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Congruence relations and direct decompositions of ordered sets

M. KOLIBIAR

Dedicated to the memory of András Huhn

1. Introduction

Given an algebra $\mathcal{A}=(A; F)$, there is a one-one correspondence between direct product decompositions $\mathcal{A}=\prod(\mathcal{A}_i \mid i\in I)$ and families $(\theta_i \mid i\in I)$ of congruence relations of $\mathcal{A}$ satisfying

(1) $\bigcap(\theta_i \mid i\in I) = \mathrm{id}_A$,

(2) $\bigvee(\theta_i \mid i\in I) = A\times A$,

(3) given a family $(x_i \mid i\in I)$ of elements of A, there exists an element $x\in A$ such that $x\theta_i x_i$ for all $i\in I$.

The situation is more complicated in the case of relational systems. A method of characterization of direct product decompositions of such systems was given in the papers [1] and [3]. For the sake of simplicity we state the result for the case of ordered sets. (We use the term "ordered set" for partially ordered set.)

There is a one-one correspondence between direct product decompositions of an ordered set $\mathcal{A}=(A; \leqq)$ and families $(\theta_i \mid i\in I)$ of equivalence relations of A satisfying (1), (2), (3) and

(4) given elements x, y, x_i, y_i $(i\in I)$ of A such that $x_i\leqq y_i$ and $x\theta_i x_i, y\theta_i y_i$ for all $i\in I$, then $x\leqq y$.

The condition (4) is a kind of "collective congruence property". Instead of a notion of an (individual) congruence relation we have to deal with a "congruence family". Recently an analogous characterization of subdirect decompositions of multialgebras was given by G. E. HANSOUL [2].

In the present note we study a notion of congruence relation in the class of ordered sets which enables the same characterization of direct decompositions of

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directed ordered sets as in the case of algebras (conditions (1), (2), (3)) whenever the number of the decomposition factors is finite or the ordered set satisfies a chain condition. Moreover, we show that the congruence relations form a distributive lattice. Also some results on subdirect decompositions are given.

2. Congruence relations

2.1. Definition. Let $\mathcal{P}=(P; \leqq)$ be a directed ordered set. An equivalence relation θ on P will be called a congruence relation on $\mathcal{P}$ if the following conditions are satisfied.

(i) For each $a\in P$, $[a]\theta$ $(=\{x\in P \mid x\theta a\})$ is a convex subset of P.

(ii) If $a, b, c\in P$, $a\leqq c$, $b\leqq c$ and $a\theta b$, then there is $d\in P$ such that $a\leqq d\leqq c$, $b\leqq d$ and $a\theta d$.

(iii) If $a, b, u, v\in P$, $u\leqq a\leqq v$, $u\leqq b\leqq v$ and $u\theta a$ $(a\theta v)$, then there is $t\in P$ such that $b\leqq t\leqq v$, $a\leqq t$, $(u\leqq t\leqq b$, $t\leqq a)$ and $b\theta t$.

If $\mathcal{P}$ is a lattice then this notion coincides with the lattice congruence relation.

2.2. It can be easily shown that the conditions (ii), (iii) are equivalent with the following one:

(iv) If $a, b, c, d\in P$, $a\leqq c\leqq d$, $b\leqq d$ $(a\geqq c\geqq d$, $b\geqq d)$ and $a\theta b$, then there is $e\in P$ such that $c\leqq e\leqq d$, $b\leqq e$ $(c\geqq e\geqq d, b\geqq e)$ and $c\theta e$.

2.3. Let Con $\mathcal{P}$ denote the set of all congruence relations of $\mathcal{P}$. Eq P will denote the lattice of all equivalence relations on P.

In what follows all ordered sets will be supposed to be (both up- and down-) directed (i.e. to any a, b there are u, v such that $u\leqq a\leqq v$, $u\leqq b\leqq v$). $\mathcal{P}$ will denote an ordered set $(P; \leqq)$. We say that $\mathcal{P}$ satisfies the restricted ascending chain condition (RACC) if every closed interval of $\mathcal{P}$ satisfies the ascending chain condition.

