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Effective constructions of cutsets for finite and infinite ordered sets

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Dedicated to the memory of András P. Huhn

Perhaps the most important result in the theory of ordered sets is the 'chain decomposition theorem' of R. P. DILWORTH [1] which states that *in a finite ordered set the minimum number of maximal chains whose union is the set equals the maximum size of an antichain.* However, this maximal antichain need not meet all of the maximal chains in the ordered set. For instance $\{a, d\}$ is an antichain of maximum size in the ordered set N illustrated in Figure 1, and yet $\{a, d\}$ does not meet the maximal chain $\{c, b\}$.

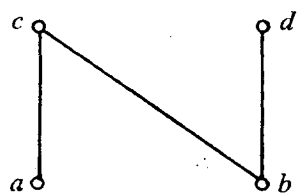

Figure 1

Call a subset K of an ordered set P a *cutset* of P if every maximal chain of P meets K. If K is an antichain then we call it an *antichain cutset* of P. If $K-\{x\}$ is not a cutset for every x in K then we call it a *minimal cutset* of P.

The N illustrated in Figure 1 is the union of the antichain cutsets $\{a, b\}$ and $\{c, d\}$. In contrast the ordered set illustrated in Figure 2 cannot be the union of antichain cutsets at all, since there is no antichain cutset which contains x.

I. RIVAL and N. ZAGUIA [4] have shown that *a finite ordered set is the union of antichain cutsets if and only if it contains no alternating-cover cycle.* For $n\geqq 2$, a

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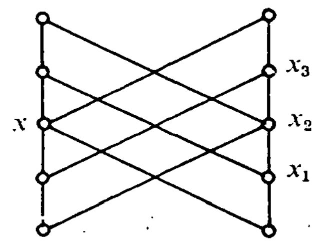


Figure 2

subset $\{x, a_1, c_1, \ldots, a_n, c_n\}$ of an ordered set P is called a *generalized alternating-cover cycle* (for x) provided that

$$c_1 > x > a_n, \quad c_1 > a_1, \quad c_2 > a_2, \ldots, c_n > a_n,$$

$$c_1 > a_n, \quad c_2 > a_1, \ldots, c_{n-1} > a_{n-2}, \quad c_n > a_{n-1}$$

and provided that

$$c_1 \succ a_1, c_2 \succ a_2, \ldots, c_n \succ a_n$$

are covering relations in P itself. If these are the only comparabilities among the elements $\{x, a_1, c_1, \ldots, a_n, c_n\}$, we call this ordered set an *alternating-cover cycle* (see Figure 3). For emphasis we sometimes indicate by 'double lines' in the figures the covering relations in an ordered set. We also say that x is contained in a generalized alternating-cover cycle. Actually I. Rival and N. Zaguia [4] prove this more general result.

Theorem 1. *In a finite ordered set an element is contained in an antichain cutset if and only if it is not contained in a generalized alternating-cover cycle.*

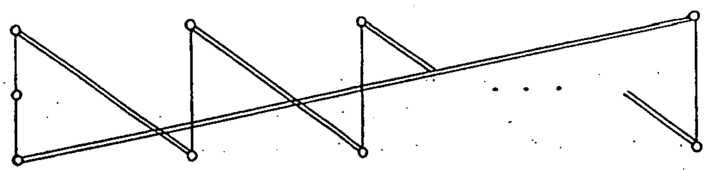


Figure 3

Here, we give an 'efficient' algorithm for the construction of an antichain cutset containing a given element, which is in effect another proof of the same theorem. From the order-theoretical point of view, the first proof in I. Rival and N. Zaguia [4] is certainly shorter and perhaps more elegant. Still, the algorithm for the construction of the antichain cutset implicit in that proof, seems on the surface at least to perform in an 'exponential' number of steps, as a function of the number of elements of the ordered set. Another proof, presented here is algorithmically much better. Indeed, the algorithm for the construction of the antichain cutset, implicit in this proof, performs in a 'polynomial' number of steps as a function of the number of elements of the ordered set.

In the ordered set illustrated in Figure 2, the element x is not contained in any antichain cutset. Still there is a minimal cutset $\{x, x_1, x_2, x_3\}$ which contains x. In a finite ordered set P, every element is contained in a minimal cutset, and thus a finite ordered set P is always the union of minimal cutsets. That is not always the case for infinite ordered sets though. For instance, in $2\times(\omega+1)$ (see Figure 4), there is no minimal cutset which contains x even though, for example every chain in $2\times(\omega+1)$ has a supremum and an infimum.

An ordered set P is *regular* if every nonempty chain C of P has a supremum and an infimum and, whenever $x<\sup C$ (respectively, $x>\inf C$), $x<c$ (respectively $x>c$), for some element c in C. We expect that regular ordered sets can be expressed as the union of minimal cutsets but we are unable to prove that yet. Here is a partial solution.

Theorem 2. *A regular ordered set satisfying a chain condition is the union of minimal cutsets.*

An ordered set is said to satisfy a *chain condition* if it does not contain either an infinite, strictly descending chain $x_1>x_2>\dots$ or it does not contain an infinite, strictly ascending chain $x_1<x_2<\dots$.

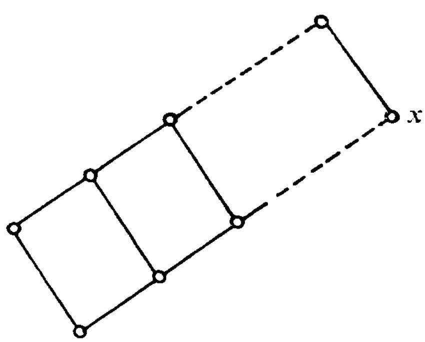


Figure 4

In general, though, a minimal cutset need not be an antichain and, of course, an antichain need not be a (minimal) cutset. I. RIVAL and N. ZAGUIA [4] have shown that, in an ordered set which contains no subset isomorphic to N, every finite, minimal cutset is an antichain. D. HIGGS [3] has extended this result to arbitrary minimal cutsets, and has also proved the converse in the case of finite ordered sets. As a consequence of Theorem 2, we can extend Higgs's result.

