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Homomorphism of distributive lattices as restriction of congruences

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Dedicated to the memory of András Huhn

1. Introduction. Let I be an ideal of a lattice L. Then the map $\varrho: \Theta \to \Theta_I$, restricting a congruence relation Θ to I is a 0 and 1 preserving lattice-homomorphism of the congruence lattice Con L into Con I. G. GRÄTZER and H. LAKSER [1] have proved the converse for finite lattices:

Theorem A. *Let D and E be finite distributive lattices, and let $\varphi: D \to E$ be a* 0 *and* 1 *preserving homomorphism of D into E. Then there exist a finite lattice L, and an ideal I of L, such that there are isomorphisms $\alpha: D \to \mathrm{Con}\, L$, $\beta: E \to \mathrm{Con}\, I$, satisfying $\beta\varphi = \varrho\alpha$, where $\varrho: \Theta \to \Theta_I$ is the restriction of $\Theta \in \mathrm{Con}\, L$ to I.* (See Figure 1.)

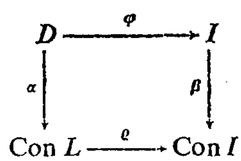

Figure 1

The purpose of this paper is twofold. Firstly, we generalize Theorem A for distributive algebraic lattices satisfying the following condition

(∗) for all compact x, $x \vee \bigwedge(x_i |\, i \in I) = \bigwedge(x \vee x_i |\, i \in I)$,

which is a weaker form of the infinite meet distributivity. Secondly, we win a short proof of Theorem A, which uses a construction given in SCHMIDT [3] and [4].

2. Dual Heyting algebras. Let L be a lattice. The dual pseudocomplement of a relative to b is an element $a*b$ of L satisfying $a \vee x \geqq b$ iff $x \geqq a*b$. A *dual Heyting algebra* is a distributive lattice with 1 in which $a*b$ exists for all $a, b \in L$. The subset

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of all compact elements of an algebraic lattice A is denoted by $K(A)$. This $K(A)$ is a join-subsemilattice with smallest element 0. A lattice A is called *arithmetic* iff it is algebraic and $K(A)$ is a sublattice of L.

Lemma 1. *Let L be a distributive arithmetic lattice, whose unit element is compact. L satisfies the condition $(*)$ if and only if $K(L)$ is a dual Heyting algebra.*

Proof. First, let $K(L)$ be a dual Heyting algebra. Then L is isomorphic to the lattice of all ideals of $K(L)$, and the compact elements of the ideal lattice are precisely the principal ideals. Therefore we have to show that

$$(x]\vee \wedge(J_i \mid i\in I) = \wedge((x]\vee J_i \mid i\in I)$$

where the J_i-s are ideals of $K(L)$. It is enough to verify that the right side is contained in the left side. Let $a\in \wedge((x]\vee J_i \mid i\in I)$, then $a\in (x]\vee J_i$ for all $i\in I$, i.e., $a\leqq x\vee y_i$ for suitable $y_i\in J_i$. $K(L)$ is a dual Heyting algebra, therefore $x*a$ exists and $x*a\leqq y_i$ implies $x*a\in J_i$, $i\in I$. Thus $x*a\in \wedge(J_i\, i\in I)$. By the definition of $x*a$ we have $a\leqq x\vee(x*a)$, i.e. $a\in(x]\vee \wedge(J_i \mid i\in I)$.

By assumption L is a distributive arithmetic lattice with compact unit element, thus $K(L)$ is a bounded distributive lattice. Consider all u_i-s such that $a\vee u_i\geqq b$. Then $b\in \wedge((a]\vee(u_i])$. Applying $(*)$ we obtain $b\in(a]\vee \wedge(u_i]$, i.e. there exists a $z\in \wedge(u_i]$ such that $a\vee z\geqq b$. Obviously $z=a*b$.

By Lemma 1, we can work with dual Heyting algebras, namely L is determined by $K(L)$.

Let L be a $\{0, 1\}$-sublattice of the Boolean lattice B. Then L is said to R-generate B if L generates B as a ring. The following lemma is due to H. M. MacNeille (see G. GRÄTZER [2]).

Lemma 2. *Let B be R-generated by L. Then every $a\in B$ can be expressed in the form*

$$a_0+a_1+\ldots+a_{n-1},\quad a_0\leqq a_1\leqq\ldots\leqq a_{n-1},\quad a_0,\ldots,a_{n-1}\in L.$$

A sublattice L' of a dual Heyting algebra L is called a *subalgebra* if for every $x\in L$ there exists a smallest $\bar{x}\in L'$ that $x\leqq\bar{x}$.