The set $\{1, 2, ..., n\}$ will be denoted by $\bar{n}$.

2.4. *A congruence relation θ on $\mathcal{P}$ has the following property.*

(ii') *If $a, b, c\in P$, $a\geqq c$, $b\geqq c$, $a\theta b$, then there exists $e\in P$ such that $a\geqq e\geqq c$, $b\geqq e$ and $a\theta e$.*

Proof. Using the fact that $\mathcal{P}$ is up-directed and (ii) we get that $d\in P$ exists such that $a\leqq d$, $b\leqq d$ and $b\theta d$. The existence of the desired element e follows by (iii).

2.5. *Let $\theta\in\mathrm{Con}\,\mathcal{P}$, $a, b\in P$, $a\leqq b$. Then to any $x\in[a]\theta$ $(y\in[b]\theta)$ there is $y\in[b]\theta$ $(x\in[a]\theta)$ such that $x\leqq y$.*

Proof. Let $x \in [a]\theta$. Since $\mathcal{P}$ is up-directed, there is $c \in P$ such that $b \leqq c$, $x \leqq c$. According to 2.2 there is $y \in P$ such that $b\theta y$ and $x \leqq y$. The second assertion is symmetric.

2.6. *Let* $\theta \in \mathrm{Con}\,\mathcal{P}$. *Given* $a, b \in P$, *set* $[a]\theta \leqq [b]\theta$ *if there are* $x, y \in P$ *such that* $x\theta a$, $y\theta b$ *and* $x \leqq y$. *Then* $(P/\Theta; \leqq)$ *is an ordered set.*

The proof is straightforward.

2.7. *Let* $\theta \in \mathrm{Con}\,\mathcal{P}$, $a, b, c, d, u \in P$, $a \leqq b$, $a\theta c$, $b\theta d$, $c \leqq u$, $d \leqq u$. *Then there is* $v \in P$ *such that* $d \leqq v \leqq u$, $c \leqq v$ *and* $b\theta v$.

Proof. According to 2.5 there exists $e \in P$ such that $e\theta a$ and $e \leqq d$. Using 2.2 we get that $v \in P$ exists such that $d \leqq v \leqq u$, $c \leqq v$ and $d\theta v$.

2.8. Theorem 1. *The congruence relations on* $\mathcal{P}$ *correspond one-one to mappings* f *of* P *onto ordered sets (given uniquely up to isomorphism) such that*

(a) f *is isotone,*

(b) *if* $x, y, u \in P$, $x \leqq u$, $y \leqq u$ $(x \geqq u,\ y \geqq u)$ *and* $f(x) \leqq f(y)$ $(f(x) \geqq f(y))$ *then there is* $z \in P$ *such that* $x \leqq z$, $y \leqq z \leqq u$ $(x \geqq z,\ y \geqq z \geqq u)$ *and* $f(y) = f(z)$.

Proof. Let $\theta \in \mathrm{Con}\,\mathcal{P}$. The mapping $f\colon P \to P/\theta$, $x \mapsto [x]\theta$, is obviously isotone. Let $f(x) \leqq f(y)$ and $x \leqq u$, $y \leqq u$. Then $a \in [x]\theta$ and $b \in [y]\theta$ exist such that $a \leqq b$. The existence of the desired element z follows by 2.7. The second assertion follows by symmetry.

Conversely, let $f\colon P \to Q$ fulfil the conditions (a) and (b) and let $\theta = \mathrm{Ker}\, f$. The property (i) is obvious. Let $a\theta b$ and $a \leqq c$, $b \leqq c$. Then $f(a) = f(b)$ and according to (b) there is $d \in P$ sucht that $a \leqq d \leqq c$, $b \leqq d$ and $f(a) = f(d)$ which proves (ii). If $u \leqq a \leqq v$, $u \leqq b \leqq v$ and $u\theta a$, then $f(a) \leqq f(b)$ and, according to (b), $t \in P$ exists such that $b \leqq t \leqq v$, $a \leqq t$ and $f(t) = f(b)$, hence $t\theta b$. The second part of (iii) follows by symmetry.

