in B$ satisfying $a \equiv b(\Phi)$. Hence,

$$a \vee x \equiv b \vee x(\Phi)$$

and $b \vee x \in B$, verifying that D is a dual ideal.

Similarly, I is an ideal of B.

Now if $a \in D$, then there is a unique $b \in I$ satisfying $a \equiv b(\Phi)$ (otherwise, $\Theta \wedge \Phi = \omega$ would be contradicted). Thus, we have a mapping φ from D to I. It is easy to verify that φ is an isomorphism. Moreover, it is clear that the Φ classes with more that one element are exactly: $\{a, a\varphi\}$, $a \in D$. Thus, A and B glued over I and D is isomorphic to K/Φ, and hence is in $\mathbf{V}$, as claimed.

Section 4. Subvarieties. In this section, we construct continuumly many distinct subvarieties of $\mathbf{D}^2$.

Let A be an atomic Boolean lattice, $|A| \geqq 8$. We construct the lattices $K(A)$ and $L(A)$ as follows (see Figure 5). We take a disjoint copy A' of A. The zero and

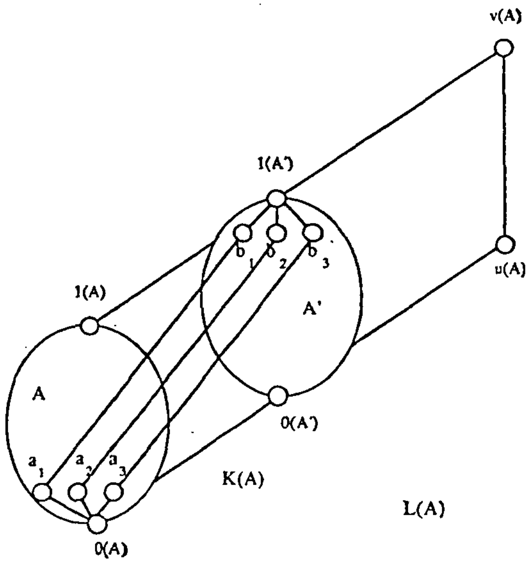

Figure 5

unit of A and A' are denoted by $0(A), 1(A), 0(A'), 1(A')$, respectively. Let $a_1(A), a_2(A), \ldots$ be the atoms of A, and $d_1(A'), d_2(A'), \ldots$ be the dual atoms of A'; if it is clear from the context, we may write a_i for $a_i(A)$ and d_i for $d_i(A')$. $K(A)$ is defined on $A \cup A'$; A and A' are subposets of $K(A)$; for $x \in A$ and $y \in A'$, $x < y$ iff $x = 0(A)$, or $y = 1(A')$, or $x = a_i(A)$, $y = b_i(A')$ for some i; for $x \in A$ and

$y \in A'$, $x > y$ never holds. $L(A)$ is defined on $K(A)$ and two new elements: $u(A)$ and $v(A)$. $v(A)$ is the unit element of $L(A)$; $0(A), 0(A') < u(A)$; $u(A)$ is only comparable to $0(A)$, $0(A')$ and $v(A)$.

Lemma 2. *Let A be an atomic Boolean lattice. Then $K(A)$ and $L(A)$ are lattices. $K(A)$ is a sublattice of $L(A)$. Both lattices are subdirectly irreducible. In both lattices, $0(A)$ and $1(A)$ form a critical edge.*

Proof. Obvious.

Lemma 3. *Let A and B be atomic Boolean lattices with A finite. If $K(A) \in \mathbf{HS}(K(B))$, then $K(A)$ is isomorphic to the sublattice of $K(B)$ generated by some subset of the form $\{a_i(A) \mid i \in I\} \cup \{d_i(A') \mid i \in I\}$ for some set I. In particular, $|A| \leqq |B|$.*

Proof. Let $K(A)$ be isomorphic under φ to S/Θ, where S is a sublattice of $K(B)$, and let Θ be a congruence of S. We claim that for any atom $a_i(A)$ of $K(A)$, $a_i(A)\varphi = \{a_j(B)\}$ for some atom $a_j(B)$ of $K(B)$. Indeed, let $a_i(A)\varphi = [x]\Theta$ for some x in S. Suppose that the claim fails. Since $a_i(A)$ is not the zero of $K(A)$, x can be chosen so that $x > a_j(B)$ for some $a_j(B)$ in $K(B)$ or $x \geqq 0(B')$. But $[x)$ is distributive in $K(B)$, which would imply that $[a_i(A))$ is distributive in $K(A)$, contradicting $|A| \geqq 8$, and verifying the claim. Similarly, for a dual atom $d_i(A')$ of $K(A)$, $d_i(A')\varphi = \{d_j(B')\}$ for some dual atom $d_j(B')$ of $K(B)$. The lemma now follows.

Lemma 4. *If A and B are atomic Boolean lattices with A finite, then $L(A) \notin \mathbf{HS}(K(B))$.*

Proof. Indeed, if $L(A) \in \mathbf{HS}(K(B))$, then $K(A) \in \mathbf{HS}(K(B))$. By Lemma 3, $K(A)$ is embedded into $K(B)$, with the unit of $K(A)$ going into the unit of $K(B)$. So there is no room for $u(A)$ and $v(A)$ in $K(B)$.

Lemma 5. *Let A and B be atomic Boolean lattices with A finite. If $L(A) \in \mathbf{HS}(L(B))$, then A and B are isomorphic.*

Proof. Let $L(A)$ be represented as S/Θ, where S is a sublattice of $L(B)$ and let Θ be a congruence relation of S. $u(B) \in S$, because otherwise S is a sublattice of $K(A)$, contradicting Lemma 4. Again, by Lemma 4, $u(B)$ cannot be congruent to an element of B' under Θ; nor can it be congruent to $v(B)$ because then the quotient could not contain $L(A)$. Thus $[u(B)]\Theta = \{u(B)\}$ represents $u(A)$; it follows, that $(S - \{u(B), v(B)\})/\Theta$ represents $K(A)$, hence $K(A) \in \mathbf{HS}(K(B))$. By Lemma 3, $K(A)$ is a specific type of sublattice of $K(B)$, the dual atoms $d_1(A'), d_2(A'), \ldots$ of A' correspond to dual atoms of B'. If A and B are not isomorphic, then A has fewer atoms, so their meet, $0(A')$ will not map onto $0(B')$, and will not be below $u(B)$, a contradiction.

Now we can state and prove the theorem of this section:

Theorem 4. $\mathbf{D}^2$ *has continuumly many subvarieties.*

Proof. Let N be a set of natural numbers $\geqq 3$. Let $\mathbf{V}(N)$ be the variety of lattices generated by the $L(A)$, where $|A|=2^n$ for some $n \in N$. We claim that for a finite Boolean lattice B, $L(B) \in \mathbf{V}(N)$ iff $|B|=2^m$ for some $m \in N$. This claim proves the theorem.

To verify the claim, let $L(B) \in \mathbf{V}(N)$. By Lemma 2, $L(B)$ is subdirectly irreducible, hence by Jónsson's Lemma, $L(B) \in \mathbf{HS}(L)$, where L is an ultraproduct of $L(A)$ with $|A|=2^n$, $n \in N$. However, the class of all $L(A)$, where A is an atomic Boolean lattice, is first-order definable. Hence, $L=L(A)$. $L(A) \in \mathbf{HS}(L(B))$ contradicts Lemma 5, unless A and B are isomorphic. This verifies the claim.

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