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On circuits of atoms in atomistic lattices

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Dedicated to the memory of Dr. András P. Huhn

1. Introduction. In [2], an atomistic lattice with the covering property is called an AC-lattice and an upper continuous AC-lattice is called a matroid lattice. (In [1], [6] and [9], a matroid lattice is called a geometric lattice.) As shown in [9], Section 3.3, a finite lattice is a matroid lattice if and only if it is isomorphic to the lattice of closed sets of a matroid. By this reason, the concept of circuits which plays an important role in matroid theory can be introduced in the theory of matroid lattices. The main purpose of this paper is to investigate lattice-theoretical properties of circuits.

In Section 2, we shall show that the set of atoms of an atomistic join-semilattice L with the finite covering property forms a simple matroid. The set $F(L)$ of all finite elements of L forms an AC-lattice, which will be called an FAC-lattice, and in Section 3, we define circuits of atoms in an FAC-lattice.

In Section 4, we shall discuss a connection between the modularity of FAC-lattices and the existence of special circuits, which will be called P-circuits. An important example of FAC-lattice is a bond lattice associated with a non-oriented finite simple graph ([6], [8]). Such a lattice has a remarkable property, that is, it has no non-trivial P-circuit.

Another important example is an affine matroid lattice whose properties are thoroughly investigated in [2], Chapter IV. This lattice always has a property called strongly planar. In Section 5, we shall show that almost all non-modular bond lattices are not strongly planar. From this we can see that, in the set of matroid lattices, there are three disjoint subsets: {non-modular affine matroid lattices}, {non-modular bond lattices} and {modular matroid lattices}.

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2. Atomistic join-semilattices and simple matroids. Let L be a join-semilattice with 0. The set of all atoms of L is denoted by $\Omega(L)$. We put

$$F(L)=\{p_1\vee\ldots\vee p_n;\ \ p_i\in\Omega(L),\ \ n=1,2,\ldots\}\cup\{0\},$$

and an element of $F(L)$ is called a *finite element*. L is called *atomistic* when every non-zero element a of L is the least upper bound of $\{p\in\Omega(L); p\leqq a\}$ (see [7]).

Let L be an atomistic join-semilattice. For any subset ω of $\Omega(L)$, the *closure* of ω is defined by

$$\mathrm{Cl}(\omega)=\{p\in\Omega(L);\ p\leqq q_1\vee\ldots\vee q_n,\ q_i\in\omega\}\ (\mathrm{Cl}(\emptyset)=\emptyset).$$

The following properties are easily verified.

(Cl 1) $\omega\subset\mathrm{Cl}(\omega)$.

(Cl 2) $\omega_1\subset\omega_2$ implies $\mathrm{Cl}(\omega_1)\subset\mathrm{Cl}(\omega_2)$.

(Cl 3) $\mathrm{Cl}(\mathrm{Cl}(\omega))=\mathrm{Cl}(\omega)$.

(Cl 4) If $p\in\mathrm{Cl}(\omega)$ then there exists a finite subset ω' of ω such that $p\in\mathrm{Cl}(\omega')$.

(Cl 5) $\mathrm{Cl}(\{p\})=\{p\}$ for $p\in\Omega(L)$.

Proposition 1. *Let L be an atomistic join-semilattice.*

(i) *The following two statements are equivalent:*

(α) *L has the finite covering property, i.e., if $p\in\Omega(L)$, $a\in F(L)$ and $p\nleqq a$ then $a\vee p$ covers a.*

(β) *If $p,q\in\Omega(L)$, $p\in\mathrm{Cl}(\omega\cup\{q\})$ and $p\notin\mathrm{Cl}(\omega)$ then $q\in\mathrm{Cl}(\omega\cup\{p\})$.*

(ii) *If L satisfies (α) (and (β)), then $F(L)$ is an AC-lattice and $M(L)=(\Omega(L),\mathrm{Cl})$ is a simple matroid. Moreover, the set $L(M(L))=\{\omega\subset\Omega(L);\ \mathrm{Cl}(\omega)=\omega\}$ forms a matroid lattice by set-inclusion, and $F(L)$ is lattice-isomorphic to $F(L(M(L)))$ by the mapping*

$$a\mapsto\omega(a)=\{p\in\Omega(L);\ p\leqq a\}.$$

Proof. (i) It follows from [7], Theorem 2.2 that (α) is equivalent to the following statement (exchange property):

(α') If $p,q\in\Omega(L)$, $a\in F(L)$, $p\leqq a\vee q$ and $p\nleqq a$ then $q\leqq a\vee p$.