Let F and G be the mappings $\theta \mapsto f$ and $f \mapsto \theta$, respectively, described above. Obviously $(GF)(\theta) = \theta$. Let $f\colon P \to Q$ be given. Then the mapping $(FG)(f)$ is the canonical mapping $P \to P/\theta$, $\theta = \mathrm{Ker}\, f$. To prove $\mathcal{Q} \cong \mathcal{P}/\theta$ let us define $h\colon P/\theta \to Q$ by setting $h([x]\theta) = f(x)$. h is surjective and well defined. If $[x]\theta \leqq [y]\theta$ then $z \in [y]\theta$ exists such that $x \leqq z$. Hence $f(x) \leqq f(z) = f(y)$. Conversely, let $f(x) \leqq f(y)$. There is $u \in P$ such that $x \leqq u$, $y \leqq u$. According to (b) $z \in P$ exists such that $y \leqq z \leqq u$, $x \leqq z$, $f(y) = f(z)$, hence $[x]\theta \leqq [z]\theta = [y]\theta$.

2.9. *Let* $x_0 \leqq x_1 \leqq \ldots \leqq x_n$, $x_0 \leqq y \leqq x_n$, $\alpha_i \in \mathrm{Con}\,\mathcal{P}$, $x_{i-1}\alpha_i x_i$ *for all* $i \in \bar{n}$. *Then there exists a sequence* $x_0 = y_0 \leqq y_1 \leqq \ldots \leqq y_n = y$ *such that* $y_i \leqq x_i$ *and* $y_{i-1}\alpha_i y_i$ *for all* $i \in \bar{n}$.

Proof. If $n = 1$, it suffices to take $y_1 = y$. Suppose the assertion holds for $n - 1 \geqq 1$. Using (iii) for the elements y, x_{n-1}, x_0, x_n we get that y_{n-1} exists such that $x_0 \leqq y_{n-1} \leqq y$, $y_{n-1} \leqq x_{n-1}$ and $y_{n-1} \alpha_n y$. By the induction assumption there are elements $y_1, \ldots, y_{n-2}$ which together with y_{n-1} give the desired sequence.

2.10. *Let* $u \leqq a \leqq v$, $u \leqq b \leqq v$, $u = u_0 \leqq u_1 \leqq \ldots \leqq u_n = a$, $u_{i-1} \alpha_i u_i$, $\alpha_i \in \mathrm{Con}\, \mathcal{P}$ *for all* $i \in \bar{n}$. *Then a sequence* $b = v_0 \leqq v_1 \leqq \ldots \leqq v_n \leqq v$ *exists such that* $a \leqq v_n$, $u_i \leqq v_i$ *and* $v_{i-1} \alpha_i v_i$ *for all* $i \in \bar{n}$.

Proof. If $n=1$, the assertion follows by (iii). Assume the assertion holds for $n-1$. By (iii) there is v_1 such that $u_1 \leqq v_1 \leqq v$, $b \leqq v_1$ and $b \alpha_1 v_1$. By the induction assumption there exists a sequence $v_1 \leqq v_2 \leqq \ldots \leqq v_n \leqq v$ such that $a \leqq v_n$, $u_i \leqq v_i$ and $v_{i-1} \alpha_i v_i$ for $i = 2, \ldots, n$.

2.11. *Let* $a = t_0, t_1, \ldots, t_n = b$ *be elements of* P, $\alpha_i \in \mathrm{Con}\, \mathcal{P}$ *and* $t_{i-1} \alpha_i t_i$ *for all* $i \in \bar{n}$. *Then there exist sequences* $a = u_0 \leqq u_1 \leqq \ldots \leqq u_n$, $b = v_0 \leqq v_1 \leqq \ldots \leqq v_n = u_n$ *such that for each* $i \in \bar{n}$, $u_{i-1} \alpha_{j(i)} u_i$, $v_{i-1} \alpha_{k(i)} v_i$ *where* $j(i), k(i) \in \bar{n}$.

