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A note on radical and semisimple classes of topological rings

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To the memory of András Huhn

In this note we apply the general Kurosh—Amitsur radical theory presented in [5] to categories of topological rings and deduce characterizations for radical and semisimple classes which go, partly, far beyond the point to which Arnautov and Vodinchar developed these aspects of the radical theory of topological rings. Furthermore, we carry over a characterization of semisimple classes of supernilpotent radicals to the topological case.

1. Let TopR denote the category of Hausdorff topological (associative) rings and all continuous homomorphisms, and $\mathscr{C}$ be a *universal class* in TopR, i.e., a subcategory such that:

(i) if $A\in\mathscr{C}^\circ$ ($\mathscr{C}^\circ$ denotes the class of objects of $\mathscr{C}$) and $B\lhd A$ (i.e., B is an ideal of A endowed with the subspace topology) with canonical embedding $\varphi\colon B\to A$, then $B\in\mathscr{C}^\circ$ and $\varphi\in\mathscr{C}$;

(ii) if $A\in\mathscr{C}^\circ$, $\psi\colon A\to C$ is a surjective morphism in TopR, and the topology of C agrees with the quotient topology corresponding to ψ, then $C\in\mathscr{C}^\circ$ and $\psi\in\mathscr{C}$.

In addition, we assume that every morphism in $\mathscr{C}$ admits a unique factorization into the composition of a surjective morphism and a morphism which is a subspace embedding; in other words, if we denote by $\mathscr{E}$ the class of all surjective morphisms and by $\mathscr{M}$ the class of all subspace embeddings in $\mathscr{C}$, then $\mathscr{C}$ admits a unique $(\mathscr{E}, \mathscr{M})$ factorization. Whenever we shall speak of a factorobject $\psi\colon A\to C$ or a subobject $\varphi\colon B\to A$ of an $A\in\mathscr{C}^\circ$, this means that $\psi\in\mathscr{E}$ or $\varphi\in\mathscr{M}$, respectively. We assume also that for every $A\in\mathscr{C}^\circ$, its factorobjects form a complete lattice and its subobjects an inductive set, the latter in the sense that any ascending chain of subobjects has

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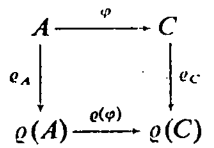
a least upper bound. By a trivial object we mean a one-element ring 0, which is necessarily in $\mathscr{C}$. Then $\mathscr{C}$ satisfies all the axioms imposed on the category in [5].

2. In accordance with the terminology in [5], by a radical we mean a mapping ϱ which assings to every $A \in \mathscr{C}^{\circ}$ a factorobject $\varrho_A\colon A \to \varrho(A)$ such that

(ϱ1) for every $\varphi\colon A \to C$ from $\mathscr{E}$, there is a $\varrho(\varphi)\colon \varrho(A) \to \varrho(C)$ in $\mathscr{E}$ which makes the diagram

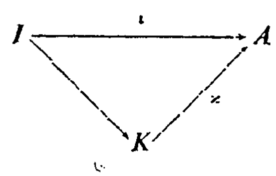


commutative;

(ϱ2) $\varrho_{\varrho(A)} = 1_{\varrho(A)}$ for all $A \in \mathscr{C}^{\circ}$;

(ϱ3) $\varrho_A = 1_A$ if and only if for all $0 \neq B \lhd A$, $\varrho(B) \neq 0$.

This notion of radical is formulated in terms of the factorization in $\mathscr{C}$. However, at least if $\mathscr{C}$ is the category of all topological rings and all continuous homomorphisms, then the notion of radical is in fact independent of the factorization. To show this, notice that this category admits two extremal factorizations: (surjective homomorphisms with quotient topology, continuous monomorphisms) and (continuous epimorphisms, extremal monomorphisms with subspace embedding). By Remark 5 to the definition of M-radicals in [5], it suffices to exhibit that these two factorizations admit the same radicals. In view of the considerations there, any ϱ_A obtained in the factorization with the larger class of epimorphisms is an epimorphism of the stricter sense — that is, ϱ_A is necessarily a surjective homomorphism with the quotient topology on $\varrho(A)$. Furthermore, if we have an M-system in the larger sense, i.e., an ideal I with a topology which is maybe finer than the subspace topology, then firstly, the embedding ι of this ideal is algebraically an extremal monomorphism, and secondly, if we consider its factorization in the other sense, then we obtain an



object K which is the same ideal I with the subspace topology, hence its embedding $\varkappa$ is a monomorphism of the strictest sense. Thus from a radical ideal in the first factorization we obtain a radical ideal in the second one. By this we have the independence we wanted to show.