Theorem 3. *Let P be a regular ordered set satisfying a chain condition. Then, every minimal cutset in P is an antichain if and only if P contains no subset isomorphic to N.*

D. Higgs [3] was the first to give an example of an ordered set which contains subsets isomorphic to N and in which every minimal cutset is an antichain.

A related question is whether every maximal antichain meets every maximal chain, that is, whether every maximal antichain is a cutset? The ordered set illustrated in Figure 5, has a maximal antichain which is not a cutset.

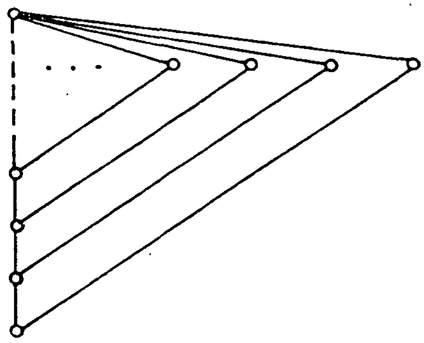
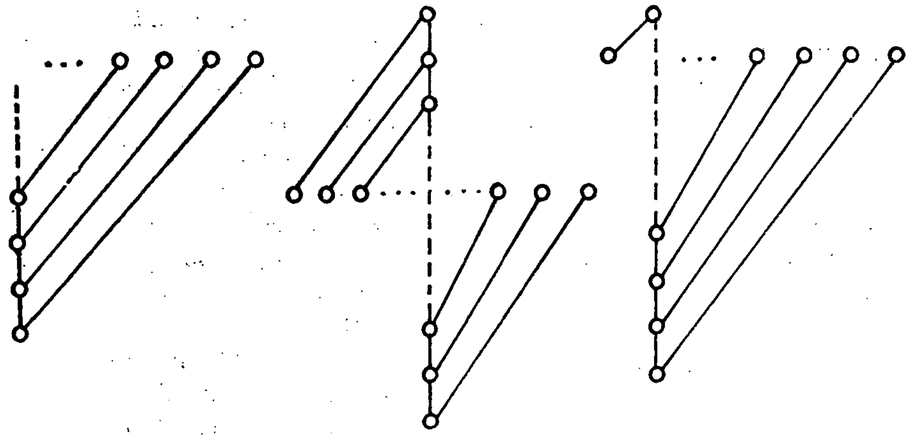


Figure 5

An early result of P. A. Grillet [2] shows that, *for a regular ordered set P, every maximal antichain meets every maximal chain if and only if P contains no subdiagram isomorphic to N.* We extend this result in terms of 'generalized' N's. This gives a characterization of those ordered sets in which every maximal antichain is a cutset.

Let A_1 and A_2 be subsets of an ordered set P. Write $A_1 < A_2$ if, for every $u \in A_1$ and $v \in A_2$, $u < v$. We say that A_2 *covers* A_1 (or A_1 is *covered* by A_2) and write $A_1 \prec A_2$ if $A_1 < A_2$ and there is no x in P such that $A_1 < \{x\} < A_2$. Also, we use the convention that $\emptyset < A$ for every subset A of P. We say that A_1 is *cofinal* for A_2 (respectively *coinitial*) provided that, for every $v \in A_2$, there exists $u \in A_1$ such that $v \leqq u$ (respectively $u \leqq v$). Let C_1, C_2, A_1 and A_2 be subsets of P such that C_1 and C_2 are chains in P and $A_1 \cup A_2$ is an antichain in P. We call the four-tuple (C_1, C_2, A_1, A_2) a *generalized N* provided that $C_1 \prec C_2$ and A_1 is cofinal for C_1 and A_2 is coinitial for C_2. In Figure 6, we illustrate basic examples of generalized N's. An N, too, is a generalized N. Also, in a regular ordered set, it is easy to see that every generalized N must be an N in the diagram itself.

Theorem 4. *In an ordered set every maximal antichain meets every maximal chain if and only if it contains no generalized N as a subdiagram.*



(a) (b) (c)

Figure 6

Proof of Theorem 1. For purposes of the proof it is convenient to speak of cutsets "for" elements. Say that a subset S is a cutset for x if $S\cup\{x\}$ is a cutset and each $s\in S$ is noncomparable with x. We shall prove, by induction on the cardinality of P, that every element has an antichain cutset provided that no element of P is contained in a generalized alternating-cover cycle. To this end let x be an element of P with no antichain cutset in P. Then x cannot be a maximal element of P for then we could choose $A(x)$ to be the set of maximal elements of P distinct from x. Let u be a maximal of P, $u>x$, and put $P'=P-\{u\}$. Notice that no element of P' is contained in a generalized alternating-cover cycle so, by the induction hypothesis, x must have an antichain cutset $A'(x)$ in P'. We may suppose that $A'(x)$ is not an antichain cutset for x in P. Then there is a maximal chain $C(u)$ of P which contains u but no element from $A'(x)\cup\{x\}$. Let u' be the lower cover of u in $C(u)$. According to the induction hypothesis any maximal chain in P' containing $C(u)-\{u\}$ must contain some element from $A'(x)\cup\{x\}$. Therefore, $u'\leqq x$ or $u'\leqq v$ for some v belonging to $A'(x)$. But $u'\not\leqq x$ since $u>x$ and $u'\prec u$; therefore, $u'\leqq v$ for some v in $A'(x)$. Our aim is to construct, starting from $A'(x)$, an antichain cutset for x in P. We cannot hope to use u in an antichain cutset for x. In order to "account" for the maximal chain $C(u)$ we may, however, try to use u', but then we could not use v in an antichain cutset for x. Then we may seek to replace v by other elements, each noncomparable to x and to the members of the "current" cutset for x.

We shall introduce and develop a "two-player game" which we use later to effect the construction of an antichain cutset. Meet our players: CHAIN — the villain— and, ANTICHAIN— our hero—.