Proof. In $n=1$, the assertion is trivial. Supposing the assertion holds for $n-1$ we shall prove it for n. Using the induction assumption for the elements $a, t_1, \ldots, t_{n-1}$ we get sequences $a = u_0 \leqq u_1 \leqq \ldots \leqq u_{n-1}$, $t_{n-1} = w_0 \leqq w_1 \leqq \ldots \leqq w_{n-1} = u_{n-1}$ such that, for each $i \in \overline{n-1}$, $u_{i-1} \alpha_{j(i)} u_i$, $w_{i-1} \alpha_{k(i)} w_i$, $j(i), k(i) \in \overline{n-1}$. Using the fact that P is up-directed and (ii), (iii), we get that $c, v_1 \in P$ exist such that $u_{n-1} \leqq c$, $t_{n-1} \leqq v_1$, $b \leqq v_1 \leqq c$, $t_{n-1} \alpha_n v_1 \alpha_n b$ and $u_{n-1} \alpha_n c$. According to 2.10 there exists a sequence $v_1 \leqq v_2 \leqq \ldots \leqq v_n \leqq c$ such that ($w_{i-1} \leqq v_i$ for $i \in \bar{n}$ and) $u_{n-1} \leqq v_n$, $v_{i-1} \alpha_{k(i)} v_i$ for $i = 2, \ldots, n$. Obviously $u_{n-1} \alpha_n u_n$, where $u_n = v_n$.

2.12. *Let* $A \subset \mathrm{Con}\, \mathcal{P}$. *Then* $\bigvee(\alpha \mid \alpha \in A) = \beta$ *has the property* (ii).

Proof. Let $a \leqq c$, $b \leqq c$, $a \beta b$. According to the proposition dual to 2.11 there exists a sequence $u_0 \leqq u_1 \leqq \ldots \leqq u_n = b$ such that $u_0 \leqq a$ and $u_{i-1} \alpha_i u_i$, $\alpha_i \in A$ for each $i \in \bar{n}$. According to 2.10 there exists a sequence $a = v_0 \leqq v_1 \leqq \ldots \leqq v_n \leqq c$ such that $b \leqq v_n$ and $v_{i-1} \alpha_i v_i$ for all $i \in \bar{n}$. Hence $a \beta v_n$ (and $b \beta v_n$).

2.13. *Let* $A \subset \mathrm{Con}\, \mathcal{P}$, $(x, y) \in \bigvee(\alpha \mid \alpha \in A)$ *and* $x \leqq y$. *Then there exists a sequence* $x = x_0 \leqq x_1 \leqq \ldots \leqq x_n = y$ *such that* $x_{i-1} \alpha_i x_i$, $\alpha_i \in A$ *for all* $i \in \bar{n}$.

Proof. According to 2.11 there exists a sequence $x = u_0 \leqq u_1 \leqq \ldots \leqq u_n$ such that $y \leqq u_n$ and $u_{i-1} \alpha_{j(i)} u_i$, $\alpha_{j(i)} \in A$ for all $i \in \bar{n}$. According to 2.9 there exists a sequence $x = t_0 \leqq t_1 \leqq \ldots \leqq t_n = y$ such that $t_{i-1} \alpha_{j(i)} t_i$ for all $i \in \bar{n}$.

2.14. *If* $A \subset \mathrm{Con}\, \mathcal{P}$ *then* $\beta = \bigvee(\alpha \mid \alpha \in A)$ *has the property* (i).

Proof. Let $x \leqq z \leqq y$ and $x \theta y$. Using 2.13 and 2.9 we get $x \theta z$.

2.15. *If* $A \subset \mathrm{Con}\, \mathcal{P}$ *then* $\bigvee(\alpha \mid \alpha \in A) \in \mathrm{Con}\, \mathcal{P}$.

