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A partial ordering for the chief factors of a solvable group

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Dedicated to the memory of András P. Huhn

Let G be a finite solvable group. The chief factors of G can be considered in a natural way as representation modules of G. If M, N are normal subgroups such that M/N is a chief factor of G then the centralizer $C_G(M/N)$ of M/N in G, consisting of all elements $g \in G$ with $x^{-1}g^{-1}xg \in N$ for each $x \in M$, is the kernel of the representation which G takes on M/N. Chief factors M_1/N_1, M_2/N_2 are G-isomorphic, denoted by $M_1/N_1 \cong_G M_2/N_2$, iff they afford equivalent representations of G over the same finite prime field. The class of chief factors of G which are G-isomorphic to the chief factor M/N will be denoted by $[M/N]$. We introduce a partial ordering for the classes of G-isomorphic chief factors of G: The class $[M_1/N_1]$ is said to be greater than the class $[M_2/N_2]$, denoted by $[M_1/N_1] > [M_2/N_2]$, if there is a chief factor M_1^*/N_1^* of G such that

$$G \geqq \ldots \geqq M_1^* > N_1^* \geqq \ldots \geqq C_G(M_2/N_2), \quad M_1^*/N_1^* \cong_G M_1/N_1.$$

The set of classes of G-isomorphic chief factors of G together with the partial ordering $\geqq$ will be denoted by $\mathfrak{H}(G)$. This paper deals with some relations between the structure of G and properties of the poset $\mathfrak{H}(G)$.

In Section 1 some basic facts are treated. They concern maximal and minimal elements of $\mathfrak{H}(G)$ in connection with the Fitting subgroup of G, a "colouring" of $\mathfrak{H}(G)$ with the primes dividing $|G|$, and the poset of classes of chief factors belonging to factor groups and direct products. In Section 2 the influence of the partial ordering on pieces of a chief series is investigated. In particular the structure of monotonic pieces of p-chief factors is clarified. In Section 3 the well known concept of a p-series, due to P. HALL and G. HIGMAN [1], is generalized to that of a $\mathfrak{P}$-series of G, where $\mathfrak{P}$ denotes any subset of $\mathfrak{H}(G)$. We are concerned with a comparison between the length $l(\mathfrak{P})$ of $\mathfrak{P}$ and the length $l_{\mathfrak{P}}(G)$ of the $\mathfrak{P}$-series of G. In general $l(\mathfrak{P}) \leqq l_{\mathfrak{P}}(G)$ holds, but in important cases we have equality here. In the concluding Section 4

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separating normal subgroups are considered. A normal subgroup N of G is called separating, if no chief factor of G between G and N is G-isomorphic to one below N. This is a generalization of the notion of a normal Hall subgroup. Let $\mathfrak{H}_G(N)$ denote the set of those elements of $\mathfrak{H}(G)$ which are represented by chief factors occurring below N. Then $N \mapsto \mathfrak{H}_G(N)$ is an isomorphism of the lattice $\mathfrak{N}_{sep}(G)$ of all separating normal subgroups of G into the lattice Low $(\mathfrak{H}(G))$ of all lower segments of the power set Pow $(\mathfrak{H}(G))$ of $\mathfrak{H}(G)$. We characterize those groups, for which this isomorphism is onto.

Notation. All groups considered are finite; G always denotes a solvable group; $H \leqq G$, $H < G$ means: H is a subgroup, proper subgroup of G, respectively; if, in addition, H is normal in G, we write $\trianglelefteqq$, $\triangleleft$, respectively; analogously $\subseteqq$, $\subset$ denotes inclusion, proper inclusion for sets; $\langle M \rangle :=$ subgroup generated by the subset M of a group; $C_G(M/N) := \{g \mid x^{-1}g^{-1}gx \in N$ for each $x \in M\}$ for normal subgroups M, N of G with $M \geqq N$ is the centralizer of M/N in G; if $U \subseteqq C_G(M/N)$, then M/N is said to be U-central; M/N is central in G means M/N is G-central; a chief factor M/N is said to be situated above or below a normal subgroup K of G, if $M > N \geqq K$ or $K \geqq M > N$, respectively; $[a, b] := a^{-1}b^{-1}ab$; $[H, K] := \langle [a, b] \mid a \in H, b \in K \rangle$; $G' := [G, G]$; p, q denote primes; $F(G) :=$ Fitting subgroup of G, i.e. the maximal nilpotent normal subgroup of G; $F_p(G) := O_{p'p}(G)$, i.e. the maximal p-nilpotent normal subgroup of G; Soc $G :=$ socle of G, i.e. the product of all minimal normal subgroups of G; $A \propto B :=$ semi-direct product of the groups A and B, where B is normal; $l(M) :=$ length of the poset M, i.e. the maximum of the lengths of all chains in M; Pow $M :=$ power set of the set M; $\mathfrak{N}(G) :=$ lattice of all normal subgroups of G. The rest of the notation is introduced in the text in so far as it is not standard (see also [2]).

1. The partially ordered set $\mathfrak{H}(G)$

Firstly we will show that the relation $\geqq$ for the classes of chief factors defined in the introduction is indeed a partial ordering.

1.1. Lemma. *The relation $\geqq$ for the classes of chief factors of a solvable group is reflexive, transitive and antisymmetric.*

Proof. Reflexivity is clear. If $[M_1/N_1] > [M_2/N_2]$ and $[M_2/N_2] > [M_3/N_3]$, then we may assume that $M_1 > N_1 \geqq C_G(M_2/N_2)$ and $M_2 > N_2 \geqq C_G(M_3/N_3)$. But $C_G(M_2/N_2) \geqq M_2$, so $M_1 > N_1 \geqq C_G(M_3/N_3)$ and transitivity is proved. In order to prove antisymmetry assume $[M_1/N_1] > [M_2/N_2]$ and $[M_2/N_2] > [M_1/N_1]$ hold simultaneously. We use the transitivity of $>$, already proved above, and obtain

$[M_1/N_1] > [M_1/N_1]$. This cannot be valid because all chief factors of G, isomorphic to M_1/N_1, are below $C_G(M_1/N_1)$.

The poset of the classes of G-isomorphic chief factors of G will be denoted by $\mathfrak{H}(G)$.

1.2. Examples. (1) If G is a group of prime power order, then $\mathfrak{H}(G)$ has only one element and vice versa.

(2) If G is nilpotent with r different primes in its order, then $\mathfrak{H}(G)$ is an antichain (i.e. the elements of $\mathfrak{H}(G)$ are pairwise incomparable) with r elements and vice versa.

(3) If G is such that $C_G(M/N)=M$ for each chief factor M/N of G, then $\mathfrak{H}(G)$ is a chain, and if in a chief series of G no two chief factors are G-isomorphic, the converse is also true.

(4) Let G be the dihedral group of order $2n$, where the number r of different odd primes in n is not zero. Then $\mathfrak{H}(G)$ has $r+1$ elements, namely a unique maximal one covering the remaining elements, which are minimal.

Obviously $[M_1/N_1] \geqq [M_2/N_2]$ yields $C_G(M_1/N_1) \geqq C_G(M_2/N_2)$. Hence $F(G)$, which is known to be the intersection of the centralizers $C_G(M/N)$ where M/N ranges over all chief factors of G, satisfies

$$F(G) = \bigcap_{\substack{[M/N] \text{ minimal} \\ \text{in } \mathfrak{H}(G)}} C_G(M/N).$$

The maximal and the minimal members of $\mathfrak{H}(G)$ are characterized in the following

1.3. Proposition. (1) *The maximal elements of* $\mathfrak{H}(G)$ *are exactly those represented by the central chief factors of* G.

(2) *The minimal elements of* $\mathfrak{H}(G)$ *are exactly those having no representative above* $F(G)$.

Proof. (1) is trivial.

(2) Assume $[M/N]$ is not a minimal element of $\mathfrak{H}(G)$. Then there are chief factors M_1/N_1, M_2/N_2 of G with $M/N \cong_G M_1/N_1$, $M_1 > N_1 \geqq C_G(M_2/N_2)$. Because of $C_G(M_2/N_2) \geqq F(G)$ it follows that there is a chief factor above $F(G)$, G-isomorphic to M/N. Let, conversely, M/N be a chief factor of G above $F(G)$. Assume $[M/N]$ is minimal in $\mathfrak{H}(G)$. Then for each chief factor M_1/N_1 of G there is no chief factor above $C_G(M_1/N_1)$ which is G-isomorphic to M/N. By the isomorphism theorem it follows that no chief factor G-isomorphic to M/N occurs outside the intersection of all centralizers $C_G(M_1/N_1)$, where M_1/N_1 runs through the chief factors of G. Since this intersection equals $F(G)$, we have a contradiction.

The elements of $\mathfrak{H}(G)$ can be "coloured" by the primes: $[M/N]$ is called a p-element of $\mathfrak{H}(G)$, and will be marked in the graph of $\mathfrak{H}(G)$ by p, if M/N is a p-group. In $\mathfrak{H}(G)$ all p-elements form a subposet $\mathfrak{H}_p(G)$. For a given set π of primes a π-element of $\mathfrak{H}(G)$ is a p-element with any prime $p\in\pi$ and $\mathfrak{H}_\pi(G)$ denotes the subposet of $\mathfrak{H}(G)$ consisting of all π-elements. Obviously $\mathfrak{H}_\pi(G)=\bigcup_{p\in\pi}\mathfrak{H}_p(G)$.