The setting for our spectacle is a finite ordered set P which contains no alternating-cover cycle. Let x belong to P and let $A(x)$ be a minimal cutset for x (that is, a cutset for x such that, for each y in $A(x)$, $A(x)-\{y\}$ is not a cutset for x in P). Notice that for each y in $A(x)$ there is a maximal chain $C(y)$ in P such that $C(y)\cap A(x)=\{y\}$. Call such a maximal chain in P *essential* for y in $A(x)$.

A *down game* between CHAIN and ANTICHAIN (in P for x) is played as follows. CHAIN is the first to move: CHAIN selects an element c_1 from $A_1=A(x)$, for which there is another element a_0 in A_1 such that $c_1>a_0$, if one exists. (In effect, CHAIN "uncovers" evidence why A_1 is not an antichain cutset for x.) If CHAIN has no such move then we say that ANTICHAIN *wins the down game* (indeed, this must mean that A_1 is an antichain cutset for x). Otherwise ANTICHAIN responds in this down game by identifying all lower covers of c_1 on essential chains for c_1 in A_1, each of which is not below x itself: call these elements $a_1^1, a_1^2, \ldots$. These elements constitute ANTICHAIN's first move in reply to CHAIN's move c_1. We now "reform" the cutset A_1 by constructing a minimal cutset A_2 for x contained in

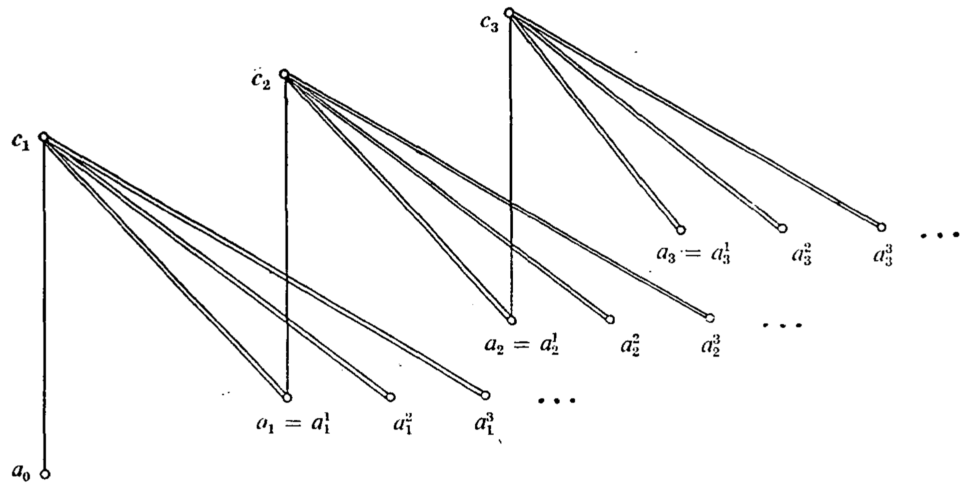
$$\{a_1^1, a_1^2, \ldots\}\cup(A_1-\{c_1\}).$$

Evidently, A_2 consists of two disjoint subsets: the first consists of the sequence $a_1^1, a_1^2, \ldots$ which is an antichain; the second is a subet of $A_1-\{c_1\}$, which together with the sequence of elements constituting ANTICHAIN's move is a minimal cutset for x. Notice that just as CHAIN may not be able to move (if A_1 is already an antichain cutset), it may be that ANTICHAIN cannot respond to CHAIN's move c_1: this would be the case if some lower cover of c_1 on an essential chain for c_1 in A_1 is itself below x, such an element cannot be in a cutset for x. If, then, ANTICHAIN cannot move we say that CHAIN *wins the down game*. If both first moves can be made then the down game continues. CHAIN selects an element c_2 from A_2 such that $c_2>a_1$, where a_1 belongs to $\{a_1^1, a_1^2, \ldots\}$. If CHAIN has no such move then ANTICHAIN wins the down game. Otherwise, ANTICHAIN again responds by selecting all lower covers of c_2 on essential chains for c_2 in A_2, each of which is not below x, say $a_2^1, a_2^2, \ldots$ CHAIN's move c_2. Again the cutset A_2 is altered to construct a minimal cutset A_3 for x contained in

$$\{a_2^1, a_2^2, \ldots\}\cup(A_2-\{c_2\})$$

(see Figure 7). Again the minimal cutset contains the antichain consisting of the elements in ANTICHAIN's second move and, as well, a subset of $A_2-\{c_2\}$. If CHAIN can now move then CHAIN will choose an element c_3 from A_3 such that $c_3>a_2$, where a_2 belongs to $\{a_2^1, a_2^2, \ldots\}$. And so on.

Furthermore, by a sequence $G_1, G_2, \ldots$ of down games we mean that each of the down games G_i begins with a comparability taken from the current cutset at the end of the preceding down game G_{i-1}, for each $i=2, 3, \ldots$.



Construction in the proof of Theorem 1.
Figure 7

We call the cutset A_k, $k=1, 2, \ldots$, the current cutset for x at CHAIN's k^{th} move, and then ANTICHAIN's k^{th} move. We say that ANTICHAIN *wins this down game* if, for some $k \leqq |P|$, CHAIN cannot make a k^{th} move in this down game; otherwise, we say that CHAIN *wins this down game*.

An up game between CHAIN and ANTICHAIN is defined dually. In an up game CHAIN's k^{th} move is to select an element c_k' from the current cutset A_k' such that $c_k' < a_{k-1}'$, where a_{k-1}' is one of the elements

$$a_{k-1}'^{1}, a_{k-1}'^{2}, \ldots$$

chosen by ANTICHAIN in the $k-1^{\text{th}}$ move.