Proof. The property (iii) of the join follows immediately from 2.13 and 2.10 (and from its duals), while (i) and (ii) were proved in 2.14 and 2.12.

2.16. *If* $\alpha, \beta \in \mathrm{Con}\,\mathscr{P}$ *then* $\alpha \cap \beta \in \mathrm{Con}\,\mathscr{P}$.

Proof. Obviously $\alpha \cap \beta$ has the property (i). The properties (ii) and (iii) can be easily checked.

2.17. From 2.15 and 2.16 we get

Theorem 2. $\mathrm{Con}\,\mathscr{P}$ *forms a complete lattice which is a sublattice of the lattice* $\mathrm{Eq}\,P$. *Moreover for any set* $A \subset \mathrm{Con}\,\mathscr{P}$, $\bigvee_{\mathrm{Con}\,\mathscr{P}}(\alpha \mid \alpha \in A) = \bigvee_{\mathrm{Eq}\,P}(\alpha \mid \alpha \in A)$.

Remark. Unfortunately, the set-theoretic intersection $\bigcap(\alpha \mid \alpha \in A)$ need not belong to $\mathrm{Con}\,\mathscr{P}$ if A is an infinite subset of $\mathrm{Con}\,\mathscr{P}$, as the following simple example shows. Let N be the set of all negative integers with the natural order and $P = \{u, a, b\} \cup N$, $u < a < n$ and $u < b < n$ for all $n \in N$. For each $n \in N$, let α_n be the equivalence relation on P with the blocks $(n]$ and $[n+1)$ ($\emptyset$ if $n = -1$). Then $\alpha_n \in \mathrm{Con}\,\mathscr{P}$ but $\bigcap(\alpha_n \mid n \in N) \notin \mathrm{Con}\,\mathscr{P}$.

Hence it can occur that $\bigwedge_{\mathrm{Con}\,\mathscr{P}}(\alpha \mid \alpha \in A) < \bigwedge_{\mathrm{Eq}\,P}(\alpha \mid \alpha \in A)$ if A is infinite.

Theorem 3. *The lattice* $\mathrm{Con}\,\mathscr{P}$ *is distributive. If* $\alpha \in \mathrm{Con}\,\mathscr{P}$ *and* $B \subset \mathrm{Con}\,\mathscr{P}$ *then* $\alpha \wedge \bigvee(\beta \mid \beta \in B) = \bigvee(\alpha \wedge \beta \mid \beta \in B)$.

Proof. Set $\varphi = \alpha \wedge \bigvee(\beta \mid \beta \in B)$, $\psi = \bigvee(\alpha \wedge \beta \mid \beta \in B)$. Obviously $\psi \leqq \varphi$. To prove the converse we first notice that $x\varphi y$ and $x \leqq y$ imply $x\psi y$. Indeed, from the assumption we get $x\alpha y$ and the existence of a sequence $x = x_0 \leqq x_1 \leqq \ldots \leqq x_n = y$, $x_{i-1}\beta_i x_i$, $\beta_i \in B$, for all $i \in \bar{n}$ (2.13). Then $x_{i-1}\ \alpha \wedge \beta_i\ x_i$ hence $x\psi y$. To get the implication for arbitrary $x, y \in P$, observe that if $x\varphi y$ then $z \in P$ exists such that $x \leqq z$, $y \leqq z$ and $x\varphi z$, $y\varphi z$.

2.18. *Let* $\mathscr{P}$ *satisfy RACC and let* $\alpha_i \in \mathrm{Con}\,\mathscr{P}$ $(i \in I)$, $\bigcap(\alpha_i \mid i \in I) = \mathrm{id}_P$. *If* a *and* a_i $(i \in I)$ *are elements of* P *such that* $a \leqq a_i$ *and* $a\,\alpha_i a_i$ *for all* $i \in I$, *then* $a = \inf(a_i \mid i \in I)$.