We shall prove (α')$\Rightarrow$(β). If $p\in\mathrm{Cl}(\omega\cup\{q\})$ and $p\notin\mathrm{Cl}(\omega)$, then there exist $r_1,\ldots,r_n\in\omega$ such that $p\leqq r_1\vee\ldots\vee r_n\vee q$ and $p\nleqq r_1\vee\ldots\vee r_n$. Hence, $q\leqq r_1\vee\ldots\vee r_n\vee p$ by (α'), and hence $q\in\mathrm{Cl}(\omega\cup\{p\})$. ($\beta$)$\Rightarrow$($\alpha'$). Let $a\in F(L)$, $p\leqq a\vee q$ and $p\nleqq a$. We put $a=r_1\vee\ldots\vee r_n$, $r_i\in\Omega(L)$ and $\omega=\{r_1,\ldots,r_n\}$. Then, $p\in\mathrm{Cl}(\omega\cup\{q\})$ and $p\notin\mathrm{Cl}(\omega)$. Hence, $q\in\mathrm{Cl}(\omega\cup\{p\})$ by (β), and hence $q\leqq a\vee p$.

(ii) If L satisfies (α), then $F(L)$ is a lattice by [7], Theorem 2.5. Evidently, $F(L)$ is atomistic and has the covering property. Moreover, $M(L)=(\Omega(L),\mathrm{Cl})$ is a matroid, since the closure operator satisfies (Cl 1)$\sim$(Cl 4) and (β) ([9], 1.2 and 20.2). $M(L)$ is simple by (Cl 5) and $\mathrm{Cl}(\emptyset)=\emptyset$ ([9], 1.4). The last statement follows from [2], (15.5) and (15.7), since ω is a subspace in the sense of [2], (15.1), if and only if $\mathrm{Cl}(\omega)=\omega$.

Definition. If an AC-lattice L satisfies $F(L)=L$, we shall call it an *FAC-lattice.*

The mapping $L\mapsto F(L)$ is a bijection between the set of matroid lattices and the set of FAC-lattices, because if L is a matroid lattice then $F(L)$ is an FAC-lattice and L is isomorphic to the lattice of all ideals of $F(L)$ by [2], (15.5) and (15.7).

Hereafter, we shall investigate properties of FAC-lattices. We remark the following facts (see [2], (8.5) and (8.14)). Each element a of an FAC-lattice L has the height $h(a)$, and $h(a)=n$ $(a\neq 0)$ if and only if there exist $p_1, \ldots, p_n \in \Omega(L)$ such that $a=p_1\vee\ldots\vee p_n$ and $(p_1\vee\ldots\vee p_{i-1})\wedge p_i=0$ for $i=2, \ldots, n$. For $a, b\in L$, we have

$$h(a\vee b)+h(a\wedge b)\leqq h(a)+h(b),$$

and equality holds if and only if (a, b) is a modular pair (denoted by $(a, b)M$).

3. Circuits of atoms. In this section, let L be an FAC-lattice.

Lemma 2. *Let $\omega=\{p_1, \ldots, p_n\}$ be a finite subset of $\Omega(L)$. The following statements are equivalent.*

(α) $(p_1\vee\ldots\vee p_{i-1})\wedge p_i=0$ *for* $i=2, \ldots, n$.

(β) *ω is a semi-orthogonal family, i.e., if ω_1, ω_2 are disjoint subsets of ω then* $\bigvee(p;\, p\in\omega_1)\perp\bigvee(p;\, p\in\omega_2)$, *where $a\perp b$ means $a\wedge b=0$ and $(a, b)M$* ([2], (2.2) and (8.12)).