Let G be a down game for x in P and let $c_1 > a_0$ be the first move for CHAIN. We say that the down game G is linked to x provided that there are sequences

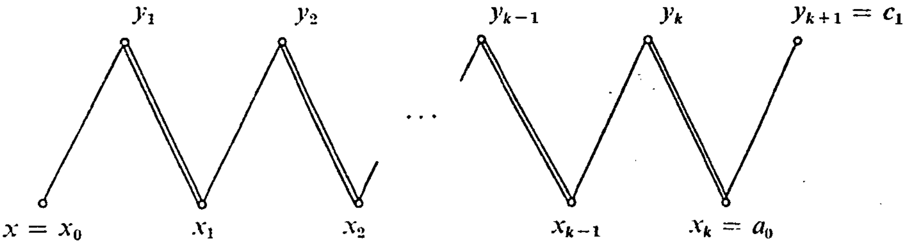
$$x = x_0, x_1, x_2, \ldots, x_k = a_0, \quad y_1, y_2, \ldots, y_k$$

in P such that

$$x_i < y_{i+1} \quad \text{and} \quad x_{i+1} \prec y_{i+1}$$

for each $i=0, 1, 2, \ldots, k-1$ (see Figure 8). The notion of an up game linked to x is defined dually.

We shall now establish two technical lemmas needed for the proof of the Theorem. The first shows that once CHAIN moves in a down game, that move can never be repeated in that down game. Moreover, in any sequence of down games, once CHAIN moves in one of the down games, that move can never be repeated in any



Construction in the proof of Theorem 1.
Figure 8

later move of any later down game. To this end we write $(m, j) < (n, k)$ for integers m, n, j, k provided that, either $m < n$, or else, $m = n$ and $j < k$ (the usual lexicographic order).

Lemma 1. *Let P be a finite ordered set and let x belong to P. Let $G_1, G_2, \ldots$ be a sequence of down games in P for x. Let $A_1^m, A_2^m, \ldots$ be the current cutsets for x in the game G_m. Then*

$$c_j^m \notin A_k^n \quad \textit{whenever} \quad (m, j) < (n, k).$$

Proof of Lemma 1. According to the rules of play, c_j^m does not belong to A_{j+1}^m. Suppose, however, that there are integers m, n, j, k such that $c_j^m \in A_k^n$. Suppose that (k, n) is chosen least with this property in the lexicographic order. This means that c_j^m is an element of the $k-1^{\text{th}}$ move of ANTICHAIN, that is, $c_j^m \prec c_{k-1}^n$. Let C_{k-1}^n denote the essential chain (containing c_j^m) for c_{k-1}^n in A_{k-1}^n and similarly, let C_j^m denote the essential chain (containing a_j^m) for c_j^m in A_j^m. We use these maximal chains C_{k-1}^n and C_j^m to construct another maximal chain C defined by

$$C = (C_{k-1}^n \cap (c_j^m]) \cup (C_j^m \cap [c_j^m))$$

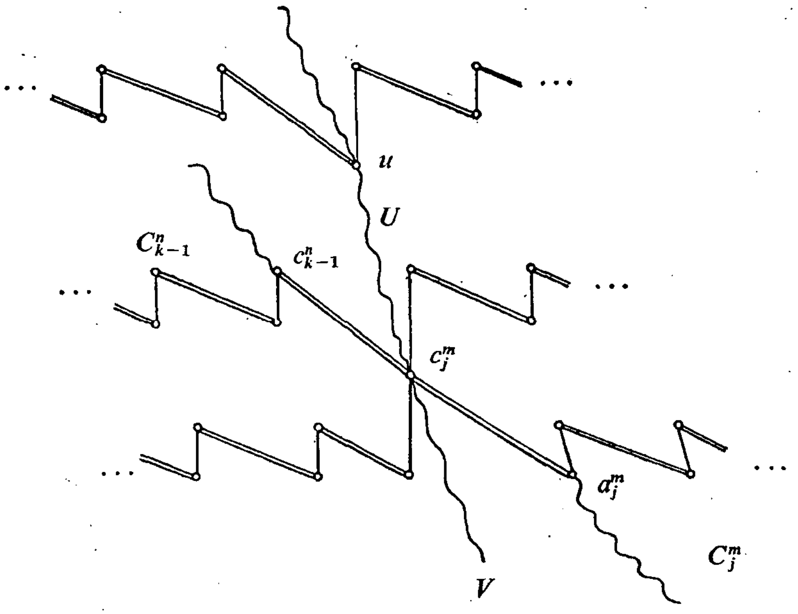
(see Figure 9). (For y in P, $(y] = \{x \in P \mid x \leqq y\}$ and $[y) = \{x \in P \mid x \geqq y\}$.) Let

$$U = C \cap [c_j^m) \quad \text{and} \quad V = C \cap (c_j^m].$$

Suppose the chain U contains an element of A_{k-1}^n. Let $m \leqq p \leqq n$ be the least integer for which there is some i such that A_i^p contains an element of U. Let i, too, be the least integer with this property and choose u a maximal element belonging to $U \cap A_i^p$. Note that u does not belong to A_j^m. Let C_{i-1}^p be the essential chain (containing u) for c_{i-1}^p in A_{i-1}^p. Now construct

$$C' = (C_{i-1}^p \cap (u]) \cup (C_j^m \cap [u)).$$

Evidently A_{i-1}^p cannot contain any element of $C_{i-1}^p \cap (u]$ and, by the maximality of u it cannot contain any element of $C_j^m \cap [u)$, either. That is impossible. It follows that U cannot contain any member of A_{k-1}^n.



Construction in the proof of Lemma 1.
Figure 9

On the other hand, V cannot contain any element of A^n_{k-1}, either. This, in turn, implies that C contains no element of A^n_{k-1}, which is a contradiction.

The next lemma indicates just how the play of games between CHAIN and ANTICHAIN is affected by generalized alternating-cover cycles. Absence of generalized alternating-cover cycles gives ANTICHAIN decided advantage.