For a radical ϱ, we denote the radical and the semisimple classes by $\mathbf{R}_\varrho$ and $\mathbf{S}_\varrho$, respectively, i.e.,

$$\mathbf{R}_\varrho = \{A\in\mathscr{C}^0\colon\ \varrho(A) = 0\},\quad \mathbf{S}_\varrho = \{A\in\mathscr{C}^0\colon\ \varrho_A = 1_A\}.$$

We know that each of the radical, the radical class, and the semisimple class determines the other two.

It follows from (ϱ1) that every radical class $\mathbf{R}_\varrho$ is $\mathscr{E}$-closed, i.e., if $\varphi\colon A\to C\in\mathscr{E}$ and $A\in\mathbf{R}_\varrho$ then $C\in\mathbf{R}_\varrho$.

Proposition 1. *Let A be a ring with two topologies σ and τ, $\sigma\leqq\tau$, in $\mathscr{C}^0$. Then $(A,\tau)\in\mathbf{R}_\varrho$ implies $(A,\sigma)\in\mathbf{R}_\varrho$ and $(A,\sigma)\in\mathbf{S}_\varrho$ implies $(A,\tau)\in\mathbf{S}_\varrho$.*

Proof. The first claim follows from the $\mathscr{E}$-closedness of $\mathbf{R}_\varrho$, and implies by (ϱ3) the second one.

For an arbitrary subclass $\mathbf{R}\subseteqq\mathscr{C}^0$ and any $A\in\mathscr{C}^0$ we put $\mathbf{R}A=\sum(B\lhd A,\ B\in\mathbf{R})$ endowed with the subspace topology. Clearly, $\mathbf{R}A\lhd A$.

In view of [5] Proposition 4.2 and the characterization II. 1—2 of radical classes in Arnautov and Vodinchar [4], our radicals are the same as those in [4]. (Notice that our definition makes no allusion to the closedness of the largest radical ideal!) Therefore $\mathbf{R}_\varrho A$ is the kernel of ϱ_A (hence a closed ideal of A) and $\mathbf{R}_\varrho A\in\mathbf{R}_\varrho$. Thus every radical in $\mathscr{C}$ is attainable in the sense of [5]. [5] Theorem 3.1 and Proposition 4.1 yield now the following characterization of radicals.

Proposition 2. *A mapping ϱ which assigns to each $A\in\mathscr{C}^0$ a factorobject $(\varrho_A, \varrho(A))$ is a radical if and only if it satisfies conditions* (ϱ1), (ϱ2), *and*

(ϱ3*) *for every $A\in\mathscr{C}^0$ there is an ideal $I\lhd A$ such that $\varrho(I)=0$, ϱ_A is the canonical factor $A\to A/\bar{I}$ (where $\bar{I}$ denotes the closure of I) and for all $J\lhd A$, $\varrho(J)=0$ implies $J\subseteqq\bar{I}$.*

Remark. Notice that in condition (ϱ3*) it is not required that the ideal I be closed; however, it can always be chosen to be closed, as was shown above.

3. From [5] we obtain now three characterizations of radical classes. The first of them is a simple transcription of [5] Proposition 3.4, the latter two follow from Remark 3.7, which is easily seen to apply in our case. Therefore we shall give no proof here. Characterization (I) is just the definition of radical classes in Arnautov and Vodinchar [4]. Notice that the closedness of the largest radical ideal is imposed only in characterization (I).

Theorem 3. *A class $\mathbf{R}\subseteqq\mathscr{C}^0$ is a radical class if and only if it satisfies*

(I) (R2) $\mathbf{R}$ *is $\mathscr{E}$-closed,* (R4) *in every* $A\in\mathscr{C}^{\circ}$ *there is an ideal* $\mathbf{R}(A)$ *such that* $\mathbf{R}(A)\in\mathbf{R}$ *and* $I\subseteq\overline{\mathbf{R}(A)}$ *for all* $I\lhd A$, $I\in\mathbf{R}$,

(R5) $\mathbf{R}(A/\overline{\mathbf{R}(A)})=0$,

and $\mathbf{R}(A)$ *in* (R4) *is closed in* A; *or*

(II) (R2), (R4), *and*

(R3) *if* $I\lhd A$, $I\in\mathbf{R}$ *and* $A/\bar{I}\in\mathbf{R}$ *then* $A\in\mathbf{R}$; *or*

(III) (R2), (R3), *and*

(R4′) *in every object, the union of any chain of ideals from* $\mathbf{R}$ *belongs to* $\mathbf{R}$.

4. Let S be a subclass of $\mathscr{C}^{\circ}$ which is closed under subdirect products (the topology on a subdirect product is the subspace topology of the product topology; of course, we consider only those subdirect products which are in $\mathscr{C}^{\circ}$). Then every $A\in\mathscr{C}^{\circ}$ has a largest factorobject in S; we denote by $S(A)$ the kernel ideal belonging to this factor (then $S(A)$ is necessarily closed).