(γ) *ω is an independent set of the matroid $M(L)=(\Omega(L), \mathrm{Cl})$, i.e.,* $p_i\notin \mathrm{Cl}(\omega-\{p_i\})$ *for every i* (see [9], 1.7).

(δ) $h(p_1\vee\ldots\vee p_n)=n$.

Proof. (γ) is equivalent to the following statement: $p_i\wedge \bigvee_{j\neq i} p_j=0$ for every i. Hence, the implications $(\beta)\Rightarrow(\gamma)\Rightarrow(\alpha)$ are evident. $(\alpha)\Rightarrow(\beta)$ follows from [2], (2.5) and (8.12). Finally, the equivalence of (α) and (δ) follows from [2], (8.4).

Definition. As in matroid theory, we call a finite subset ω of $\Omega(L)$ a *circuit* when ω is a minimal dependent set, i.e., $\omega-\{p\}$ is independent and $p\leqq \bigvee(q;\, q\in\omega-\{p\})$ for any $p\in\omega$. For instance, if p, q, r are different atoms and $p\leqq q\vee r$ then $\{p, q, r\}$ is a circuit. The cardinality $|C|$ of a circuit C is not less than 3. The set of all circuits of $\Omega(L)$ is denoted by $\mathscr{C}(L)$.

Proposition 3. *Let a, b be elements of an FAC-lattice L. The following statements are equivalent.*

(α) *a and b are semi-orthogonal.*

(β) $h(a\vee b) = h(a)+h(b)$.

(γ) $\omega(a)\cap\omega(b)=\emptyset$ *and there is no* $C\in\mathscr{C}(L)$ *such that* $C\subset\omega(a)\cup\omega(b)$, $C\not\subset\omega(a)$ *and* $C\not\subset\omega(b)$.

Proof. $(\beta)\Rightarrow(\alpha)$. By ($\beta$) we have $h(a\vee b)+h(a\wedge b)\leqq h(a)+h(b)=h(a\vee b)$. Hence, $h(a\wedge b)=0$ and $(a, b)M$, so that $a\perp b$.

$(\alpha)\Rightarrow(\gamma)$. $\omega(a)\cap\omega(b)=\emptyset$ is evident. Let $C\in\mathscr{C}(L)$ with $C\subset\omega(a)\cup\omega(b)$. If both $\omega(a)\cap C$ and $\omega(b)\cap C$ were independent sets, then $C=(\omega(a)\cap C)\cup(\omega(b)\cap C)$ would be a semi-orthogonal family by (α) and [2], (2.4). This contradicts that C is a dependent set. Hence, for instance, $\omega(a)\cap C$ is dependent. Since C is minimal dependent, we have $C=\omega(a)\cap C\subset\omega(a)$.

$(\gamma)\Rightarrow(\beta)$. We may assume that $a\neq 0$ and $b\neq 0$. We put $h(a)=m$ and $h(b)=n$. Since $\omega(a)\cap\omega(b)=\emptyset$ by (γ), there are disjoint independent sets $\{p_1, \dots, p_m\}$, $\{q_1, \dots, q_n\}$ of $\Omega(L)$ with $a=p_1\vee\dots\vee p_m$, $b=q_1\vee\dots\vee q_n$. If $h(a\vee b)<m+n$, then $\{p_1, \dots, p_m, q_1, \dots, q_n\}$ would be dependent, and hence there is $C\in\mathscr{C}(L)$ such that

$$C\subset \{p_1, \dots, p_m, q_1, \dots, q_n\}\subset \omega(a)\cup\omega(b).$$

But, $C\not\subset\{p_1, \dots, p_m\}$ since $\{p_1, \dots, p_m\}$ is independent. Hence, C contains some q_i, so that $C\not\subset\omega(a)$. Similarly, $C\not\subset\omega(b)$, a contradiction. Therefore, $h(a\vee b)= =m+n$.

Lemma 4. *Let* $\omega\subset\Omega(L)$ *and* $p\in\Omega(L)-\omega$. $p\in\mathrm{Cl}(\omega)$ *if and only if there exsits* $C\in\mathscr{C}(L)$ *such that* $p\in C\subset\omega\cup\{p\}$.