An arbitrary finite group H is said to be π-nilpotent, if it has a normal π'-subgroup N such that H/N is a nilpotent π-group. In any finite group G all π-nilpotent normal subgroups generate a normal subgroup which again is π-nilpotent. This normal subgroup will be designated by $F_\pi(G)$. Clearly, it generalizes the notion of the greatest p-nilpotent normal subgroup $F_p(G)$ of G. We note that

$$F_p(G)=\bigcap_{\substack{[M/N]\text{ minimal}\\ \text{in } \mathfrak{H}_p(G)}} C_G(M/N),$$

$$F_\pi(G)=\bigcap_{\substack{[M/N]\text{ minimal}\\ \text{in } \mathfrak{H}_\pi(G)}} C_G(M/N).$$

Similarly to Proposition 1.3 we have

1.4. Proposition. (1) *The maximal elements of* $\mathfrak{H}_p(G)$ *are exactly those represented by the* $O^{p'}(G)$*-central p-chief factors of* G.

(2) *The minimal elements of* $\mathfrak{H}_p(G)$ *are those elements of* $\mathfrak{H}_p(G)$, *which have no representative above* $F_p(G)$.

1.5. Proposition. (1) *The maximal elements of* $\mathfrak{H}_\pi(G)$ *are exactly those represented by the* $O^{\pi'}(G)$*-central* π*-chief factors of* G.

(2) *The minimal elements of* $\mathfrak{H}_\pi(G)$ *are those elements of* $\mathfrak{H}_\pi(G)$, *which have no representative above* $F_\pi(G)$.

The results stated in Proposition 1.5 will be generalized in Lemma 3.5.

The poset belonging to a direct product can be established from those of the factors in a simple manner.

1.6. Proposition. *Let* $G=G_1\times G_2$. *Then* $\mathfrak{H}(G)$ *arises from* $\mathfrak{H}(G_1)$ *and* $\mathfrak{H}(G_2)$ *by identifying the maximal elements of* $\mathfrak{H}(G_1)$ *and* $\mathfrak{H}(G_2)$ *which are marked with the same prime.*

Proof. The chief factors of G_1 and G_2 can be considered in a natural way as chief factors of G. Let M_1/N_1 be a chief factor of G_1, M_2/N_2 one of G_2 such that $[M_1/N_1]$ and $[M_2/N_2]$ are comparable in $\mathfrak{H}(G)$. If $[M_1/N_1]>[M_2/N_2]$, then there is a chief factor M_1^*/N_1^* of G with

$$M_1^*>N_1^*\geqq C_G(M_2/N_2),\quad M_1^*/N_1^*\cong_G M_1/N_1.$$

Because of $C_G(M_2/N_2)\cong G_1$ we see that M_1/N_1 is G-isomorphic to a chief factor of G_2. Hence M_1/N_1 is centralized by G_1, and since obviously M_1/N_1 is centralized by G_2, we see that M_1/N_1 is both a central chief factor of G_1 and G-isomorphic to a central chief factor of G_2. An analogous consideration works for M_2/N_2 if $[M_1/N_1]\prec\prec[M_2/N_2]$. If $[M_1/N_1]=[M_2/N_2]$ in $\mathfrak{H}(G)$, then clearly M_i/N_i is central in G_i $(i=1, 2)$ and further M_1/N_1, M_2/N_2 are isomorphic. Conversely, central chief factors of G_1 and G_2 are G-isomorphic if they are isomorphic.

Proposition 1.6 shows that $\mathfrak{H}(G_1\times G_2)$ decomposes into disjoint subposets $\mathfrak{H}(G_1)$, $\mathfrak{H}(G_2)$ if G_1, G_2 have coprime orders. However, $\mathfrak{H}(G)$ can decompose in this way even if G does not decompose directly. An example is the group $SL(2, 3)$ (see 1.13 (1) (b)).

We often have to consider the appearance or non-appearance of certain chief factors of G between given normal subgroups of G. For brevity we introduce the following notation: For normal subgroups M, N of G with $M\succ N$ let $\mathfrak{H}_G(M/N)$ denote the subset of those elements of $\mathfrak{H}(G)$, which have a representative between M and N; further, let $\mathfrak{K}_G(M/N)$ denote the subset of those elements of $\mathfrak{H}(G)$, which do not have representatives above M or below N. We put $\mathfrak{H}_G(M/N)=\mathfrak{K}_G(M/N)=\emptyset$, when $M=N$. Furthermore, $\mathfrak{H}_G(M/1)=:\mathfrak{H}_G(M)$, $\mathfrak{K}_G(M/1)=:\mathfrak{K}(M)$. Obviously,

$$\mathfrak{K}_G(M/N)\subseteqq\mathfrak{H}_G(M/N),\quad \mathfrak{K}_G(M)\subseteqq\mathfrak{H}_G(M).$$

If, for a given subset $\mathfrak{P}\subseteqq\mathfrak{H}(G)$ and normal subgroups M, N of G, $\mathfrak{H}_G(M/N)\subseteqq\mathfrak{P}$ holds, then M/N is said to be a $\mathfrak{P}$-factor of G (or M a normal $\mathfrak{P}$-subgroup in case $N=1$).

If $N\lhd G$ then each chief factor of G lying above N can in a natural way be considered as a chief factor of G/N and vice versa.

1.7. Lemma. (1) *Let $K\lhd G$ and let M/N run through the non-equivalent chief factors of G occurring above K. Then $[M/N]\mapsto[(M/K)/(N/K)]$ is a bijection of $\mathfrak{H}_G(G/K)$ onto $\mathfrak{H}(G/K)$ preserving the partial ordering.*

(2) *$\mathfrak{H}(G/F(G))$ arises from $\mathfrak{H}(G)$ by deleting the minimal elements of $\mathfrak{H}(G)$.*

Proof. (1) is obvious.

(2) Choose $K=F(G)$ in (1) and note that according to Proposition 1.3 (2) $\mathfrak{H}_G(G/K)$ consists of all non-minimal elements of $\mathfrak{H}(G)$.

1.8. Lemma. *$[M_1/N_1]$ is an upper neighbour of $[M/N]$ in $\mathfrak{H}(G)$ iff it has a representative between F and $C_G(M/N)$ but none above F; here F is such that $F/C_G(M/N)$ is the Fitting subgroup of $G/C_G(M/N)$.*

Proof. $[M_1/N_1]\succ[M/N]$ holds iff M_1/N_1 has a G-isomorphic copy between G and $C_G(M/N)=:K$, and by Lemma 1.7 (1) all those $[M_1/N_1]$ form a subposet of

$\mathfrak{H}(G)$ isomorphic to $\mathfrak{H}(G/K)$. $[M_1/N_1]$ is an upper neighbour of $[M/N]$ iff $[(M_1/K)/(N_1/K)]$ is a minimal element of G/K and by Lemma 1.3 (2) this happens iff there are no chief factors above F/K which are G/K-isomorphic to $(M_1/K)/(N_1/K)$, i.e. iff there is no G-chief factor above F which is G-isomorphic to M_1/N_1.

A condition for the colouring of $\mathfrak{H}(G)$ is in

1.9. Lemma. (1) *Let M_1/N_1 and M/N be chief factors of G such that M_1/N_1 is between F and $C_G(M/N)$, where F is as in Lemma* 1.8. *Then $|M_1/N_1|$ and $|M/N|$ are relatively prime.*

(2) *In $\mathfrak{H}(G)$ different maximal elements as well as neighbouring elements bear different primes.*

Proof. (1) Let M/N be a p-group, say. Then G/K with $K:=C_G(M/N)$ induces on M/N à faithful irreducible representation over $GF(p)$. Therefore it has no normal p-subgroup $\neq 1$ ([2], p. 485, Satz 5.17). Hence $p \nmid |F: K|$, which yields $p \nmid |M_1/N_1|$.

(2) Since the maximal elements of $\mathfrak{H}(G)$ are the classes of central chief factors, they produce the 1-representations of G. Thus they afford equivalent representations of G, when they have the same prime order. The statement on neighbouring elements comes from (1) in view of Lemma 1.8.

An immediate consequence of the isomorphism theorems is

1.10. Lemma. *Let N_1, N_2 be normal subgroups of G. Then*

(1) $\mathfrak{H}_G(N_1N_2) = \mathfrak{H}_G(N_1) \cup \mathfrak{H}_G(N_2)$.

(2) $\mathfrak{K}_G(N_1 \cap N_2) = \mathfrak{K}_G(N_1) \cap \mathfrak{K}_G(N_2)$.

(3) $\mathfrak{H}_G(G/N_1 \cap N_2) = \mathfrak{H}_G(G/N_1) \cup \mathfrak{H}_G(G/N_2)$.

(4) $\mathfrak{K}_G(G/N_1N_2) = \mathfrak{K}_G(G/N_1) \cap \mathfrak{K}_G(G/N_2)$.