Lemma 2. *Let P be a finite ordered set and let x be an element of P which is contained in no generalized alternating-cover cycle. Then* ANTICHAIN *wins every down game (in P for x) linked to x.*

Proof of Lemma 2. Let G be a down game. The first move for CHAIN is an element c_1 for which there is an element a_0 in the current cutset A_1 for x. Suppose that CHAIN wins some down game G. Then, according to Lemma 1, there are (finitely many) distinct elements

$$c_1, c_2, \ldots, c_j$$

(the sequence of CHAIN's moves) and there are elements

$$a_1, a_2, \ldots, a_{j-1}$$

(the sequence of ANTICHAIN's moves) such that CHAIN wins this down game in j moves. Therefore, there must be a lower cover a_j on an essential chain for c_j in A_j such that $a_j < x$. Let

$$x = x_0, x_1, x_2, \ldots, x_k = a_0, \quad y_1, y_2, \ldots, y_{k+1} = c_1$$

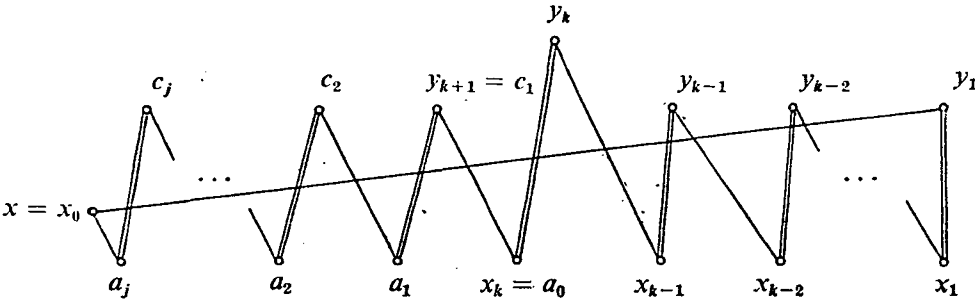
be elements satisfying

$$x_i < y_{i+1} \quad \text{and} \quad x_{i+1} \prec y_{i+1}, \quad i = 0, 1, \ldots, k-1.$$

Then x is contained in the generalized alternating-cover cycle

$$x = x_0, x_1, y_1, x_2, y_2, \ldots, x_k, y_k, a_1, c_1 = y_{k+1}, a_2, c_2, \ldots, a_j, c_j$$

a contradiction (see Figure 10).



Construction in the proof of Lemma 2.
Figure 10

We are now ready to proceed directly with the proof of Theorem 1.

Let P be a finite ordered set and let x be an element of P contained in no generalized alternating-cover cycles. We shall show by induction on the cardinality of P, that x has an antichain cutset. If, for instance, x is a maximal element of P then the remaining maximal elements of P distinct from x constitute an antichain cutset (possibly empty) for x. Let us suppose, then, that x is not a maximal element of P. Let u be a maximal element of P satisfying $u > x$ and put $P' = P - \{u\}$. Of course, P', too, does not contain any generalized alternating-cover cycles for x, so we may apply the induction hypothesis to P' to obtain an antichain cutset A' for x in P. We may suppose that A' is not an antichain cutset for x in P. Then there are maximal chains in P, each containing u and each disjoint from A'. Let u' be the unique lower cover of u on some such maximal chain C. Now $C - \{u\}$ is not a maximal chain in P', for otherwise $C - \{u\}$, whence C itself, contains an element of A'. Then, for every such u' there is some v in A' satisfying $u' < v$.

Since $u > x$, no lower cover of u can lie below x. In fact

$$A' \cup \{u' \mid u' < v \text{ for some } v \text{ in } A'\}$$

is a cutset for x in P. Let A be a minimal cutset for x in P contained in this set. Notice that A contains each of these elements u'. We may suppose that A is not an antichain.

Our aim now is to successively construct new cutsets, at each stage eliminating comparabilities of the form $u'<z$ where $u'\prec u$, $u'<v$ for some v in A' and z belongs to the current cutset. As there are only finitely many such comparabilities and there can be no repetitions this construction must terminate with an antichain cutset.

Here is the first step in the construction. We begin with the comparabilities $u_1'<z_1$ in the current cutset $A=A^{1,0}$. Put $c_1^{1,1}=z_1$ and $a_0^{1,1}=u_1'$. Since $x<u$ and $u\succ u_1'$, $u_1'<z_1$, then according to Lemma 2, ANTICHAIN wins a down game G_1^1 which begins with $a_0^{1,1}<c_1^{1,1}$. Let $A^{1,1}$ be the current cutset at the end of such a down game. Let $a_0^{1,2}\in(A^{1,1}-A^{1,0})$ and $c_1^{1,2}\in A^{1,1}$ satisfy $a_0^{1,2}<c_1^{1,2}$ and let G_2^1 be a down game which begins with $a_0^{1,2}<c_1^{1,2}$. As there are sequences

$$x,\ u_1' = a_0^{1,1}, a_1^{1,1}, a_2^{1,1}, \dots, a_0^{1,2}$$

and

$$u,\ z_1 = c_1^{1,1}, c_2^{1,1}, \dots, c_1^{1,2}$$

G_2^1 is linked to x and so ANTICHAIN wins G_2^1. In general, let $a_0^{1,i}\in A^{1,i}-A^{1,0}$ and $c_1^{1,i}\in A^{1,i}$ satisfy $a_0^{1,i}<c_1^{1,i}$ where $A^{1,i}$ is the current cutset at the end of the down game G_{i-1}^1, and let G_i^1 be a down game which begins with $a_0^{1,i}<c_1^{1,i}$. Again, G_i^1 is linked to x, so ANTICHAIN wins G_i^1. By Lemma 1, this sequence terminates after $m(1)$ such successive games. Let $A^{1,m(1)}$ be the current cutset at the end of this sequence of down games.