Lemma 3. *A subalgebra of a dual Heyting algebra is a dual Heyting algebra.*

Proof. Let L' be a subalgebra of L and let $a, b\in L'$, $a\leqq b$. Then $a*b$ exists in L, and it is easy to verify that $\overline{a*b}$ is the dual pseudocomplement of a relative to b in L'. It is clear that if the dual pseudocomplement exists for comparable pairs then there exists for arbitrary pairs.

For a bounded distributive lattice L we shall denote by $B(L)$ the Boolean lattice R-generated by L.

Lemma 4. *Let L be a dual Heyting algebra. Then L is a subalgebra of $B(L)$.*

Proof. Let L be a dual Heyting algebra. Then by Lemma 2 every x can be expressed in the form $x=a_0+\ldots+a_{n-1}$. We prove the existence of $\bar{x}$ by induction on n. If $n=1$, i.e. $x=a_0$ then $x\in L$ hence $\bar{x}=x$. For $n=2$, i.e. $x=a_0+a_1$, x is the relative complement of a_0 in the interval $[0, a_1]$. Then $a_0\vee y\geqq a_1$ and $y\in L$ imply $y\geqq a_0+a_1$, thus $\overline{a_0+a_1}$ exists and $\overline{a_0+b_1}=a_0*a_1$. Let us assume that $n>2$. $a_{n-2}+a_{n-1}$ is the relative complement of a_{n-2} in the interval $[0, a_{n-1}]$, hence $a_{n-3}\wedge(a_{n-2}+a_{n-1})\leqq a_{n-2}\wedge(a_{n-2}+a_{n-1})=0$. Obviously $a_0+\ldots+a_{n-3}\leqq a_{n-2}$, thus $(a_0+\ldots+a_{n-3})\wedge(a_{n-2}+a_{n-1})=0$. This implies that $a_0+\ldots+a_{n-1}=(a_0+\ldots+a_{n-3})+$ $+(a_{n-2}+a_{n-1})=(a_0+\ldots+a_{n-3})\vee(a_{n-2}+a_{n-1})$.

Let x, y be arbitrary elements of $B(L)$ such that $\bar{x}$ and $\bar{y}$ exist. We prove that $\overline{x\vee y}$ exists and $\overline{x\vee y}=\bar{x}\vee\bar{y}$. Let $x\vee y\leqq z\leqq \bar{x}\vee\bar{y}$, $z\in L$. Then we get from $x, y\leqq x\vee y$ that $x\leqq\bar{x}\wedge z\in L$, $y\leqq\bar{y}\wedge z\in L$, and we conclude that $\bar{x}\leqq z$, $\bar{y}\leqq z$, i.e. $z=\bar{x}\vee\bar{y}$, which proves $\overline{x\vee y}=\bar{x}\vee\bar{y}$. Applying this equality for $x=a_0+\ldots+a_{n-3}$ and $y=$ $=a_{n-2}+a_{n-1}$ we obtain that $\overline{x\vee y}=\overline{x+y}$ exists.

Lemma 5. *Let Θ be a compact congruence relation of a dual Heyting algebra L. Then L/Θ is a dual Heyting algebra.*

Proof. The compact congruence relations are exactly the finite joins of principal congruence relations. To prove the lemma, by the Second Isomorphism Theorem we may assume that Θ is a principal congruence relation, i.e. $\Theta=\Theta(u, v)$, $u\leqq v$.

Let L be a dual Heyting algebra. We prove that each congruence class of $\Theta(u, v)$ contains a smallest element. In distributive lattices $\Theta(u, v)$ has the following description (see [2], p. 74): $a\equiv b\ (\Theta(u, v))$ iff $v\vee a=v\vee b$ and $u\wedge a=u\wedge b$. Let b be a fixed element of L. Then $v\vee a=v\vee b$ implies that $a\geqq v*(v\vee b)$. Therefore $v*(v\vee b)$ is the least element of the $\Theta(u, v)$-class containing b. Now, let $a<b$ and let c denote the least element of the $\Theta(u, v)$-class containing b. Let $[x]$ denote the $\Theta(u, v)$-class containing x. Then obviously $[a]*[b]=[a*c]$.

Corollary. *Every Θ-class of a compact congruence relation Θ of a dual Heyting algebra contains a smallest element.*

3. The main theorem. In this section we formulate our main theorem and then we give two special representations of dual Heyting algebras.