1.11. Corollary. *Suppose $\mathfrak{H}^*$ and $\mathfrak{K}^*$ are subsets of $\mathfrak{H}(G)$. Then the set*

$$\{N \mid N \trianglelefteq G, \mathfrak{H}_G(N) \subseteqq \mathfrak{H}^*, \mathfrak{K}_G(N) \supseteqq \mathfrak{K}^*\}, \tag{1.1}$$

if nonvoid, is a lattice, which is an interval of $\mathfrak{N}(G)$.

This enables us to introduce the following normal subgroups. For $\mathfrak{H}^* = \mathfrak{H}(G)$ and arbitrary $\mathfrak{K}^*$ let $N_{\min}(\mathfrak{K}^*)$ be the minimal element of (1.1), and for arbitrary $\mathfrak{H}^*$ and $\mathfrak{K}^* = \emptyset$ let $N_{\max}(\mathfrak{H}^*)$ be the maximal element of (1.1). In particular let $\mathfrak{K}^*$ consists of all minimal elements of $\mathfrak{H}(G)$. Then $N_{\min}(\mathfrak{K}^*) =: N_{\min}(G)$ is a characteristic subgroup of G. On the other hand let $\mathfrak{H}^*$ consist of all non-maximal elements of $\mathfrak{H}(G)$. Then again $N_{\max}(\mathfrak{H}^*) =: N_{\max}(G)$ is a characteristic subgroup of G. $N_{\min}(G)$ is by definition the least normal subgroup of G such that the corresponding factor group has no representative of a minimal element of $\mathfrak{H}(G)$. This characterization from above is accompanied by one from below: namely $N_{\min}(G)$ is also the greatest normal subgroup of G such that each chief factor $N_{\min}(G)/M$ of G represents a minimal element of $\mathfrak{H}(G)$. By Proposition 1.3 (2) $N_{\min}(G) \leqq F(G)$. $N_{\max}(G)$ is by definition the greatest normal subgroup of G which does not include a central chief factor of G. On the other hand, $N_{\max}(G)$ is the least normal subgroup with the property that the corresponding factor group has only central minimal normal subgroups. Obviously, $N_{\max}(G)$ is contained in the nilpotent residual, i.e. the coradical of G with respect to nilpotence. The subgroups $N_{\min}(G)$ and $N_{\max}(G)$ seem not to have been considered yet.

The well-known inclusion $C_G(F(G)) \leqq F(G)$ can be sharpened to

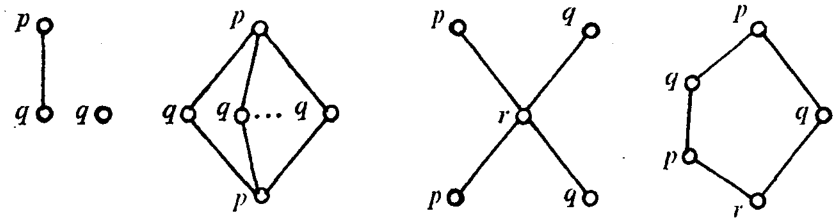
1.12. Proposition. *If N is a normal subgroup of G with $N_{\min}(G) \leqq N \leqq F(G)$, then $C_G(N) \leqq F(G)$.*

Proof. Let $M_1/N_1, \ldots, M_k/N_k$ be representatives below $N_{\min}(G)$ of all minimal elements of $\mathfrak{H}(G)$. If $x \in C_G(N)$, then x centralizes each M_i/N_i $(i=1, \ldots, k)$ and hence $x \in \bigcap_{i=1}^{k} C_G(M_i/N_i) = F(G)$.

We conclude this section with several examples. The following remarks can sometimes be helpful for the construction of a group G with $\mathfrak{H}(G)$ isomorphic to a given prime-coloured poset: If H is a solvable group, which possesses a faithful irreducible representation ∂ over a field $GF(p)$ with representation module M, then the poset $\mathfrak{H}(G)$ of $G = H \propto_\partial M$ arises from $\mathfrak{H}(H)$ by adding a new "minimal" element $[M]$ marked by p; that is to say, $\mathfrak{H}(G) = \mathfrak{H}(H) \dot{\cup} [M]$ and $[M]$ as a p-element is covered by all minimal elements of $\mathfrak{H}(H)$. This construction can be generalized by taking pairwise inequivalent faithful irreducible representation modules $M_1, M_2, \ldots, M_k$ of H over arbitrary prime fields. Then for the group $G = H \propto (M_1 \times M_2 \times \ldots \times M_k)$ we get $\mathfrak{H}(G)$ from $\mathfrak{H}(H)$ by adding k new minimal elements $[M_1], [M_2], \ldots, [M_k]$, each of which is covered by every minimal element of $\mathfrak{H}(H)$ and should be marked by the characteristic of the corresponding ground field. These considerations can be modified for the case where the faithful irreducible representations are replaced by irreducible representations with given kernels.

A sufficient condition for a solvable group H to possess a faithful irreducible representation is the existence of an irreducible representation of $\operatorname{Soc} H$, which has a kernel not containing a normal subgroup of G besides 1 (see also [3]). Here the ground field can be arbitrary. This condition is satisfied, if the characteristic of the ground field does not divide $|\operatorname{Soc} H|$ and $\operatorname{Soc} H$ is the direct product of minimal normal subgroups of G no two of which are G-isomorphic.

1.13. Examples. The following posets, coloured with different primes p, q, r are realized by the groups G mentioned below.



(1) (2) (3) (4)

(1) Here two sorts of groups are constructed.

(a) $G=H\times\langle b\rangle$, where H is a minimal non-abelian group of order pq^n with normal Sylow q-subgroup and b has order q.

(b) $G=\langle a\rangle\propto H$, where a has order p and H is an extra-special q-group such that $\langle a\rangle$ acts irreducibly on H/H'. An example in case $p=3$, $q=2$ is the group $G=SL(2,3)=\langle a\rangle\propto H$ with $\operatorname{ord} a=3$ and H the quaternion group. Further groups of this kind of order pq^{2n+1} do exist if $n\geqq 1$, q is odd, and $p \mid q^{2n}-1$, $p\nmid q^i-1$ for $0<i<2n$ (see [6], pp. 14—15).

(2) We start with the cyclic group $\langle a\rangle$ of order p^n and assign to each i with $1\leqq i\leqq n$ an irreducible representation of $\langle a\rangle$ over $GF(q)$ $(q\neq p)$ with kernel $\langle a^{p^i}\rangle$ and corresponding representation module M_i. These representations give rise to a semi-direct product $H=\langle a\rangle\propto(M_1\times M_2\times\ldots\times M_n)$ in which $M_1M_2\ldots M_n$ is the socle. Let $1\neq b_i\in M_i$ for $1\leqq i\leqq n$. Then there is an irreducible representation of the socle over $GF(p)$, which maps $\langle b_1b_2\ldots b_n\rangle$ faithfully. Since its kernel does not contain a normal subgroup of H, there exists a faithful irreducible representation of H over $GF(p)$ with representation module M, say. It produces a semi-direct product $G=H\propto M$, where H acts on M according to this representation. Now $\mathfrak{H}(G)$ has the desired form.

(3) Here we start with the cyclic group $\langle a\rangle$ of order pq and represent it faithfully and irreducibly over $GF(r)$ with representation module M. The associated semi-direct product $H=\langle a\rangle\propto M$ has a faithful irreducible representation over $GF(p)$ as well as over $GF(q)$. The corresponding representation modules M_1, M_2 give rise to a semi-direct product $G=H\propto(M_1\times M_2)$ for which $\mathfrak{H}(G)$ is as desired.

(4) Let $H=\langle a\rangle\propto(M_1\times M_2)$ be the group defined in (2) above with $n=2$. We represent it irreducibly over $GF(p)$ such that M_2 is the kernel. Denote the representation module with M_3 and form the corresponding semi-direct product $K=H\propto M_3$. Obviously $\operatorname{Soc} K=M_2\times M_3$. Hence there is a faithful irreducible representation module M_4 of K over $GF(r)$ and the corresponding semi-direct product $G=K\propto M_4$ has $\mathfrak{H}(G)$ as desired.

2. Monotonic pieces of chief series

2.1. Definition. We will call a piece

$$N_0 > N_1 > \ldots > N_k \tag{2.1}$$

of a chief series of G *monotonic*, if

$$[N_0/N_1] \geqq [N_1/N_2] \geqq \ldots \geqq [N_{k-1}/N_k] \tag{2.2}$$

in the poset $\mathfrak{H}(G)$. If $>$ holds everywhere instead of $\geqq$, then the piece is called *strongly monotonic.*

2.2. Proposition. *G has a strongly monotonic chief series iff G is abnilpotent.*

Recall that a solvable group G is said to be abnilpotent, if $C_G(M/N)=M$ for each chief factor M/N of G (see [7]).

Proof. If G is abnilpotent then there is a chief series (2.1) with $N_0=G$ and $N_k=1$ such that $C_G(N_{i-1}/N_i)=N_{i-1}$ for $i=1, \ldots, k$. This shows the chain to be strongly monotonic. Conversely, let the chief series above be strongly monotonic. Assume there is an index i with $C_G(N_{i-1}/N_i)>N_{i-1}>N_i$. Then in $\mathfrak{H}(G)$ there are less than $i-1$ elements greater than $[N_{i-1}/N_i]$, while otherwise the property of the chain requires that at least $i-1$ members of $\mathfrak{H}(G)$ are greater than $[N_{i-1}/N_i]$. This contradiction yields $C_G(N_{i-1}/N_i)=N_{i-1}$ for each $i=1, \ldots, k$, and therefore G is abnilpotent.