Proof. Let $b \leqq a_i$ for all $i \in I$. Choose $i(1) \in I$. According to (iii) there is $b_1 \in P$ such that $b_1 \leqq a$, $b_1 \leqq b$ and $b\alpha_{i(1)}b_1$. Choose $i(2) \in I - \{i(1)\}$. Then there exists $b_2 \in P$ such that $b_1 \leqq b_2 \leqq a$, $b_2 \leqq b$ and $b\alpha_{i(2)}b_2$. By induction we get a sequence $b_1 \leqq b_2 \leqq \ldots$, $b_j \leqq a$, $b_j \leqq b$, $b\alpha_{i(j)}b_j$, which ends with some member b_m by virtue of RACC. Then $b\alpha_j b_m$ for each $j \in I$ hence $b = b_m$ so that $b \leqq a$.

By an analogous argument we get the following proposition.

Let $\mathscr{P}$ *be an arbitrary directed ordered set. If there are given* $a, a_1, \ldots, a_n \in P$ *and* $\alpha_i \in \mathrm{Con}\,\mathscr{P}$ *with* $\alpha_1 \wedge \ldots \wedge \alpha_n = \mathrm{id}_P$ *such that* $a \leqq a_i$ *and* $a\,\alpha_i a_i$ *for each* $i \in \bar{n}$ *then* $a = \inf(a_1, \ldots, a_n)$.

Remark. Without the condition RACC the first proposition would not be true as the following example shows.

Let P be the set $A\cup B\cup\{a, b\}$ where A (B) is the set of all positive (negative) integers with its natural order and for any $m\in A$ and $n\in B$ let $m<a<n$, $m<b<n$. Then $\mathcal{P}=(P; \leqq)$ is a directed ordered set. For each $m\in A$, let α_m be the equivalence relation on P in which the only non-singleton blocks are the intervals $[a, -m]$ and $[m, b]$. Then $\alpha_m\in\text{Con}\,\mathcal{P}$, $\bigcap(\alpha_m \mid m\in A)=\text{id}_P$, $a\leqq n$, $b\leqq n$ and $a\alpha_{-n}n$ for each $n\in B$ but $b\leqq a$ does not hold.

3. Direct and subdirect representations

3.1. Definition. A subdirect product $\mathcal{P}\to\prod(\mathcal{P}_i \mid i\in I)$ will be called a full subdirect product whenever to each $i\in I$ and any $a, b\in P$ there is $c\in P$ such that $c_i=a_i$ and $c_j=b_j$ for all $j\neq i$.

3.2. Theorem 4. *Let $\mathcal{P}\to\prod(\mathcal{P}_i \mid i\in I)$ be a full subdirect product of ordered sets and let, for each $i\in I$, θ_i be the kernel of the projection $\mathcal{P}\to\mathcal{P}_i$. Then $\theta_i\in\text{Con}\,\mathcal{P}$.*

Proof. Obviously θ_i fulfils (i). Let $a\leqq c$, $b\leqq c$ and $a\theta_i b$, i.e., $a_i=b_i$. Let d be the element of P with $d_i=a_i$ and $d_j=c_j$ for $j\neq i$. Then d is the element needed for (ii). The dual part of (ii) is analogous. Finally, let the elements $a, b, u, v\in P$ satisfy $u\leqq a\leqq v$, $u\leqq b\leqq v$, $u\theta_i a$ and let d be the element fulfilling $d_i=b_i$ and $d_j=v_j$ for $j\neq i$. Then d dulfils the condition of (iii). The dual part is analogous.

Remark. The theorem would not be true if the word "full" was omitted. This is shown by the following example. Let $L=\{o, i, a, b, c\}$ be the five-element modular and non-distributive lattice and C the chain $0<1<2$. Then $f\colon L\to C\times C$, where $o\mapsto(0, 0)$, $a\mapsto(0, 2)$, $b\mapsto(1, 1)$, $c\mapsto(2, 0)$, $i\mapsto(2, 2)$, gives a subdirect decomposition of the ordered set L but the kernels of the corresponding projections do not fulfil condition (iii).