We may suppose that $A^{1,m(1)}$ is not an antichain. We show that any comparability $y<z$ in $A^{1,m(1)}$ satisfies $y=u'$ for some $u'\prec u$ and $u'<v$ for some $v\in A'$. By Lemma 1, $z\neq z_1$. If $y\notin A^{1,0}$ then this sequence

$$G_1^1, G_2^1, \dots, G_{m(1)}^1$$

can be extended by a down game $G_{m(1)+1}^1$ which begins with $a_0^{1,m(1)+1}=y<z=c_1^{1,m(1)+1}$. We may suppose therefore that $y\in A^{1,0}\cap A^{1,m(1)}$. Next observe that each element in every move for ANTICHAIN is below some element in $A=A^{1,0}$. To see this we proceed by induction. Evidently, $a_0^{1,0}=u_1'<z_1$ and $z_1\in A=A^{1,0}$. In general, let

$$a_k^{1,i} \prec c_k^{1,i}.$$

If $c_k^{1,i}\in A^{1,0}$ then we are done. Otherwise, $c_k^{1,i}=a_1^{1,j}$ for $(j, 1)<(i, k)$ in the lexicographic order. By induction $a_1^{1,j}$ is below some element of $A^{1,0}$ and so, $a_k^{1,i}$, too, is below some element of $A^{1,0}$. This means that, in particular, there is $t\in A^{1,0}$ satisfying $z\leqq t$ and so $y<t$ in $A^{1,0}$. That in turn, implies that y is a lower cover of u and so y is some u', where $u'<v$ for some $v\in A^{1,0}$.

Let us suppose that we have completed $k-1$ steps in this construction. Let $A^{k,0}=A^{k-1,m(k-1)}$ be the current cutset, let $u'_k<z_k$ be a comparability in $A^{k,0}$ and let G_1^k be a down game beginning with

$$a_0^{k,1} = u'_k < z_k = c_1^{k,1}.$$

Now $x<u$, $u\succ u'_k$ and so G_1^k is linked to x whence ANTICHAIN wins this down game. Let $A^{k,1}$ be the current cutset at the end of G_1^k. Let

$$a_0^{k,2}\in(A^{k,1}-A^{k,0}), \quad c_1^{k,2}\in A^{k,1}$$

satisfy $a_0^{k,2}<c_1^{k,2}$. Again ANTICHAIN wins any down game G_2^k beginning with $a_0^{k,2}<c_1^{k,2}$.

In general, let

$$a_0^{k,i}\in(A^{k,i-1}-A^{k,0}), \quad c_1^{k,i}\in A^{k,i-1}$$

satisfy $a_0^{k,i}<c_1^{k,i}$ where $A^{k,i}$ is the current cutset at the end of the down game G_{i-1}^k. As before the down game G_i^k beginning with $a_0^{k,i}<c_1^{k,i}$ is linked to x, so ANTICHAIN wins this down game. By Lemma 1, this sequence

$$G_1^k, G_2^k, \ldots, G_i^k, \ldots$$

terminates after finitely many such successive down games, $m(k)$ say.

We may suppose that $A^{k,m(k)}$, the current cutset at the end of the down game $G_{m(k)}^k$, is not an antichain. Let $y<z$ in $A^{k,m(k)}$. As the sequence

$$G_1^k, G_2^k, \ldots, G_{m(k)}^k$$

cannot be extended, $y\in(A^{k,0}\cap A^{k,m(k)})$. Again as above, each element in every move for ANTICHAIN in every game G_i^k, is below some element in $A^{k,0}$. Therefore, there is $t\in A^{k,0}$ satisfying $z\leq t$, so $y<t$ in $A^{k,0}$. By induction, y must be some u', where $u'\prec u$ and $u'<v$ for some v in $A^{1,0}$.

By Lemma 1, there can be no repetition of the comparabilities $y<z$ where y is of the form u' with $u'\prec u$ and $u'<v$ for some v in $A^{1,0}$. As there are only finitely many comparabilities of this type the process must end and the current cutset at the end of this construction must be an antichain. This completes the proof.

Implicit in this proof of Theorem 1 is an effective procedure to construct an antichain cutset. We do this, as in the proof, by a sequence of 'moves'. Every move begins with a comparability in P. According to Lemma 1, two different moves always begin with different comparabilities; thus, at most n^2 moves are needed to produce an antichain cutset. It remains, therefore, to prove that a move can be effected in a polynomial (in n) number of steps. This is the outline of a move.

(i) Find a comparability $a<b$ in the current cutset.

(ii) Replace b in the current cutset by all of its lower covers.

(iii) Remove, from among these lower covers those, nonessential to the new cutset.

(iv) Remove any further elements nonessential to the new cutset.

(v) The new minimal cutset is the current cutset for the next move.

The only outstanding item is how to decide effectively whether or not an element is essential, in a cutset.

Let K be a cutset of an ordered set P and let $x \in K$. Then x is *essential* in K if there is a maximal chain C in P such that $C \cap K = \{x\}$. Let $I(x) = \{y \in P \mid \text{either } y < x \text{ or } y > x\}$. Then x is essential in K if and only if $K \cap I(x)$ is not a cutset in $I(x)$. Our problem therefore reduces (is polynomially equivalent to) the following.

Given a subset K in P is there an effective procedure to decide whether or not K is a cutset of P?

If the subset K is an antichain, then, as is well known, K is a cutset in P if and only if P does not contain an $N = \{a < c,\ b < d,\ b \prec c\}$, such that $\{a, d\} \subseteq K$. Obviously this can be decided in a polynomial number of steps too.

If K is not an antichain, we can transform (polynomially) P to an ordered set P' and K to an antichain K' of P' such that K is a cutset of P if and only if K' is a cutset of P'. To see this we consider several cases. Let $x, y \in K$ satisfy $x \prec y$ in K. If $x \prec y$ in P too then we may delete the covering edge $x \prec y$ to obtain P' and K'. Let us suppose that $x \prec z < y$ in P for some z and suppose that there is no $t < z$ with $t \not\geqq x$. In this case we construct P' by only removing the element z and we choose K' in P' to be the same set as K. If, for each $x \prec z < y$, there is $t < z$ with $t \not\geqq x$, then we remove the edge $x \prec z$ again to produce the ordered set P', and K' is the same set as K.

Proof of Theorem 2. The proof consists in showing that every element in P is in some minimal cutset of P. Let $x \in P$ and let C_x be a maximal chain of P such that $x \in C_x$.