2.3. Proposition. *If G has a monotonic chief series then $\mathfrak{H}(G)$ is a chain.*

Proof. Let (2.1) be a monotonic chief series of G with $N_0=G$, $N_k=1$. When in (2.2) among equal members all but one are deleted, then $\mathfrak{H}(G)$ is seen to be a chain.

The converse of Proposition 2.3 is not true in general. For instance $GL(2, 3)= = S_3 \propto Q$ with Q a quaternion group is a counter-example.

The following proposition gives an insight into the structure of a piece of a chief series in the case when all factor groups are p-groups.

2.4. Proposition. *Let $N_0>N_1>\ldots>N_k$ be a strongly monotonic piece of a chief series of G such that N_0/N_k is a p-group. Then the following hold.*

(1) *N_0/N_k is elementary abelian and $|N_{i-1}: N_i|<|N_i: N_{i+1}|$ for $i=1, \ldots, k-1$.*

(2) *There is a chain*

$$C_G(N_{k-1}/N_k) = M_1 \geqq M_2 \geqq \ldots \geqq M_k = C_G(N_0/N_k)$$

of normal subgroups M_i of G such that $[M_i, N_j] \leqq N_{i+j}$ for any i, j (here $N_l := N_k$ for $l \geqq k$) and each factor group M_i/M_{i+1} is abelian with an exponent dividing p.

(3) *If N_0/N_k, as a G-group, is completely reducible, then $C_G(N_{i-1}/N_i)=C_G(N_0/N_i)$ for $i=1, \ldots, k$.*

Proof. (1) To see that the indices $|N_{i-1}\colon N_i|$ increase recall that the p-rank of an irreducible linear group over a field of characteristic p is less than its degree ([5], p. 56, Satz 12). Since N_{i-1}/N_i appears, up to equivalence, above $C_G(N_i/N_{i+1})$, we always have

$$(2.3) \qquad |N_{i-1}\colon N_i| < |N_i\colon N_{i+1}|$$

for $i=1, \ldots, k-1$.

In proving the rest of assertion (1) we proceed by induction on k. If $k=1$, there is nothing to prove. Suppose $k>1$ and that the assertion is valid for pieces of chief series which are shorter than the given one. Assume N_0/N_k is non-abelian. Then $N_{k-1}/N_k=(N_0/N_k)'\leqq Z(N_0/N_k)$. Since all $[N_{i-1}/N_i]$ form in $\mathfrak{H}(G)$ a descending chain, we have

$$(2.4) \qquad C_G(N_0/N_1) > C_G(N_1/N_2) > \ldots > C_G(N_{k-1}/N_k).$$

Imagining the representation of G on N_0/N_{k-1} and having in mind that by (2.4) $C_G(N_{k-2}/N_{k-1})$ centralizes all N_{i-1}/N_i for $i=1, \ldots, k-1$, we find that

$$C_G(N_{k-2}/N_{k-1})/C_G(N_0/N_{k-1})$$

is a p-group. If $C_G(N_{k-1}/N_k)\geqq C_G(N_0/N_{k-1})$, then

$$C_G(N_{k-2}/N_{k-1})/C_G(N_{k-1}/N_k)$$

is a non-trivial normal p-subgroup in the group $G/C_G(N_{k-1}/N_k)$. The latter group has a faithful irreducible representation on N_{k-1}/N_k. This is impossible, and hence there is an $a\in C_G(N_0/N_{k-1})\setminus C_G(N_{k-1}/N_k)$. When acting on N_0/N_k the element a multiplies each element of N_0/N_k with an element of the center of this group, however, it does not fix the commutator subgroup elementwise. This cannot happen. Thus we conclude that N_0/N_k is commutative. By the induction hypothesis N_1/N_k is elementary abelian and so is N_0/N_1. Assume that N_0/N_k is not elementary abelian. Then $\Omega_1(N_0/N_k)=N_1/N_k$ and consequently $|\mho_1(N_0/N_k)|=|N_0\colon N_1|$. Since $\mho_1(N_0/N_k)\leqq N_1/N_k$, the group N_1/N_k has a G-invariant subgroup, the order $|N_0\colon N_1|$ of which is by (2.3) less than the order of each G-chief factor between N_1 and N_k, a contradiction.

(2) For $i=1, \ldots, k$ we define

$$M_i := \{g \mid g\in G,\ [g, N_j] \leqq N_{i+j} \text{ for } j = 0, \ldots, k-1\}.$$

Then $M_i \trianglelefteqq G$ and $M_1\geqq M_2\geqq \ldots \geqq M_k$, as can easily be checked. Obviously, $M_1\leqq C_G(N_{k-1}/N_k)$, and by (2.4) $C_G(N_{k-1}/N_k)\leqq M_1$. On the other hand, $M_k\leqq \leqq C_G(N_c/N_k)\leqq M_k$ is clear. By the definition of M_i we have $[M_i, N_j]\leqq N_{i+j}$. As to the commutativity and the exponent of M_i/M_{i+1}, we notice that M_i centralizes N_j/N_{i+j} as well as N_{i+j}/N_{i+j+1}. Therefore in view of (1) $M_i/C_{M_i}(N_j/N_{i+j+1})$ is abelian with exponent dividing p. But $C_{M_i}(N_j/N_{i+j+1})=M_{i+1}$ holds.

(3) If N_0/N_k, as a G-group, is completely reducible, then it is a direct product of subgroups which are admissible with respect to G and G-isomorphic to N_0/N_1, $N_1/N_2, \ldots, N_{k-1}/N_k$. It follows by (2.4) that

$$C_G(N_0/N_i) = \bigcap_{j=1}^{i} C_G(N_{j-1}/N_j) = C_G(N_{i-1}/N_i).$$

The group N_0/N_k mentioned in Proposition 2.4 (3) can be completely reducible or not. Both cases do happen as will be demonstrated in the following

2.5. Example. We start with the group A of order pq^r, where $r = \operatorname{ord} q \bmod p$ and $|A'| = q^r$. It has a faithful irreducible representation ∂ of degree n, say, over $GF(p)$. Let B be the group of all matrices

$$\begin{pmatrix} 1 & \alpha_1 \; \alpha_2 \ldots \alpha_n \\ 0 & \\ \vdots & \partial(a) \\ 0 & \end{pmatrix}, \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in GF(p), \quad a \in A. \tag{2.5}$$

Obviously $|B| = pq^r p^n$. B has the set of all matrices with $a=1$ as a normal subgroup N of order p^n; namely it is the kernel of the homomorphism mapping the matrix (2.5) onto the element $a \in A$. We can write $B = A_1 \propto N$ with $A_1 \cong A$. Let N_0 denote a module of dimension $n+1$ over $GF(p)$, on which B acts according to (2.5), and define G to be the appropriate semi-direct product $G := B \propto N_0$. Thus N_0 appears as a normal subgroup of order p^{n+1} of G and contains a normal subgroup N_1 of order p^n of G, on which B acts via ∂. Now G has the chief series

$$G > A_1' N N_0 > N N_0 > N_0 > N_1 > N_2 = 1.$$

We look at the piece $N_0 > N_1 > N_2 = 1$. Since B is represented faithfully on N_0, we have $C_G(N_0) = C_G(N_0/N_2) = N_0$. Further, N_1 is centralized by N_0 and by those elements of B, for which $a=1$ holds in (2.5), i.e. by the elements of N. Hence $C_G(N_1) = C_G(N_1/N_2) = NN_0$, and we obtain $C_G(N_1/N_2) > C_G(N_0/N_2)$. Obviously, N_0/N_1 is a central chief factor of G isomorphic to $G/A_1'NN_0$. Because of $G > A_1'NN_0 > > C_G(N_1/N_2)$ we have $[N_0/N_1] > [N_1/N_2]$. So the piece $N_0 > N_1 > N_2 = 1$ is monotonic. By Proposition 2.4 (3) N_0/N_2 does not decompose as a G-group.

To get an example with decomposing factor group N_0/N_2 we can proceed in a similar way. However we take in (2.5) only those matrices with $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. Then $N=1$ and the chief series now is

$$G > A_1' N_0 > N_0 > N_1 > N_2 = 1.$$

We have $C_G(N_1/N_2) = C_G(N_0/N_2) = N_0$ and $G/A_1'N_0 \cong_G N_0/N_1$, implying that $[N_0/N_1] > [N_1/N_2]$. Here N_0/N_2 decomposes into two minimal normal subgroups of G, as it is immediately seen by the shape of (2.5) in view of the vanishing α_i.

3. $\mathfrak{P}$-series

One of the central notions in the theory of solvable groups is that of a p-series introduced by P. Hall and G. Higman [1]. Using the poset $\mathfrak{H}(G)$ we can define a more general concept, namely that of a $\mathfrak{P}$-series, where $\mathfrak{P}$ is an arbitrary subset of $\mathfrak{H}(G)$. As in the classical theory, we are interested in the connections between the members of the $\mathfrak{P}$-series and the intersections of the centralizers of chief factors representing elements of $\mathfrak{P}$. Furthermore, the length $l(\mathfrak{P})$ of $\mathfrak{P}$ as a poset is related to the length of the $\mathfrak{P}$-series, and finally Sylow-tower-like theorems are formulated by means of the ordering in $\mathfrak{H}(G)$.