3.3. Theorem 5. *Let, for each $i\in\bar{n}$, $\alpha_i\in\text{Con}\,\mathcal{P}$ and $\alpha_1\wedge\alpha_2\wedge\ldots\wedge\alpha_n=\text{id}_P$. Then $\mathcal{P}$ is a subdirect product of the ordered sets $\mathcal{P}/\alpha_i$, where $x\mapsto([x]\alpha_i \mid i\in\bar{n})$.*

Proof. It suffices to show that $[x]\alpha_i\leqq[y]\alpha_i$ for all $i\in\bar{n}$ implies $x\leqq y$. For each $i\in\bar{n}$ there exists $y_i\in P$ such that $y\leqq y_i$, $y\alpha_i y_i$ and $x\leqq y_i$ (see 2.5). According to the second proposition in 2.18, $y=\inf(y_1, \ldots, y_n)$, hence $x\leqq y$.

By an analogous argument (using the first proposition 2.18) we get

Theorem 6. *Let $\mathcal{P}$ satisfy the RACC and let A be a subset of Con $\mathcal{P}$ such that $\bigcap(\theta \mid \theta\in A)=\text{id}_P$. Then $\mathcal{P}$ is a subdirect product of the ordered sets $\mathcal{P}/\theta$ $(\theta\in A)$, where $x\mapsto([x]\theta \mid \theta\in A)$.*

Theorem 7. *Let $\mathcal{P}$ be a (directed) ordered set. There is a one-one correspondence between direct product decompositions of $\mathcal{P}$ into finitely many (say n) factors and the families $(\theta_i \mid i \in \bar{n})$ of congruence relations of $\mathcal{P}$ satisfying* (1), (2) *and* (3) (see the introduction).

Proof. The theorem is an easy consequence of Theorems 4 and 5.

Analogously (using Theorem 6) the following theorem can be proved.

Theorem 8. *Let $\mathcal{P}$ satisfy RACC.*

(a) *There is a one-one correspondence between the direct product decompositions $\mathcal{P} \to \prod(\mathcal{P}_i \mid i \in I)$ and the families $(\theta_i \mid i \in I)$ of congruence relations satisfying* (1), (2) *and* (3).

(b) *There is a one-one correspondence between the full subdirect product decompositions $\mathcal{P} \to \prod(\mathcal{P}_i \mid i \in I)$ and the families $(\theta_i \mid i \in I)$ of congruence relations satisfying* (1) *and*

(5) *for each $i \in I$, $\theta_i \circ \bigcap(\theta_j \mid j \in I,\ j \neq i) = P \times P$.*

Theorem 9. *Let $\mathcal{P}$ satisfy RACC and let $(\theta_i \mid i \in I)$ be a family of congruence relations of $\mathcal{P}$ satisfying* (1) *and* (5). *Then, for any subset $J \subset I$, $\bigcap(\theta_j \mid j \in J)$ (set-theoretic intersection) belongs to* Con $\mathcal{P}$.

Proof. According to Theorem 8 the family $(\theta_i \mid i \in I)$ gives a full subdirect product decomposition of $\mathcal{P}$. Then $\varphi = \bigcap(\theta_j \mid j \in J)$ and $\psi = \bigcap(\theta_k \mid k \in I - J)$ are equivalence relations corresponding to the direct product $\mathcal{P}/\varphi \times \mathcal{P}/\psi$, hence they belong to Con $\mathcal{P}$ (see Theorem 4).

Added in proof. 1. Recently J. Jakubík showed that the condition RACC in Theorem 8 cannot be omitted.

2. The condition (i) in 2.1 is an easy consequence of (iii) (this was observed by Mrs. J. Lihová). Also the condition (2) in the introduction may be omitted.

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KATEDRA ALGEBRY A TÉORIE ČÍSEL
UNIVERZITA KOMENSKÉHO
842 15 BRATISLAVA, CZECHOSLOVAKIA