By Theorem 1 in Arnautov [3] every radical in $\mathscr{C}$ has the A—D—S property, i.e., for any radical class $\mathbf{R}$ and any $A, B\in\mathscr{C}^{\circ}$, $B\lhd A$, we have $\mathbf{R}B\lhd A$. Consequently, every semisimple class in $\mathscr{C}$ is hereditary (with respect to ideals).

We also have the obvious characterization (see Arnautov and Vodinchar [4]): $S\subseteq\mathscr{C}^{\circ}$ is a semisimple class if and only if, for all $A\in\mathscr{C}^{\circ}$,

$$A\in S \Leftrightarrow \forall B\lhd A\colon\ B\neq 0\Rightarrow B \text{ has a non-zero factor in } S.$$

Theorem 3.6 from [5] translates into the following:

A class $S\subseteq\mathscr{C}^{\circ}$ *is a semisimple class if and only if*

(S3) S *is closed under subdirect products,*

(S4′) S *is regular, i.e., if* $A\in S$ *and* $0\neq B\lhd A$ *then* B *has a non-zero factor in* S,

(S6) *if* $\psi\colon A\to B$ *is a surjective continuous homomorphism with* $\operatorname{Ker}\psi\in S$ *and* $B\in S$ *then* $A\in S$,

(S7) $S(S(A))\lhd A$ *for all* $A\in\mathscr{C}^{\circ}$.

In fact, here (S7) follows from the other conditions, and (S3) can be weakened to the coinductive property

(S3′) if (I_α) is a descending chain of closed ideals in $A\in\mathscr{C}^{\circ}$ such that $A/I_\alpha\in S$ for all α, then $A/\bigcap I_\alpha\in S$.

Theorem 4. *A class* $S\subseteq\mathscr{C}^{\circ}$ *is a semisimple class if and only if it satisfies* (S3′), (S4′) *and* (S6).

Lemma 5. *Suppose that* $S\subseteq\mathscr{C}^{\circ}$ *satisfies* (S3′), (S4′), (S6). *If* $I\lhd A\in S$ *and* $I^2=0$, *then also* $I\in S$.

Proof. At first we shall prove the validity of the weaker statement: if $I\lhd A\in S$ and $A^2=0$, then $I\in S$. By condition (S3′) Zorn's lemma is applicable, so there exists a closed ideal J of I which is minimal with respect to the property $I/J\in S$. If $J\neq 0$ then $J\lhd A$ because $A^2=0$, hence by (S4′) there is a closed ideal $K\lhd J$ such that $0\neq J/K\in S$. Again we have $K\lhd I$, K is closed in I, and so $(I/K)/(J/K)\cong$ $\cong I/J\in S$ holds, hence condition (S6) yields $I/K\in S$. By the minimality of J it follows now $K=J$, i.e., $J/K=0$, a contradiction. Thus $J=0$ and $I\in S$.

Now we turn to the proof of the general case of Lemma 5, and choose J again as before. If $J\lhd A$ then, as above, we conclude $I\in S$. Suppose therefore that J is not an ideal of A. Then there exists an element $a\in A$ such that, say, $aJ\nsubseteq J$. Now we have $0\neq(aJ+J)/J\lhd I/J\in S$ and $(I/J)^2=0$, hence the foregoing consideration yields that $(aJ+J)/J\in S$. Furthermore, it is easy to check that the mapping

$$\varphi\colon J\to(aJ+J)/J \quad \text{defined by} \quad j\mapsto aj+J$$

is a continuous surjective homomorphism (J and $(aJ+J)/J$ have the subspace topology induced by I and I/J, respectively) and that Ker φ is a closed ideal not only of J but also of I. Also, the algebraic isomorphism $J/\text{Ker}\,\varphi\to(aJ+J)/J$ is easily seen to be continuous, therefore $J/\text{Ker}\,\varphi\in S$ by (S6). Now $(I/\text{Ker}\,\varphi)/(J/\text{Ker}\,\varphi)\cong$ $\cong I/J\in S$, hence by (S6) we conclude that $I/\text{Ker}\,\varphi\in S$. Then by the minimality of J we have $J=\text{Ker}\,\varphi$ and so $aJ+J=J$, a contradiction. Hence $J\lhd A$, and the lemma is proven.