We suppose that the ordered set P has no infinite decreasing chains. (Otherwise, if P has no infinite increasing chains then we apply dual arguments.) Let

$$B_1 = \{y \in P - C_x \mid y \succ z \text{ for some } z \text{ in } C_x \text{ and } z < x\}$$

and

$$B_2 = \{y \in P - C_x \mid y \text{ is minimal in } P\}.$$

The subset $A = B_1 \cup B_2 \cup \{x\}$ of P is a cutset. Indeed, let C be a maximal chain in P. If $C \cap B_2 = \emptyset$ then $\inf C_x \in C$. Therefore $C_x \cap C \neq \emptyset$. We set

$$u = \sup \{t \in C_x \cap C \mid t < x\}.$$

Since P is chain complete, $u \in C_x \cap C$. If $u = x$ then $x \in C$, otherwise there is v an upper cover of u such that $v \in C - C_x$. Thus $v \in C \cap B_1$. Also, x is essential in A

since $A \cap C_x = \{x\}$. The ordered set P contains no infinite descending chains, so we can consider a well ordering of

$$B_1 \cup B_2 = \{x_1, x_2, \ldots, x_\alpha, \ldots\}_{\alpha<\lambda}$$

which is an extension of the order on $B_1 \cup B_2$. Thus $x_i < x_j$ in P implies $i<j$.

Now, we define an algorithm which transforms the cutset A to a minimal cutset of P containing x. Let $\{A_1, A_2, \ldots, A_\alpha, \ldots\}_{\alpha \leqq \lambda}$ be a sequence of cutsets of P, defined inductively as follows. If x_1 is essential in A then $A = A_1$. Otherwise $A_1 = A - \{x_1\}$. Assume we have already defined $(A_i)_{i<\alpha}$. If the ordinal α is isolated $\alpha = \beta + 1$, and x_α is essential in A_β then $A_\alpha = A_\beta$. Otherwise $A_\alpha = A_\beta - \{x_\alpha\}$. If the ordinal α is a limit $\alpha = \sup_{\beta<\alpha} \beta$, then $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$.

First of all, we prove by induction on α that A_α is a cutset of P for every $\alpha < \lambda$. Suppose that α is the least ordinal such that A_α is not a cutset of P. Let C be a maximal chain in P such that $C \cap A_\alpha = \emptyset$. If $\alpha = \beta + 1$, then A_β is a cutset of P, thus $A_\beta \cap C \neq \emptyset$. From the inductive construction of A_α, $A_\alpha - A_\beta \subseteqq \{x_\alpha\}$, therefore $A_\beta \cap C = \{x_\alpha\}$ which means that x_α is essential in A_β. So $x_\alpha \in A_\alpha$, which is a contradiction. If $\alpha = \sup_{\beta<\alpha} \beta$, then $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$ and $C \cap A_\beta \neq \emptyset$ for every $\beta < \alpha$. (Thus $|A \cap C|$ is infinite, for otherwise, let x_μ be in $A \cap C$ with a largest index μ, thus $x_\mu \notin A_\mu$ which gives $A_\mu \cap C = \emptyset$.) Also $A \cap C$ contains infinitely many elements in B_1, since C cannot contain more than one element in B_2 (B_2 is an antichain). Let

$$C \cap B_1 = \{y_1, y_2, \ldots, y_i, \ldots\}.$$

Then for every i, y_i covers t_i for some t_i in C_x and $t_i < x$. Let $y = \sup y_i$ and $t = \sup t_i$. Since $y > t_i$ for every i and P is chain complete, $y \geqq t$. If $y > t$, then from the regularity of P, $t < y_j$ for some j. Therefore $t_j < t < y_j$ which contradicts $t_j \prec y_j$. Thus $y = t$ and $y \in C_x \cap C$. Obviously $y \neq x$ since $C \cap A_\alpha = \emptyset$ and $x \in A_\alpha$. Also if $y > x$, then from the regularity of P, $x < y_j$ for some j, thus $t_j < x < y_j$, which contradicts $y_j \succ t_j$. Therefore $y < x$. Now consider the maximal chain

$$K = (C_x \cap (y]) \cup (C \cap [y))$$

of P. Obviously $K \cap A = \emptyset$, which contradicts that A is a cutset of P.

The subset A_λ is a minimal cutset. Indeed let $x_\alpha \in A_\lambda$ then $x_\alpha \in A_\beta$, for every $\beta < \lambda$. In particular $x_\alpha \in A_\alpha$, which implies the existence of a maximal chain C of P such that $C \cap A_\alpha = \{x\}$. Since $A_\lambda \subseteqq A_\alpha$, $C \cap A_\lambda = \{x_\alpha\}$. This completes the proof.

The proof of Theorem 2, does not extend to the case that P does not satisfy a chain condition. Indeed, in the example illustrated in Figure 11, $\{x\} \cup B_1 \cup B_2$ is not a cutset.

In general, it need not be the case that a cutset always contains a minimal one, even for regular ordered sets. For instance, the ordered set illustrated in Figure 12

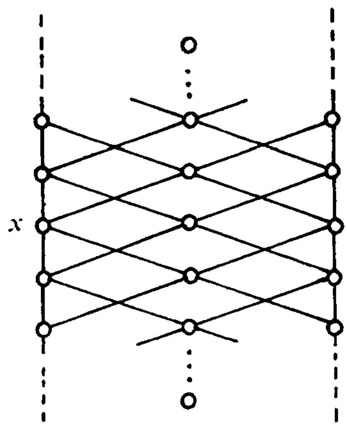


Figure 11

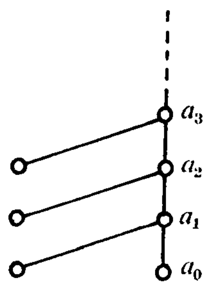


Figure 12

is regular and $\{a_1, a_2, ...\}$ is a cutset which does not contain a minimal one. A related question is this. Which ordered sets contain at least one minimal cutset?

Proof of Theorem 3. Suppose that P contains a subset $\{a, b, c, d\}$ isomorphic to N, that is, $a<c$, $b<c$, $b<d$ are the only comparabilities among the elements $\{a, b, c, d\}$. Without loss of generality, we may assume that $c\succ a$ and $d\succ b$. Consider maximal chains C and D of P such that $\{a, c\}\subseteqq C$ and $\{b, d\}\subseteqq D$. Obviously $C\cap\{b, d\}=\emptyset$ and $D\cap\{a, c\}=\emptyset$. Assume that for every x in D and $x\geqq d$, x does not cover in P any element y such that $y\in C$ and $y\leqq a$. Thus $\{c, b\}$ is a cutset in $P_0=C\cup D$.