Proof. If $p\in\mathrm{Cl}(\omega)$, then there exist $q_1, \dots, q_n\in\omega$ such that $p\leqq q_1\vee\dots\vee q_n$. Let ω_0 be a minimal subset of $\{q_1, \dots, q_n\}$ such that $p\leqq\bigvee(q;\, q\in\omega_0)$. Then, ω_0 must be independent by the minimality. Moreover, for any $q_i\in\omega_0$, $(\omega_0-\{q_i\})\cup\{p\}$ is independent since $p\not\leqq\bigvee(q;\, q\in\omega_0-\{q_i\})$. Therefore, $C=\omega_0\cup\{p\}$ is a circuit and $p\in C\subset\omega\cup\{p\}$.

Conversely, if $C\in\mathscr{C}(L)$ and $p\in C\subset\omega\cup\{p\}$, then we have $p\leqq\bigvee(q;\, q\in C-\{p\})$, and then $p\in\mathrm{Cl}(\omega)$, since $C-\{p\}\subset\omega$.

4. Modularity of *FAC*-lattices. Let L be an atomistic lattice. For $n=1, 2, \dots$, we put

$$\Omega^n = \{p_1\vee\dots\vee p_n;\ p_i\in\Omega(L)\}.$$

Evidently, $\Omega^1=\Omega(L)$, $\Omega^n\subset\Omega^{n+1}$ for every n, and $\bigcup_{n=1}^{\infty}\Omega^n=F(L)-\{0\}$.

For two subsets A, B of L, we write $(A, B)M$ (resp. $(A, B)M^*$) if (a, b) is modular (resp. dual-modular) for all $a\in A$, $b\in B$. The following equivalences are proved in [4] (or [7]).

(1) $(A, \Omega^n)M \Leftrightarrow (A, \Omega^{n-1})M^* \quad (n = 2, 3, \ldots), \quad (A, L)M \Leftrightarrow (A, L)M^*.$

(2) $(\Omega^n, \Omega)M^* \Leftrightarrow (\Omega^{n-1}, \Omega^2)M^* \Leftrightarrow \ldots \Leftrightarrow (\Omega^2, \Omega^{n-1})M^* \quad (n = 3, 4, \ldots).$

(3)
$$(F(L), \Omega^n)M^* \Leftrightarrow (F(L), F(L))M^* \quad (n = 1, 2, \ldots),$$
$$(\Omega^n, F(L))M^* \Leftrightarrow (F(L), F(L))M^* \quad (n = 2, 3, \ldots).$$

If L is an FAC-lattice, then $F(L)=L$, and $(\Omega, L)M^*$ holds by the covering property. For L, we have the following implications:

$(*)$ $(L, L)M^* \Rightarrow \ldots \Rightarrow (\Omega^{m+1}, \Omega)M^* \Rightarrow (\Omega^m, \Omega)M^* \Rightarrow \ldots \Rightarrow (\Omega^2, \Omega)M^*.$

We remark that each of $(\Omega^m, \Omega^n)M^*$, $(\Omega^m, L)M^*$ and $(L, \Omega^n)M^*$ $(m \geqq 2$ and $n \geqq 1)$ is equivalent to some member of $(*)$ by (2) and (3).

Lemma 5. *Let L be an FAC-lattice and let $p \in \Omega(L)$ and $m \geqq 2$. The following statements are equivalent.*

(α) *$(a, p)M^*$ for every $a \in \Omega^m$.*

(β) *If $p \in C \in \mathscr{C}(L)$ and $|C| \leqq m+2$, then for any $q \in C - \{p\}$ there exists $r \in \mathrm{Cl}(C - \{p, q\})$ such that $\{p, q, r\} \in \mathscr{C}(L)$.*

Proof. It follows from [3], Lemma 2 that (α) is equivalent to

(α') $(a, p)P$ for every $a \in \Omega^m$.