Proof of Theorem 4. We have already seen that the conditions (S3′), (S4′), (S6) are necessary. In view of an observation made at the beginning of section 4, the sufficiency will be proven if we exhibit that the converse of (S4′) holds. So, let $A\in\mathscr{C}^0$ be such that every non-zero ideal of A has a non-zero factor in S. Then by (S3′) and (S4′) there exists a closed ideal $I\lhd A$ such that $A/I\in S$ and I is minimal with respect to this property. We shall show that $I=0$ and so $A\in S$. Assume that $I\neq 0$. Applying (S4′) to $I\lhd A$ and (S3′) to I, we obtain a closed ideal $J\lhd I$ such that $0\neq I/J\in S$ and J is minimal with respect to this property. We claim that $J\lhd A$. Assume that this is not the case and that $aJ\nsubseteq J$ for an element $a\in A$. Then we have, as in Lemma 5, a continuous surjective homomorphism

$$\varphi\colon J\to(aJ+J)/J$$

with Ker $\varphi\lhd I$, and by Lemma 5 we have $(aJ+J)/J\in S$. Now we proceed exactly as in the proof of the general case in Lemma 5, and arrive at $J\lhd A$. Then (S6) together with $(A/J)/(I/J)\cong A/I\in S$ and $I/J\in S$ yields $A/J\in S$. By the minimality of I we have now $I=J$, contrary to the assumption $I/J\neq 0$. Thus the case $I\neq 0$ is impossible, and the proof of the theorem is complete.

Remark. If all rings in $\mathscr{C}$ are compact or linearly compact in the narrow sense, then by Ánh [2] we also know that S is a semisimple class if and only if it satisfies (S6), is hereditary, and is closed under inverse limits.

5. We also have the following characterization for pairs of corresponding radical and semisimple classes, which is a transcription of [5] Theorem 3.5.

Theorem 6. *A pair* $(\mathbf{R}, \mathrm{S})$ *of subclasses of* $\mathscr{C}^\circ$ *is a pair of corresponding radical and semisimple classes if and only if*

(α) $\mathbf{R} \cap \mathrm{S} = \{0\}$,

(β'') *if* $A \in \mathbf{R}$ *then* A *has no non-zero factors from* S,

(γ) *if* $A \in \mathrm{S}$ *then* A *has no non-zero ideals from* $\mathbf{R}$,

(δ) *each* $A \in \mathscr{C}^\circ$ *has a closed ideal* I *such that* $I \in \mathbf{R}$ *and* $A/I \in \mathrm{S}$.

6. Finally we are going to characterize semisimple classes of supernilpotent radicals.

Lemma 7. *Let* A *be a topological ring and* $I \lhd A$ *(not necessarily closed). Further, let* K *be an ideal of* A *which is maximal relative to* $I \cap K = 0$. *Then* $I \cap \overline{K}$ *is contained in the annihilator of* I *in* A.

Proof. If K is closed then there is nothing to prove. If K is not closed then let $a \in I \cap \overline{K}$ be any non-zero element. Now every neighbourhood U_a of a such that $0 \notin U_a$ contains an element $x \in K$, so we have

$$Ix + xI \subseteqq IK + KI \subseteqq I \cap K = 0.$$

Hence each neighbourhood of a contains a two-sided annihilator of I. By the continuity of multiplication also a annihilates I. Since a was arbitrary, we are done.

Corollary 8. *Let* S *be a regular class of topological rings which contains no non-trivial zero-rings. If* $I \lhd A$ *and* $I \in \mathrm{S}$, *then any ideal* K *of* A *which is maximal relative to* $I \cap K = 0$, *is closed in* A.

Proof. If K is not closed then by Lemma 7 $0 \neq I \cap \overline{K} \subseteqq \mathrm{ann}_A I$. Hence $I \cap K$ is a zero-ring, and at the same time an ideal of I. Since S is regular, a non-trivial homomorphic image of $I \cap \overline{K}$ is in S, a contradiction.

Recall that a radical class is said to be *supernilpotent* if it is hereditary and contains all nilpotent rings, and that a class $\mathbf{C}$ of (topological) rings is said to be *closed under essential extensions* if $A \in \mathbf{C}$ whenever $\mathbf{C}$ contains an essential ideal of A.

Theorem 9. *A class* $\mathrm{S} \subseteqq \mathscr{C}^\circ$ *is the semisimple class of a supernilpotent radical if and only if* S *is regular, closed under subdirect products and essential extensions, and consists of semiprime rings.*

Proof. The standard proof for the discrete case (see Anderson and Wiegandt [1]) works, as the ideal K in the Corollary is closed.

In [4] ARNAUTOV and VODINCHAR proved the following strong result: in the universal class of all (Hausdorff) topological rings a hereditary radical class is either supernilpotent or subidempotent (that is, it consists of idempotent rings). This gives rise to the following

Problem. Characterize the semisimple classes of hereditary radicals of topological rings (by characterizing the semisimple classes of subidempotent radicals and using the above quoted result of Arnautov and Vodinchar).

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