From the proof of Theorem 2, there is a minimal cutset K of P containing c such that $K\subseteqq\{c\}\cup B_1\cup B_2$ with

$$B_1 = \{y\in P-C \mid y \succ z \text{ for some } z \text{ in } C \text{ and } z < c\}$$

and

$$B_2 = \{y\in P-C \mid y \text{ is minimal in } P\}.$$

Since K is a cutset, $K \cap D \neq \emptyset$. Let $x \in K \cap D$. If $x \in B_1$, then $x \succ z$ in P for some z in C and $z < c$. By assumption $x \leqq b$, so $x < c$. If $x \in B_2$, then $x = \inf D$ thus $x < c$. Therefore K is not an antichain.

Assume that there exists x in D with $x \geqq d$ and such that x covers y in P for some $y \in C$ and $y \leqq a$. Necessarily, either $x \neq d$ or $y \neq a$. So, without loss of generality, we can assume that $x \neq d$ and c covers b in P (otherwise we start with $\{d, y, x, c\}$ as an N in P). From Theorem 2, d is contained in a minimal cutset K of P. Suppose that K is an antichain and let

$$X = \{U(x) \cap D\} \cup \{D(y) \cap C\} \quad \text{and} \quad Y = \{U(c) \cap C\} \cup \{D(b) \cap D\}.$$

Since C and D are maximal chains and $c \succ b$, $x \succ y$ in P, the chains X and Y are maximal in P. Therefore $K \cap X \neq \emptyset$ and $K \cap Y \neq \emptyset$. Let $u \in K \cap X$ and $v \in K \cap Y$. Since K is an antichain, $v \geqq c$ and $u \leqq a$. Thus $v < u$. This contradiction completes the proof.

Proof of Theorem 4. Suppose that P is an ordered set which contains a generalized $N = (C_1, C_2, A_1, A_2)$. Let A be a maximal antichain in P, containing $A_1 \cup A_2$ and let C be a maximal chain in P such that $C_1 \cup C_2 \subseteq C$. Since $C_1 < C_2$ and there is no x such that $C_1 < \{x\} < C_2$, for every element y in $C - (C_1 \cup C_2)$, either $y > c_2$ for some element c_2 in C_2 or $y < c_1$ for some element c_1 in C_1. But A_2 is coinitial in C_2 and A_1 is cofinal in C_1, thus either $y > a_2$ for some a_2 in A_2 or $y < a_1$ for some a_1 in A_1. Therefore $y \notin A$ and $C \cap A = \emptyset$, which contradicts that A is an antichain cutset in P.

To prove the converse assume that P contains a maximal antichain A and a maximal chain C such that $A \cap C = \emptyset$. Let

$$C_1 = \{x \in C \mid x < a \text{ for some } a \text{ in } A\}$$

and let

$$C_2 = \{x \in C \mid x > a \text{ for some } a \text{ in } A\}.$$

Since $A \cap C = \emptyset$ and A is a maximal antichain, $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = C$. If $C_1 = \emptyset$ then $(\emptyset, C_2, \emptyset, A)$ is a generalized N, and the dual argument applies if $C_2 = \emptyset$. So, we assume that $C_1 \neq \emptyset \neq C_2$.

Let α and β be ordinals such that $\alpha = \mathrm{cf}(C_1)$ and $\beta = \mathrm{ci}(C_2)$. (The *cofinality* of a chain C of order type γ, denoted by $\mathrm{cf}(C)$ or $\mathrm{cf}(\gamma)$ too, is the least ordinal α such that there is a subchain C' of C of order type α and cofinal in C. The *coinitiality* of a chain C of order type γ, denoted by $\mathrm{ci}(C)$ or $\mathrm{ci}(\gamma)$ too, is the least ordinal β such that there is a subchain C' of C of order type β^d, the dual of β, and coinitial in C.) Let

$$F = \{x_0 < x_1 < \ldots < x_i < \ldots\}_{i<\alpha}$$

be a cofinal subset of C_1 and

$$I = \{y_0 > y_1 > \ldots > y_i \ldots\}_{i<\beta}$$

be a coinitial subset of C_2. Now, we construct, simultaneously, the antichains $(A_1^i)_{i<\gamma}$ and $(A_2^i)_{i<\gamma}$ in P, where $\gamma=\min(\alpha, \beta)$, as follows.

Let a_0 be in A such that $a_0>x_0$, and let b_0 in $A-\{a_0\}$ such that $b_0<y_0$. We set $A_1^0=\{a_0\}$ and $A_2^0=\{b_0\}$.

Suppose we have already constructed $(A_1^i)_{i<\delta}$ and $(A_2^i)_\delta$. If $\delta=\delta'+1$, an isolated ordinal, then let $a_\delta \in A-(A_1^{\delta'}\cup A_2^{\delta'})$ such that $a_\delta>x_\delta$. And let $b_\delta\neq a_\delta$ in $A-(A_1^{\delta'}\cup A_2^{\delta'})$ such that $b_\delta<y_\delta$. We set

$$A_1^\delta = A_1^{\delta'}\cup\{a_\delta\} \quad \text{and} \quad A_2^\delta = A_2^{\delta'}\cup\{b_\delta\}.$$

If δ is a limit ordinal then we set

$$A_1^\delta = \bigcap_{i<\delta} A_1^i \quad \text{and} \quad A_2^\delta = \bigcap_{i<\delta} A_2^i.$$

Since the antichain A is cofinal in C_1 and coinitial in C_2, this construction is possible until A_1^γ and A_2^γ. Without loss of generality we can assume that $\gamma=\alpha\leqq\beta$. Let $A_1=A_1^\alpha$ and $A_2=A-A_1$. The four-tuple (C_1, C_2, A_1, A_2) is a generalized N in P.

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