Theorem B. *Let D and E be dual Heyting algebras, and let $\varphi\colon D\to E$ be a 0 and 1 preserving homomorphism of D into E such that the congruence kernel* Ker φ *is a compact congruence relation and $F=\operatorname{Im}\varphi$ is a subalgebra of E. Then there exist a lattice L, and a principal ideal I of L, such that there are isomorphisms $\alpha\colon D\to K(\operatorname{Con} L)$, $\beta\colon E\to J(\operatorname{Con} I)$ satisfying $\beta\varphi=\varrho\alpha$, where $\varrho\colon\Theta\to\Theta_I$ is the restriction of $\Theta\in\operatorname{Con} L$ to I.*

If L_1 and L_2 are lattices with zero elements 0_1 resp. 0_2 then in the direct product $L_1 \times L_2$ the elements $\langle x, 0_2 \rangle$ $(x \in L_1)$ form an ideal L_1' isomorphic to L_1. Therefore we can identify L_1 with L_1' and similarly L_2 with the ideal $L_2' = \{\langle 0_1, x \rangle\}$.

Let Θ be the congruence kernel of the homomorphism $\varphi: D \to E$. By our assumption Θ is a compact congruence relation of D. On the other hand D is a bounded distributive lattice, therefore the unit of Con D is compact. The compact elements of Con D form a Boolean lattice (see [2], p. 86, Exercise 41), consequently Θ has a complement Θ' in Con D. Then D is a subdirect product of D/Θ and D/Θ', therefore $D \subseteq D/\Theta \times D/\Theta'$.

$F = \mathrm{Im}\,\varphi$ is a $\{0, 1\}$-sublattice of E and F is isomorphic to D/Θ; we identify D/Θ and F. Hence we may consider D as a $\{0, 1\}$-sublattice of $E \times D/\Theta'$. Let e be the unit of E and $\pi_1(x) = x \wedge e$ denotes the projection map of $E \times D/\Theta'$ onto E. Observe, that the restriction of π_1 to D $(\subseteq E \times D/\Theta')$ gives the homomorphism φ (see Figure 2).

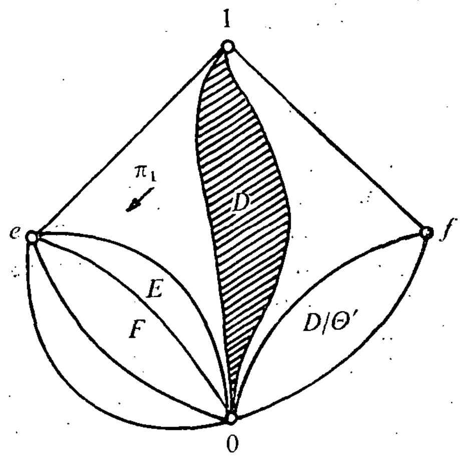

Figure 2

Lemma 6. *$E \times D/\Theta'$ is a dual Heyting algebra and D is a subalgebra of $E \times D/\Theta'$.*

Proof. By our assumptions D and E are dual Heyting algebras and Θ' is a compact congruence relation. Hence by Lemma 5 D/Θ' and thus $E \times D/\Theta'$ are dual Heyting algebras. $F = \mathrm{Im}\,\varphi$ is a subalgebra of E, hence by Lemma 3 F and $F \times D/\Theta'$ are dual Heyting algebras.

We have seen that D is a subdirect product of F and D/Θ'. First we show that D is a subalgebra of the dual Heyting algebra $F \times D/\Theta'$. An arbitrary element of $F \times D/\Theta'$ can be written in the form $x = \langle [a]\Theta, [b]\Theta' \rangle$ where $a, b \in D$. By Corollary of Lemma 5 the congruence classes of Θ and Θ' have smallest elements. Let now a_0 and b_0 be the smallest elements of $[a]\Theta$ resp. $[b]\Theta'$. Then $\langle [a_0 \wedge b_0]\Theta, [a_0 \wedge b_0]\Theta' \rangle \in D$. Obviously this element is in D, the smallest one which is greater or equal than x, i.e. $\bar{x}$ exist. This proves that D is a subalgebra of $F \times D/\Theta'$. On the other hand F is a subalgebra of E, consequently $F \times D/\Theta'$ is a subalgebra of $E \times D/\Theta'$, which proves finally that D is a subalgebra of $E \times D/\Theta'$ (namely a subalgebra of a subalgebra is again a subalgebra).

In [3] (or see [4]) there was given a special lattice construction to prove that the lattice of all ideals of a dual Heyting algebra is isomorphic to the congruence lattice of some lattice. The most important properties of this construction are summarized in the following lemma.