($(a, p)P$ means that if $q \in \Omega(L)$ and $q \leqq a \vee p$ then there exists $r \in \Omega(L)$ such that $q \leqq r \vee p$ and $r \leqq a$.) We shall prove (α')$\Rightarrow$(β). Let $p \in C \in \mathscr{C}(L)$, $|C| \leqq m+2$ and $q \in C - \{p\}$. We put $a = \bigvee(r;\ r \in C - \{p, q\})$. Then, $a \in \Omega^m$. We have $p \wedge a = 0$, since $C - \{q\}$ is independent. Similarly, $q \wedge a = 0$. Since $q \in C \subset \omega(a \vee p) \cup \{q\}$, it follows from Lemma 4 that $q \in \mathrm{Cl}(\omega(a \vee p)) = \omega(a \vee p)$, so that $q \leqq a \vee p$. By (α') there exists $r \in \Omega(L)$ such that $q \leqq r \vee p$ and $r \leqq a$. We have $r \neq p, q$ since $p \wedge a = q \wedge a = 0$. Hence, $\{p, q, r\}$ is a circuit and $r \in \omega(a) = \mathrm{Cl}(C - \{p, q\})$.

(β)$\Rightarrow$(α'). Let $a \in \Omega^m$. There is an independent set $\{r_1, \ldots, r_n\}$ with $a = r_1 \vee \ldots \vee r_n$, and then $n \leqq m$. Let $q \leqq a \vee p$ $(q \in \Omega(L))$ and we shall show the existence of $r \in \Omega(L)$ with $q \leqq r \vee p$, $r \leqq a$. We may assume $q \nleqq a$ and $q \neq p$. The set $\{p, q, r_1, \ldots, r_n\}$ is dependent since $q \leqq p \vee r_1 \vee \ldots \vee r_n$. Hence, there is a circuit C such that $\{p, q\} \subset C \subset \{p, q, r_1, \ldots, r_n\}$. By ($\beta$) there exists $r \in \mathrm{Cl}(C - \{p, q\})$ such that $\{p, q, r\} \in \mathscr{C}(L)$. Then, $q \leqq p \vee r$, and we have $r \leqq a$ since $C - \{p, q\} \subset \omega(a)$.

Definition. Let L be a FAC-lattice. A circuit $C \in \mathscr{C}(L)$ is called a *P-circuit* if for every $p, q \in C$ $(p \neq q)$ there exists $r \in \mathrm{Cl}(C - \{p, q\})$ such that $\{p, q, r\} \in \mathscr{C}(L)$. Evidently, if $|C| = 3$ then C is a P-circuit.

Theorem 6. *Let L be an FAC-lattice, and let $m \geqq 2$.*

(i) *L satisfies $(\Omega^m, \Omega)M^*$ if and only if every $C \in \mathscr{C}(L)$ with $|C| \leqq m+2$ is a P-circuit.*

(ii) *L is modular* (i.e., $(L, L)M^*$) *if and only if every $C \in \mathscr{C}(L)$ is a P-circuit.*

Proof. (i) directly follows from Lemma 5. Since $(L, \Omega)M^* \Leftrightarrow (L, L)M^*$, (ii) follows from (i).

Definition. Let L be a lattice and let $a, b \in L$. (a, b) is called a *distributive pair* (a join-distributive pair in [5]), denoted by $(a, b)D$, when

$$(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) \quad \text{for every} \quad x \in L.$$

If L is atomistic, it is easy to verify that $(a, b)D$ is equivalent to the following condition:

If $p \in \Omega(L)$ and $p \leqq a \vee b$ then $p \leqq a$ or $p \leqq b$. Hence, $(a, b)D \Leftrightarrow \omega(a \vee b) = \omega(a) \cup \omega(b)$.

We shall now be interested in an FAC-lattice L satisfying the following condition:

(D) If $C \in \mathscr{C}(L)$ and $|C| \geqq 4$ then for any $p \in C$ there exists $q \in C - \{p\}$ such that $(p, q)D$.

Lemma 7. *If an FAC-lattice L satisfies* (D), *then $C \in \mathscr{C}(L)$ is a P-circuit only when $|C| = 3$.*

Proof. Let $C \in \mathscr{C}(L)$ with $|C| \geqq 4$, and let $p \in C$. By (D) there is $q \in C - \{p\}$ such that $(p, q)D$. Then, $\{p, q, r\}$ is not a circuit for any $r \in C - \{p, q\}$, because $r \leqq p \vee q$ implies $r = p$ or $r = q$. Hence, C is not a P-circuit.