3.1. Definition. Let $\mathfrak{P} \subseteqq \mathfrak{H}(G)$ and put $\mathfrak{P}' := \mathfrak{H}(G) \setminus \mathfrak{P}$. The *upper $\mathfrak{P}$-series* of G is defined as

$$(3.1) \qquad 1 = P_0 \leqq Q_0 < P_1 \leqq Q_1 < \ldots < P_l \leqq Q_l = G,$$

where Q_i is the greatest normal subgroup of G containing P_i such that $\mathfrak{H}_G(Q_i/P_i) \subseteqq \mathfrak{P}'$ while P_{i+1} is the greatest normal subgroup containing Q_i such that P_{i+1}/Q_i is nilpotent and $\mathfrak{H}_G(P_{i+1}/Q_i) \subseteqq \mathfrak{P}$; this works upwards inductively when starting with $P_0 = 1$. The number l in (3.1) is called the $\mathfrak{P}$-length of G and will be denoted by $l_{\mathfrak{P}}(G)$.

As was already mentioned, the chain (3.1) coincides with the upper p-series if $\mathfrak{P} = \mathfrak{H}_p(G)$ and it is exactly the upper nilpotent series if $\mathfrak{P} = \mathfrak{H}(G)$. Hence in these cases $l_{\mathfrak{P}}(G)$ is the p-length or the nilpotent length of G, respectively. In case $\mathfrak{P} = \mathfrak{H}_\pi(G)$ the chain (3.1) will be called the upper π-chain and the corresponding length $l_{\mathfrak{P}}(G)$ the π-length of G, denoted by $l_\pi(G)$. The members P_i, Q_i of the chain (3.1) are normal, but need not be characteristic in G. P_1 is called the greatest $\mathfrak{P}$-nilpotent normal subgroup of G and is denoted by $F_{\mathfrak{P}}(G)$.

The upper $\mathfrak{P}$-length is in a certain sense minimal. This is shown in

3.2. Lemma. *Let $\mathfrak{P} \subseteqq \mathfrak{H}(G)$, $\mathfrak{P}' := \mathfrak{H}(G) \setminus \mathfrak{P}$ and let*

$$1 = P_0^* \leqq Q_0^* \leqq P_1^* \leqq Q_1^* \leqq \ldots \leqq P_k^* \leqq Q_k^* = G$$

be a chain of normal subgroups of G such that $\mathfrak{H}_G(Q_i^*/P_i^*) \subseteqq \mathfrak{P}'$ $(i=0, \ldots, k)$ and P_{i+1}^*/Q_i^* is nilpotent with $\mathfrak{H}_G(P_{i+1}^*/Q_i^*) \subseteqq \mathfrak{P}$ $(i=0, \ldots, k-1)$. Then, for the members P_i, Q_i of the chain (3.1) *we have*

$$P_i^* \leqq P_i, \quad Q_i^* \leqq Q_i \quad \textit{for all } i.$$

In particular, $l_{\mathfrak{P}}(G) \leqq k$.

Proof. $P_0^* \leqq P_0$ holds trivially. Let $P_i^* \leqq P_i$ for a certain i. Then $Q_i^* \leqq Q_i$ can be proved in the following way. Since

$$Q_i^* P_i / P_i \cong_G Q_i^* / Q_i^* \cap P_i \quad \text{and} \quad P_i^* \leqq Q_i^* \cap P_i,$$

the factor group $Q_i^* P_i/P_i$ includes only chief factors of G representing elements of $\mathfrak{P}'$. So $Q_i^* P_i/P_i \leqq Q_i/P_i$ and hence $Q_i^* \leqq Q_i$ holds. In a similar manner we get $P_{i+1}^* \leqq P_{i+1}$. The assertion now follows by induction.

3.3. Corollary. *For* $\mathfrak{P} \subseteqq \mathfrak{H}(G)$ *let*

$$[M_1/N_1] < [M_2/N_2] < \ldots < [M_r/N_r]$$

be a chain in $\mathfrak{P}$. *For each* $i=1, \ldots, r$ *let* $j(i)$ *be the maximal* j *with* $[M_i/N_i] \in \mathfrak{H}_G(P_j/Q_{j-1})$, *where the meaning of* P_j, Q_{j-1} *is as in* (3.1). *Then*

(1) $1 \leqq j(1) < j(2) < \ldots < j(r) \leqq l_{\mathfrak{P}}(G)$.

(2) $l(\mathfrak{P}) \leqq l_{\mathfrak{P}}(G)$.

Proof. (1) Since $P_{j(i)}/Q_{j(i)-1}$ is nilpotent, $P_{j(i)} \leqq C_G(M_i/N_i)$. This yields, in view of $[M_i/N_i] < [M_{i+1}/N_{i+1}]$, that there occurs a chief factor above $P_{j(i)}$ which is G-isomorphic to M_{i+1}/N_{i+1}, hence $j(i) < j(i+1)$.

(2) is a consequence of (1).

3.4. Remark. It can happen that $l(\mathfrak{P}) < l_{\mathfrak{P}}(G)$. Take for instance $G := GL(2,3) = S_3 \propto H$, where S_3 denotes the symmetric group of degree 3 and H denotes the quaternion group, and let $\mathfrak{P}$ consist of the unique class of central chief factors of G. Then $l(\mathfrak{P})=1$ and the $\mathfrak{P}$-series of G has $P_0=1$, $Q_0=1$, $P_1=Z(G)$, $Q_1=A_3 \propto H$, $P_2=G$, $Q_2=G$ with A_3 the alternating group of degree 3 in S_3. Hence $l_{\mathfrak{P}}(G)=2$.

Next we are looking for conditions, which guarantee that $l(\mathfrak{P})=l_{\mathfrak{P}}(G)$ holds instead of $l(\mathfrak{P}) \leqq l_{\mathfrak{P}}(G)$.

For a given subset $\mathfrak{P} \subseteqq \mathfrak{H}(G)$ we define

$$O_{\mathfrak{P}}(G) := \langle N \mid N \trianglelefteq G, \mathfrak{H}_G(N) \subseteqq \mathfrak{P} \rangle,$$

$$O^{\mathfrak{P}}(G) := \bigcap_{\substack{N \trianglelefteq G \\ \mathfrak{H}_G(G/N) \subseteqq \mathfrak{P}}} N,$$

$$C_G(\mathfrak{P}) := \bigcap_{[M/N] \in \mathfrak{P}} C_G(M/N).$$

If $\mathfrak{P}=\emptyset$, then by the definition of $\mathfrak{H}_G$ and $\mathfrak{K}_G$ we have $O_{\mathfrak{P}}(G)=1$, $O^{\mathfrak{P}}(G)=G$; additionally we define in this case $C_G(\mathfrak{P}) := G$.

The groups $O_{\mathfrak{P}}(G)$, $O^{\mathfrak{P}}(G)$ are generalizations of the characteristic subgroups $O_\pi(G)$, $O^\pi(G)$ for a set of primes π; they appear with $\mathfrak{P}=\mathfrak{H}_\pi(G)$. They are also connected with the groups $N_{\min}$ and $N_{\max}$ introduced in Section 1, namely

$$N_{\min}(\mathfrak{P}) = O^{\mathfrak{P}'}(G), \quad N_{\max}(\mathfrak{P}) = O_{\mathfrak{P}}(G).$$

As a common generalization of Propositions 1.3, 1.4 and 1.5 we obtain

3.5. Lemma. *An element* $[M/N]$ *of* $\mathfrak{H}(G)$ *is not surpassed by any element of* $\mathfrak{P}$ *iff* M/N *is* $O^{\mathfrak{P}'}(G)$*-central, and it does not surpass any element of* $\mathfrak{P}$ *iff* $[M/N]\in\mathfrak{K}_G(C_G(\mathfrak{P}))$.

Proof. $[M/N]$ is not surpassed by any element of $\mathfrak{P}$ iff $\mathfrak{H}_G(G/C_G(M/N))\subseteq\mathfrak{P}'$ and this happens iff $C_G(M/N)\supseteq O^{\mathfrak{P}'}(G)$. On the other hand, $[M/N]$ does not surpass any element of $\mathfrak{P}$ iff for each $[M_1/N_1]\in\mathfrak{P}$ we have $[M/N]\notin\mathfrak{H}_G(G/C_G(M_1/N_1))$, which means $[M/N]\in\mathfrak{K}_G(C_G(M_1/N_1))$. By Lemma 1.10 (2) this is equivalent to $[M/N]\in\mathfrak{K}_G(C_G(\mathfrak{P}))$.

We define an ascending centralizer chain with respect to a given non-empty subset $\mathfrak{P}\subseteq\mathfrak{H}(G)$. Let

$$\emptyset=\mathfrak{P}_0\subset\mathfrak{P}_1\subset\ldots\subset\mathfrak{P}_l=\mathfrak{P}$$

be the unique chain of subsets of $\mathfrak{P}$ such that $\mathfrak{P}_i\setminus\mathfrak{P}_{i-1}$ consists of the minimal elements of $\mathfrak{P}\setminus\mathfrak{P}_{i-1}$ $(i=1,\ldots,l)$. Obviously, l coincides with the length $l(\mathfrak{P})$ of $\mathfrak{P}$. The equality $C_G(\mathfrak{P}\setminus\mathfrak{P}_{i-1})=C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})$ is easily seen, too.

3.6. Lemma. *We have, for* $\emptyset\neq\mathfrak{P}\subseteq\mathfrak{H}(G)$,

(1) $1<C_G(\mathfrak{P}_1\setminus\mathfrak{P}_0)<C_G(\mathfrak{P}_2\setminus\mathfrak{P}_1)<\ldots<C_G(\mathfrak{P}_l\setminus\mathfrak{P}_{l-1})\leqq G$.
Here the chief factors of G *between* $C_G(\mathfrak{P}_l\setminus\mathfrak{P}_{l-1})$ *and* G *represent only elements of* $\mathfrak{P}'$, *i.e.,* $\mathfrak{H}_G(G/C_G(\mathfrak{P}_l\setminus\mathfrak{P}_{l-1}))\subseteq\mathfrak{P}'$.

(2) $P_i\leqq C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})$ for $i=1,\ldots,l$, *where the* P_i *come from the chain in* (3.1) *related to* $\mathfrak{P}$.

Proof. (1) Since $C_G(\mathfrak{P}_1\setminus\mathfrak{P}_0)$ contains the Fitting subgroup of G, it differs from 1. Next we prove that $C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})\neq C_G(\mathfrak{P}_{i+1}\setminus\mathfrak{P}_i)$. By Lemma 3.5 $\mathfrak{P}_i=\mathfrak{P}\cap\mathfrak{K}_G(C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1}))$. Here on the left hand side with different values of i always different sets $\mathfrak{P}_i$ occur. Hence the corresponding groups $C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})$ must be different. If $C_G(\mathfrak{P}_l\setminus\mathfrak{P}_{l-1})\leqq N<M\leqq G$, where M/N is a chief factor of G, then $[M/N]\notin\mathfrak{K}_G(C_G(\mathfrak{P}_l\setminus\mathfrak{P}_{l-1}))\supseteq\mathfrak{P}_l=\mathfrak{P}$. Hence we obtain $[M/N]\notin\mathfrak{P}$.

(2) Assume $P_i\not\leqq C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})$ for some $i\geqq 1$ and take i minimal. There is an $[M/N]\in\mathfrak{P}_i\setminus\mathfrak{P}_{i-1}$ which is not centralized by P_i. Therefore $[M/N]\notin\mathfrak{H}_G(G/P_i)$. Because of the nilpotence of P_i/Q_{i-1}, we have $[M/N]\notin\mathfrak{H}_G(P_i/Q_{i-1})$. Consequently $i\geqq 2$ and $[M/N]\in\mathfrak{K}_G(Q_{i-1})$. Since $[M/N]\in\mathfrak{P}$, it follows that $[M/N]\in\mathfrak{K}_G(P_{i-1})$. By the minimality of i, $P_{i-1}\leqq C_G(\mathfrak{P}_{i-1}\setminus\mathfrak{P}_{i-2})$. Now we get that $[M/N]$ does not surpass an element of $\mathfrak{P}_{i-1}\setminus\mathfrak{P}_{i-2}$, a contradiction.

Lemma 3.6 (2) yields, for $i=1$, the following

3.7. Corollary. $F_{\mathfrak{P}}(G)\leqq C_G(\mathfrak{P})$ *for* $\emptyset\neq\mathfrak{P}\subseteq\mathfrak{H}(G)$.

3.8. Corollary. $l(\mathfrak{P})=l_{\mathfrak{P}}(G)$ *holds if in Lemma* 3.6 (2) $P_l=C_G(\mathfrak{P}\setminus\mathfrak{P}_{l-1})$ *with* $l=l(\mathfrak{P})$.

Proof. By Lemma 3.6 (1) the assumption yields that $\mathfrak{H}(G/P_l)\subseteq\mathfrak{P}'$ and therefore $Q_l=G$. Hence, in view of Lemma 3.2, $l_{\mathfrak{P}}(G)\leqq l(\mathfrak{P})$, and by Corollary 3.3 (2), $l(\mathfrak{P})=l_{\mathfrak{P}}(G)$.

3.9. Remark. In Lemma 3.6 (2) equality as well as inequality can happen. The case of equality is discussed in the next theorem. Inequality holds for the group $G:=SL(2,3)=\langle a\rangle\propto H$ which was already mentioned in Example 1.13 (1) (b), if we take $\mathfrak{P}:=\{[Z(H)/1]\}$. Then $P_1=F_{\mathfrak{P}}(G)=Z(H)$, $C_G(\mathfrak{P}_1\setminus\mathfrak{P}_0)=C_G(\mathfrak{P})=G$. This shows also that the converse of Corollary 3.8 need not be true, since $l(\mathfrak{P})=l_{\mathfrak{P}}(G)$.

3.10. Theorem. *Suppose in Lemma 3.6 (2) we have $C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})>P_i$ for a certain index $i\geqq 1$, and choose i minimal with this property. Let T/P_i be the product of all minimal normal subgroups of G/P_i contained in $C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})/P_i$. Then T/Q_{i-1} is nilpotent and $\mathfrak{H}_G(T/P_i)\subseteq\mathfrak{P}'$, $\mathfrak{H}_G(P_i/Q_{i-1})\subseteq\mathfrak{P}\setminus\mathfrak{P}_{i-1}$. Furthermore, to each element $[L_1/M_1]\in\mathfrak{H}_G(T/P_i)$ there exists a piece $L>M>N$ of a chief series of G such that the following conditions* (1) *through* (4) *are satisfied:*

(1) $T\geqq L,\quad P_i\geqq M>N\geqq Q_{i-1}$.

(2) $[L/M]=[L_1/M_1]$.

(3) *L/N has prime power order.*

(4) *L/N is indecomposable as a G-group.*

Proof. By the definition of i we have $C_G(\mathfrak{P}_j\setminus\mathfrak{P}_{j-1})=P_j$ for $j=1,\ldots,i-1$. Hence, in view of Lemma 3.5, $\mathfrak{P}_j\setminus\mathfrak{P}_{j-1}\subseteq\mathfrak{K}_G(P_j)$ for $j=1,\ldots,i-1$ and so $\mathfrak{P}_{i-1}\subseteq\mathfrak{K}_G(P_{i-1})$. This yields $\mathfrak{H}_G(P_i/Q_{i-1})\subseteq\mathfrak{P}\setminus\mathfrak{P}_{i-1}$. Hence T centralizes all G-chief factors between P_i and Q_{i-1}. Those between T and P_i are centralized by T, too. It follows that T/Q_{i-1} centralizes all of its own chief factors, and therefore T/Q_{i-1} is nilpotent. By the definition of P_i no chief factor of G between T and P_i can belong to $\mathfrak{P}$. So $\mathfrak{H}_G(T/P_i)\subseteq\mathfrak{P}'$.

Assume now that to a given L_1/M_1 no piece $L>M>N$ with the properties (1) through (4) exists. We can find a piece $K_1>F_1>F_2$ of a chief series of G with

$$T\geqq K_1>F_1>F_2\geqq Q_{i-1},\quad F_1=P_i,\quad L_1/M_1\cong_G K_1/F_1.$$

Since $[F_1/F_2]\in\mathfrak{P}\setminus\mathfrak{P}_{i-1}$, K_1 centralizes F_1/F_2.

If $|K_1/F_1|$ is coprime to $|F_1/F_2|$, then by Schur—Zassenhaus' Theorem there exists a normal subgroup K_2 of G such that $K_1>K_2>F_2$ and $K_1/F_1\cong_G K_2/F_2$; $F_1/F_2\cong_G K_1/K_2$.

If K_1/F_2 has prime power order, then by assumption K_1/F_2 decomposes as a G-group completely. Hence, again there is a normal subgroup K_2 of G with $K_1>K_2>F_2$, $K_1/F_1\cong_G K_2/F_2$, $F_1/F_2\cong_G K_1/K_2$.

If $F_2>Q_{i-1}$, take a new chief factor F_2/F_3 of G with $F_3\geqq Q_{i-1}$. Considering the chain $K_2>F_2>F_3$ we can find, as above, a normal subgroup K_3 of G with $K_2>$

$>K_3>F_3$, $K_2/F_2\cong_G K_3/F_3$, $F_2/F_3\cong_G K_2/K_3$. Continuing in this way we find chains

$$K_1>K_2>\ldots>K_t,\quad P_i=F_1>F_2>\ldots>F_t=Q_{i-1}$$

such that always $K_{i-1}/F_{i-1}\cong_G K_i/F_i$. Hence $K_1/F_1\cong_G K_t/F_t=K_t/Q_{i-1}$. However, $[K_1/F_1]=[L_1/M_1]\notin\mathfrak{P}$, contradicting the construction of Q_{i-1}.