Example 8. Let G be a non-oriented finite simple graph, and let $E(G)$ be the set of all edges of G. The cycle matroid $M(G)$ is defined by the collection of independent subsets of $E(G)$, where S is an independent subset if and only if S does not contain a cycle of G ([9], 1.3). A subset C of $E(G)$ is minimal dependent if and only if C is a cycle, and the closure operator in $M(G)$ is defined as follows:

$$x \in \mathrm{Cl}(S) \Leftrightarrow x \in S \quad \text{or there exists a cycle } C \text{ such that} \quad x \in C \subset S \cup \{x\}.$$

It is easy to verify in the same way as in Proposition 1 (ii) that the set

$$L(G) = \{S \subset E(G);\ \mathrm{Cl}(S) = S\}$$

forms an AC-lattice (cf. [9], 3.3). Since $E(G)$ is a finite set, $L(G)$ is an FAC-lattice, and we call it a *bond lattice* associated with G ([6], [8]). We remark that the set $\mathscr{C}(L(G))$ is just the set of all cycles of G.

We shall show that

(G) $(S_1, S_2)D$ in $L(G)$ if S_1 and S_2 has no common vertex.

Let $x \in S_1 \vee S_2 = \mathrm{Cl}(S_1 \cup S_2)$. If $x \notin S_1 \cup S_2$ then there is a cycle C such that $x \in C \subset S_1 \cup S_2 \cup \{x\}$. Since S_1 and S_2 has no common vertex, we have $C \subset S_i \cup \{x\}$ for some i. Then, $x \in \mathrm{Cl}(S_i) = S_i$. Therefore, $(S_1, S_2)D$ holds.

It is easy to show by (G) that any bond lattice satisfies the condition (D). In fact, if C is a cycle with $C \geqq |4|$ and if $x \in C$, then there exists $y \in C$ such that x and y have no common vertex.

Theorem 9. *Let L be an FAC-lattice satisfying* (D) *(for instance, a bond lattice), and let $m \geqq 2$.*

(i) *L satisfies $(\Omega^m, \Omega)M^*$ if and only if there is no $C \in \mathscr{C}(L)$ such that $4 \leqq |C| \leqq m+2$.*

(ii) *L satisfies $(\Omega^m, \Omega)M^*$ but does not satisfy $(\Omega^{m+1}, \Omega)M^*$ if and only if there is $C_0 \in \mathscr{C}(L)$ with $|C_0| = m+3$ and there is no $C \in \mathscr{C}(L)$ such that $4 \leqq |C| \leqq m+2$.*

(iii) *L does not satisfy $(\Omega^2, \Omega)M^*$ if and only if there is $C_0 \in \mathscr{C}(L)$ with $|C_0| = 4$.*

(iv) *L is modular if and only if there is no $C \in \mathscr{C}(L)$ such that $|C| \geqq 4$.*

Proof. Evidently, the statements (i) and (iv) follow from Theorem 6 and Lemma 7, and (ii) and (iii) follow from (i). (The statement (iv) was proved in [8], in case $L = L(G)$.)

We remark that if a graph G is a cycle with $n+3$ edges then $L(G)$ is isomorphic to the lattice given in [4], Example 3.

5. Strongly planar lattices. An AC-lattice L is called *strongly planar* when it satisfies the following condition ([2], (14.3)):

(SP) If $p, q, r \in \Omega(L)$, $a \in L$ and if $p \leqq q \vee a$ and $r \leqq a$ then there exists $s \in \Omega(L)$ such that $p \leqq q \vee r \vee s$ and $s \leqq a$.

It follows from [2], (14.4) that an AC-lattice is strongly planar if either L is modular or the length of L is 3 (i.e., L has 1 and $h(1) = 3$). We call such a lattice a trivial strongly planar lattice. It is well-known that non-modular affine matroid lattices are non-trivial strongly planar lattices (see [2], (18.3) and (14.5)). Here we shall show that if a bond lattice $L(G)$ is strongly planar then it is a trivial one. (Hence, the set of non-modular affine matroid lattices and the set of bond lattices have no common element.)