An application of Theorem 3.10 is

3.11. Theorem. *With the notation of Lemma* 3.6 *we have* $C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})=P_i$ *for* $i=1,\ldots,l(\mathfrak{P})$, *and hence also* $l(\mathfrak{P})=l_{\mathfrak{P}}(G)$, *in each of the following cases:*
(1) $\mathfrak{P}=\mathfrak{H}_p(G)$.
(2) $\mathfrak{P}=\mathfrak{H}_\pi(G)$.
(3) $\mathfrak{P}=\mathfrak{H}(G)$.
(4) $\mathfrak{P}$ *is such that* $C_G(M_1/N_1)=M_1$ *holds for every* $[M_1/N_1]\in\mathfrak{P}$.
(5) $\mathfrak{P}$ *is arbitrary and in each chief series of* G *neighbouring factors have coprime orders.*

Proof. (1) and (3) are special cases of (2). Conditions (2), (4), (5) lead to $C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})=P_i$ for $i=1,\ldots,l(\mathfrak{P})$. Assume not, and choose i minimal with $C_G(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})>P_i$. We utilize Theorem 3.10. The existence of a series $L>M>N$ described there is impossible, because L/N has prime power order and
in case (2) $[L/M]\in\mathfrak{H}_{\pi'}(G)$ and $[M/N]\in\mathfrak{H}_\pi(G)$ hold simultaneously;
in case (4) $M/N\leqq Z(L/N)$, whence $L\leqq C_G(M/N)=M$;
in case (5) $|L/M|$ and $|M/N|$ are relatively prime.

As a consequence of Theorem 3.11 (1)—(3) we have

3.12. Corollary. *The π-length, p-length, and nilpotent length of* G *coincides with the length of* $\mathfrak{H}_\pi(G)$, $\mathfrak{H}_p(G)$, $\mathfrak{H}(G)$, *respectively.*

In particular, we have

3.13. Corollary. $l_\pi(G)=1$ *iff* $\mathfrak{H}_\pi(G)$ *is an antichain,* $l_p(G)=1$ *iff* $\mathfrak{H}_p(G)$ *is an antichain.* G *is nilpotent iff* $\mathfrak{H}(G)$ *is an antichain.*

We close this section with a Sylow-tower-like property.

3.14. Theorem. *Let* $\pi(G)=\pi_1\dot\cup\pi_2\dot\cup\ldots\dot\cup\pi_r$ *be a partition of* $\pi(G)$ *into non-empty subsets* π_i. *Suppose an element of* $\mathfrak{H}_{\pi_i}(G)$ *surpasses an element of* $\mathfrak{H}_{\pi_j}(G)$ *only if* $i<j$. *Then* G *has a chain*

$$G=N_0>N_1>\ldots>N_r=1$$

of normal subgroups in which N_{i-1}/N_i *is a nilpotent* π_i*-group, and vice versa.*

Proof. Let $\mathfrak{H}(G)$ satisfy the assumption. We start with $N_0 := G$ and suppose that $N_0, N_1, \ldots, N_{i-1}$ have been constructed for a certain $i \geqq 1$ as normal Hall subgroups of G such that N_{j-1}/N_j is a nilpotent π_j-group for $j = 1, \ldots, i-1$. We consider the $\mathfrak{P}$-series of G with $\mathfrak{P} = \mathfrak{H}_{\pi_i}(G)$. Since $\mathfrak{P}$ is an antichain, $l(\mathfrak{P}) = 1$ and hence by Theorem 3.11 the $\mathfrak{P}$-series is shaped like this:

$$1 = P_0 \leqq Q_0 < P_1 \leqq Q_1 = G.$$

Here P_1/Q_0 is a π_i-group, while Q_0/P_0 and Q_1/P_1 are π_i'-groups. Because of the assumption on the partial ordering in $\mathfrak{H}(G)$, for each π_i-chief factor M/N the factor group $G/C_G(M/N)$ is a $(\pi_1 \dot{\cup} \ldots \dot{\cup} \pi_{i-1})$-group. By Theorem 3.11

$$\bigcap_{[M/N] \in \mathfrak{P}} C_G(M/N) = C_G(\mathfrak{P}) = C_G(\mathfrak{P}_1 \setminus \mathfrak{P}_0) = P_1,$$

so that G/P_1 is a $(\pi_1 \dot{\cup} \ldots \dot{\cup} \pi_{i-1})$-group and therefore $N_{i-1} \leqq P_1$. Put $N_i := N_{i-1} \cap Q_0$. Then $N_{i-1}/N_i \cong_G N_{i-1}Q_0/Q_0 \leqq P_1/Q_0$, so N_{i-1}/N_i is a nilpotent π_i-group. Further, N_i being a subgroup of Q_0 is a π_i'-group. Because of $N_i \leqq N_j$ for $j = 1, \ldots, i-1$, N_i is also a π_j'-group for $j = 1, \ldots, i-1$. Hence N_i is a Hall subgroup. The converse is trivial.

3.15. Corollary. *G has a Sylow tower belonging to the ordering $p_1 \succ p_2 \succ \ldots \ldots \succ p_r$ of $\pi(G)$ iff $\mathfrak{H}(G)$ is such that $[M_1/N_1] > [M_2/N_2]$, M_1/N_1 a p_i-group and M_2/N_2 a p_j-group, imply $p_i \succ p_j$.*

4. Separating normal subgroups

4.1. Definition. A normal subgroup N of a solvable group G is said to be *separating*, if no chief factor of G has a G-isomorphic copy above N as well as below N.

For instance each normal Hall subgroup is separating. For a normal subgroup N of G the property of being separating is characterized by each of the equations $\mathfrak{H}_G(N) = \mathfrak{K}_G(N)$, $\mathfrak{H}_G(G/N) = \mathfrak{K}_G(G/N)$, $\mathfrak{H}(G) = \mathfrak{H}_G(G/N) \dot{\cup} \mathfrak{H}_G(N)$. All separating normal subgroups of G constitute a sublattice $\mathfrak{N}_{sep}(G)$ of the lattice $\mathfrak{N}(G)$ of all normal subgroups of G.

Recall the notion of a lower segment in a poset M. It is defined as a subset of M, which contains with each of its elements x all elements of M which are surpassed by x. Upper segments are defined analogously.

4.2. Proposition. *If $N \in \mathfrak{N}_{sep}(G)$ then $\mathfrak{H}_G(N)$ is a lower and consequently $\mathfrak{H}_G(G/N)$ is an upper segment of $\mathfrak{H}(G)$.*

Proof. Let $[K/L]\notin\mathfrak{H}_G(N)$ and assume $K\leq LN$. Then K/L is a chief factor between LN and L, which is G-isomorphic to one between N and $N\cap L$. But then $[K/L]\in\mathfrak{H}_G(N)$. Hence $K\nleq LN$ and so $LN\cap K=L$. This yields that K/L and LN/L centralize mutually in G/L and so $N\leq C_G(K/L)$, i.e. each M_1/N_1 with $[M_1/N_1]>$ $>[K/L]$ fulfils $[M_1/N_1]\notin\mathfrak{H}_G(N)$.

All lower segments of $\mathfrak{H}(G)$ form a sublattice of the power set Pow $\mathfrak{H}(G)$; we denote it by Low $\mathfrak{H}(G)$.

4.3. Lemma. *$N\mapsto\mathfrak{H}_G(N)$ $(N\in\mathfrak{N}_{sep}(G))$ is a lattice isomorphism of $\mathfrak{N}_{sep}(G)$ into* Low $\mathfrak{H}(G)$. *In particular, $\mathfrak{N}_{sep}(G)$ is a distributive lattice.*

Proof. Assume there are $N_1, N_2\in\mathfrak{N}_{sep}(G)$ with $N_1\neq N_2$ and $\mathfrak{H}_G(N_1)=\mathfrak{H}_G(N_2)$. Then $N_1\neq 1$ and $N_2\neq 1$ and $N_1N_2>N_2>1$. Any G-chief factor between N_1N_2 and N_2 represents an element of $\mathfrak{H}_G(N_1)$. Because of $\mathfrak{H}_G(N_1)=\mathfrak{H}_G(N_2)$ it must also have a copy below N_2, which is impossible. Thus the mapping $N\mapsto\mathfrak{H}_G(N)$ under consideration is injective. From Lemma 1.10 (1) (2) we obtain that it preserves union and intersection. Now, since $\mathfrak{N}_{sep}(G)$ is isomorphic to a sublattice of the distributive lattice Pow $\mathfrak{H}(G)$, it is distributive, too.

4.4. Corollary. *If $\mathfrak{H}(G)$ is a chain, then $\mathfrak{N}_{sep}(G)$ is a chain.*

We will characterize those groups G for which the mapping in Lemma 4.3 is onto Low $\mathfrak{H}(G)$. A key role is played by the homogeneous socle of a group introduced below.

4.5. Definition. A normal subgroup N of G is said to be *homogeneous*, if all chief factors of G below N are G-isomorphic. The product of all homogeneous normal subgroups is called the *homogeneous socle* of G and will be denoted by Hos G.

It is easy to show that Hos G is the direct product of maximal homogeneous normal subgroups of G. Since each homogeneous normal subgroup has prime power order and each minimal normal subgroup is homogeneous, we have

$$F(G)\geqq \operatorname{Hos} G\geqq \operatorname{Soc} G.$$

Each normal subgroup of G contained in Hos G is a direct product of homogeneous normal subgroups.

The notion of homogeneous factor groups and the homogeneous cosocle can be defined in an analogous manner; but we do not need them in the present paper.