Firstly we remark that any bond lattice $L = L(G)$ satisfies the following two conditions by the property (G):

(D′) If $C \in \mathscr{C}(L)$ and $|C| \geqq 5$ then there exist three different elements $p, q, r \in C$ such that $(p, q \vee r)D$.

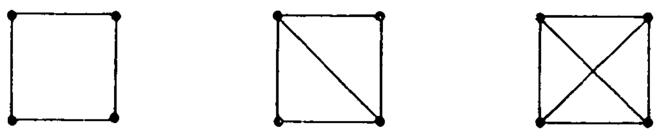
(D″) If $p \in \Omega(L)$, $C \in \mathscr{C}(L)$, $|C| = 4$ and $p \nleqq \bigvee(q;\ q \in C)$ then there exist $q_1, q_2 \in C$ $(q_1 \neq q_2)$ such that $(p \vee q_1, q_2)D$.

Theorem 10. *Let L be a FAC-lattice satisfying* (D′) *and* (D″) *(for instance, a bond lattice). If L is strongly planar and non-modular then the length of L is* 3.

Proof. Since L is non-modular, by Theorem 6 (ii) there is $C \in \mathscr{C}(L)$ which is not a P-circuit. Then, $|C| \geqq 4$. Suppose $|C| \geqq 5$, then by (D′) there exist three different elements $p, q, r \in C$ such that $(p, q \vee r)D$. Put $a = \bigvee(t;\ t \in C - \{p, q\})$. Since $p \leqq q \vee a$ and $r \leqq a$, by (SP) there exists $s \in \Omega(L)$ such that $p \leqq q \vee r \vee s$ and $s \leqq a$. The set $\{p, q, r\}$ is independent, since C is minimal dependent. Hence, $p \not\leqq q \vee r$, and then we have $s \leqq q \vee r \vee p$ by (α') in the proof of Proposition 1. Since $C - \{q\}$ is independent, we have $p \not\leqq a$, so that $s \neq p$. Hence, by $(p, q \vee r)D$ we have $s \leqq q \vee r$. This implies $p \leqq q \vee r$, a contradiction. Therefore, we obtain that $|C| = 4$.

Next we shall show that $\bigvee(t;\ t \in C) = 1$. Suppose there is $p \in \Omega(L)$ such that $p \not\leqq \bigvee(t, t \in C)$. By (D″) there exist $q_1, q_2 \in C$ such that $(p \vee q_1, q_2)D$. We put $C = \{q_1, q_2, q_3, q_4\}$ and $b = p \vee q_3 \vee q_4$. Since $q_2 \leqq q_1 \vee q_3 \vee q_4 \leqq q_1 \vee b$ and $p \leqq b$, by (SP) there exists $s \in \Omega(L)$ such that $q_2 \leqq q_1 \vee p \vee s$ and $s \leqq b$. We have $q_2 \not\leqq p \vee q_1$, since the set $\{p, q_1, q_2\}$ is independent by $p \not\leqq q_1 \vee q_2$. Hence, we have $s \leqq p \vee q_1 \vee q_2$ by (α'), and then either $s = q_2$ or $s \leqq p \vee q_1$ by $(p \vee q_1, q_2)D$. But, $s \leqq p \vee q_1$ implies $q_2 \leqq q_1 \vee p$, a contradiction. Moreover, since $\{q_2, q_3, q_4\}$ is independent and $p \not\leqq q_2 \vee q_3 \vee q_4$, $\{p, q_2, q_3, q_4\}$ is independent and hence $q_2 \not\leqq b$. Thus, $s = q_2$ contradicts that $s \leqq b$. Therefore, we obtain $\bigvee(t;\ t \in C) = 1$, and then $h(1) = |C| - 1 = 3$.

Corollary 11. *If a bond lattice $L(G)$ is strongly planar and non-modular then G is isomorphic to one of the following three graphs:*



Proof. By the proof of the theorem, there is a cycle $C \subset E(G)$ such that $|C| = 4$ and $\mathrm{Cl}(C) = E(G)$.

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