4.6. Theorem. *The following statements are equivalent:*

(1) *$\mathfrak{N}_{sep}(G)$ is isomorphic to* Low $\mathfrak{H}(G)$, *the lattice of all lower segments of* Pow $\mathfrak{H}(G)$.

(2) *Whenever* $L>M>N$ *is a piece of a chief series of* G, *then either* L/N *decomposes completely as a G-group or* $[L/M]\geqq[M/N]$ *in* $\mathfrak{H}(G)$.

(3) $N_{\min}(G/N)\leqq \mathrm{Hos}\,(G/N)$ *holds for each normal subgroup* N *of* G. (For the definition of $N_{\min}$ see Section 1.)

Proof. (1)⇒(2). Let $L>M>N$ be a piece of a chief series of G such that L/N as a G-group does not decompose completely and L/M, M/N are not G-isomorphic. Assume $[L/M]\not\geq[M/N]$. Let $\mathfrak{K}$ be the set of all $[M_1/N_1]\in\mathfrak{H}(G)$ with $[M_1/N_1]\leqq[L/M]$. Then $\mathfrak{K}$ is a lower segment of $\mathfrak{H}(G)$ and by assumption there is a separating normal subgroup K of G such that $\mathfrak{H}_G(K)=\mathfrak{K}$. We consider the series

$$(4.1)\qquad KL \underset{1}{\geqq} KN \underset{2}{\geqq} K \underset{3}{\geqq} K\cap L \underset{4}{\geqq} K\cap N,$$

$$(4.2)\qquad KL \underset{3}{\geqq} L \underset{1}{\geqq} KN\cap L \geqq (K\cap L)N \underset{4}{\geqq} N \underset{2}{\geqq} K\cap N,$$

where equally numbered factors are G-isomorphic. This yields the equality $KN\cap L= =(K\cap L)N$. In (4.2) we look at the piece $L\geqq(K\cap L)N\geqq N$ and realize, in addition, that up to G-isomorphy L/M does not occur above K and M/N does not occur below K. We obtain in view of (4.1) that L/M cannot be G-isomorphic to a chief factor of G between L and $(K\cap L)N$; further, M/N cannot be G-isomorphic to one between $(K\cap L)N$ and N. It follows

$$L/M \cong_G (K\cap L)N/N, \quad M/N\cong_G L/(K\cap L)N.$$

This leads to

$$L/N\cong_G (L/M)\times(M/N),$$

i.e. L/N decomposes into two minimal normal G-subgroups, contrary to our assumption.

(2) ⇒ (3). Suppose neighbouring chief factors of G always have the property described in (2). Let $[M_1/N_1]$ be a minimal element of $\mathfrak{H}(G)$ and let M be a normal subgroup of G such that there is no chief factor between G and M which is G-isomorphic to M_1/N_1; moreover let us choose M minimal with this property. We will show that M is homogeneous in G. Assume not, then there is a chief factor M_2/N_2 of G with $M>M_2>N_2$ such that between M and M_2 there are only chief factors which are G-isomorphic to M_1/N_1 and $M_2/N_2\not\cong_G M_1/N_1$. In particular, M/M_2 is a group of prime power order. Let us choose M_2, N_2 such that M/M_2 is as small as possible. If $M\not\leqq C_G(M_2/N_2)$, then $[M_1/N_1]>[M_2/N_2]$, so $[M_1/N_1]$ would not be minimal. Hence $M\leqq C_G(M_2/N_2)$. If the orders of M/M_2 and M_2/N_2 were relatively prime, then $M/N_2\cong_G(M/M_2)\times(M_2/N_2)$ and there would exist a normal subgroup K of G with $M/K\cong_G M_2/N_2$, contradicting the choice of M. Hence M/N_2 has prime power order. There exists L with $M\geqq L>M_2>N_2$ such that L/M_2 and M_2/N_2 are chief factors of G. Since $[L/M_2]=[M_1/N_1]\not\geqq[M_2/N_2]$, by assumption L/N_2 de-

composes completely. But then L/N_2 has a factor group which is G-isomorphic to M_2/N_2, contradicting the choice of M_2 and N_2. This proves that M is homogeneous.

Let $\mathfrak{M}$ be the set of minimal elements of $\mathfrak{H}(G)$. As was shown above, there exists to each element $[M_0/N_0]$ of $\mathfrak{M}$ a homogeneous separating normal subgroup of G involving only chief factors of G which are G-isomorphic to M_0/N_0. The product of these homogeneous normal subgroups belonging to all elements of $\mathfrak{M}$ coincides with $N_{\min}(G)$ and is contained in Hos G. Thus the assertion (3) is proved for $N=1$. Since the assumption (2) is hereditary to factor groups, so is the assertion (3), and we are done.

(3)$\Rightarrow$(1). The assertion that each lower segment $\mathfrak{L}$ of $\mathfrak{H}(G)$ belongs to a separating normal subgroup L of G such that $\mathfrak{H}_G(L)=\mathfrak{L}$ is proved by induction on $|G|$. Suppose $|G|>1$ because in case $|G|=1$ there is nothing to prove. Let $\mathfrak{L}_0$ be the set of minimal elements of $\mathfrak{L}$. Assumption (3) yields that to each element $[M_0/N_0]$ of $\mathfrak{L}_0$ there exists a homogeneous separating normal subgroup of G involving only chief factors which are G-isomorphic to M_0/N_0. Let L_0 be the product of these homogeneous normal subgroups belonging to all elements of $\mathfrak{L}_0$. The factor group G/L_0 has $\mathfrak{L}\setminus\mathfrak{L}_0$ as a lower segment in a natural way. By the induction hypothesis G/L_0 has a separating normal subgroup L/L_0 with $\mathfrak{H}_{G/L_0}(L/L_0)=\mathfrak{L}\setminus\mathfrak{L}_0$. Now L is separating in G and satisfies $\mathfrak{H}_G(L)=\mathfrak{L}$.

We will formulate some conditions, which are sufficient for the properties mentioned in Theorem 4.6.

4.7. Theorem. *The following conditions are equivalent and they imply the properties described in Theorem* 4.6:

(1) *Whenever* $L>M>N$ *is a piece of a chief series of* G *such that* L/N *has prime power order and* $L/M\not\cong_G M/N$, *then* L/N *as a* G-*group decomposes completely.*

(2) $F(G/N)=\text{Hos}\,(G/N)$ *holds for each normal subgroup* N *of* G.

Proof. (1)$\Rightarrow$(2). Assume $F(G)>\text{Hos}\,G$, and let

$$F(G)\geqq L>\text{Hos}\,G\geqq K\geqq 1$$

be a series of normal subgroups of G such that all chief factors between L and K are G-isomorphic to $L/\text{Hos}\,G$ but no chief factor below K is such. Choose K_1 with $L\geqq K_1>K$ where K_1/K is a G-chief factor. By the nilpotency of L and by (1) the chief factor K_1/K can be "permuted" with each one below K. Similarly we can proceed with a chief factor K_2/K_1 below L and so on. Finally we obtain a homogeneous normal subgroup M of G with $M\cong_G L/K$. But then $L=M\times K$ is a product of homogeneous normal subgroups of G which is impossible. Since (1) is hereditary to factor groups, (2) follows.

(2)⇒(1). L/N is nilpotent and therefore contained in $F(G/N) = \mathrm{Hos}\,(G/N)$. But then L/N is a direct product of homogeneous normal subgroups which yields the assertion (1).

Finally we have $N_{\min}(G/N) \leqq F(G/N)$ for each $N \trianglelefteqq G$. Hence condition (2) yields $N_{\min}(G/N) \leqq \mathrm{Hos}\,(G/N)$, and this is condition (3) of Theorem 4.6.

4.8. Proposition. *Let G be a solvable group, which has only non-G-isomorphic chief factors in a chief series. Then $\mathfrak{N}(G) \cong \mathrm{Low}\,\mathfrak{H}(G)$ iff in each factor group of G the Fitting group coincides with the socle.*

Proof. Obviously in each factor group of G the homogeneous socle coincides with the socle; further, $\mathfrak{N}_{\mathrm{sep}}(G) = \mathfrak{N}(G)$.

If $F(G/N) = \mathrm{Soc}\,(G/N)$ for each normal subgroup N of G, then in view of Theorems 4.7 and 4.6 we obtain $\mathfrak{N}(G) \cong \mathrm{Low}\,\mathfrak{H}(G)$.

Conversely let $\mathfrak{N}(G) \cong \mathrm{Low}\,\mathfrak{H}(G)$. We take a piece $L > M > N$ of a chief series of G such that L/N is a prime power group. Then $C_G(M/N) \geqq L$, so that no chief factor G-isomorphic to L/M can appear above $C_G(M/N)$. Hence $[L/M] \not\equiv [M/N]$. By Theorem 4.6 L/N decomposes completely as a G-group. This implies in view of Theorem 4.7 the coincidence of the Fitting group and the socle in each factor group of G.

Remark. Recently P. P. Pálfy has shown in [4] that the property (2) in Lemma 1.9 completely describes the prime-coloured poset of a solvable group; i.e. each finite poset, endowed with primes such that (2) of Lemma 1.9 holds, can be represented as the coloured poset $\mathfrak{H}(G)$ of a suitable finite solvable group G.

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