Lemma 7. *Let K be a $\{0, 1\}$-subalgebra of a Boolean lattice A, and let $\varepsilon\colon K \to A$ be the identy map. There exists a bounded lattice M with the following properties:*

(i) *M contains three elements u, v, w such that $\{0, u, v, w, 1\}$ form a sublattice isomorphic to the diamond M_3. There are isomorphisms $\mu\colon (u] \to K$ and $\tau\colon (v] \to A$. If for $x \in K$ $\pi(x)$ means $(x \vee w) \wedge v$ then $\tau\pi = \varepsilon\mu$.*

(ii) *The map $x \to x \vee u$ $(x \leqq v)$ is an isomorphism of $(v]$ onto the filter $[u)$.*

(iii) *A congruence relation $\Theta(0, x)$ of $(v]$ can be extended to M iff $\tau(x) \in \varepsilon(K)$, and every compact congruence relation of M is the extension of a congruence relation $\Theta(0, x) \in \mathrm{Con}\,(v]$.*

Remark. A is a Boolean lattice, therefore every compact congruence relation of A $(\cong (v])$ can be written in the form $\Theta(0, x)$. Condition (iii) implies that $\mathrm{Con}\, M \cong I(K)$, i.e. $K(\mathrm{Con}\, M) \cong K$.

4. The proof of Theorem B. We apply Lemma 7 twice to get two lattices M_1 and M_2. Then we use the so called Hall—Dilworth gluing construction which yields a lattice L having the properties required in the theorem.

By Lemma 6 D is a subalgebra of the dual Heyting algebra $E \times D/\Theta'$ and by Lemma 4 $E \times D/\Theta'$ is a subalgebra of $B(E \times D/\Theta')$. Consequently D is a subalgebra of $B(E \times D/\Theta')$. Then we can choose in Lemma 7 $K = D$ and $A = B(E \times D/\Theta')$. We obtain the lattice M_1 with a diamond $\{0_1, u_1, v_1, w_1, 1_1\}$ given in condition (i) of Lemma 7. In the second case we consider $K = E \times B(D/\Theta')$ and $A = B(E \times D/\Theta')$. By Lemma 4 E is a subalgebra of $B(E)$ hence $E \times B(D/\Theta')$ is a subalgebra of $B(E) \times B(D/\Theta') = B(E \times D/\Theta')$. The resulting lattice is M_2 with the diamond $\{0_2, u_2, v_2, w_2, 1_2\}$.

By condition (i) of Lemma 7 the ideal $(v_1]$ of M_1 is isomorphic to $B(E \times D/\Theta')$. On the other hand by condition (ii) the filter $[u_2)$ of M_2 is isomorphic to $B(E \times D/\Theta')$. Consequently we have an isomorphism $\delta\colon [u_2) \to (v_1]$. We apply the Hall—Dilworth gluing construction which gives a lattice L having M_1 as a filter and M_2 as an ideal.

(L is the set of all $x \in M_1 \cup M_2$, we identify x with $\delta(x)$ for all $x \in [u_2)$; $x \leqq y$ has unchanged meaning if $x, y \in M_1$ or $x, y \in M_2$ and $x \prec y$, $x, y \notin [u_2) = (v_1]$ iff $x \in M_2$, $y \in M_1$ and there exists a $z \in [u_2)$ such that $x \prec z$ in M_2 and $z < y$ in M_1.) The lattice L is given by Figure 3 where $B = B(E \times D/\Theta')$:

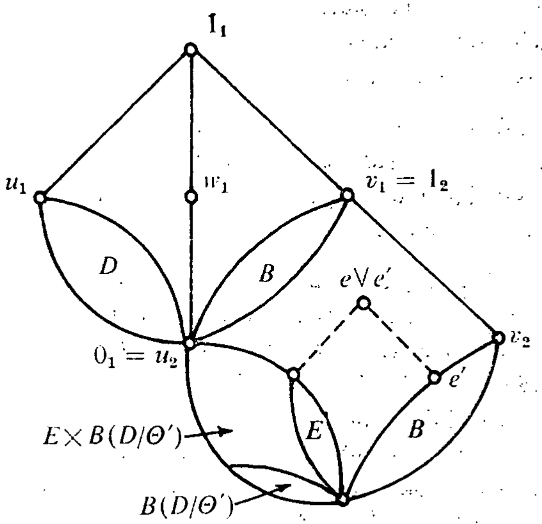

Figure 3

The function $\pi(x) = (x \vee w_2) \wedge v_2$ yields the element $e' = \pi(e) = (e \vee w_2) \wedge v_2 \leqq v_2$. Let I be the principal ideal generated by $e \vee e'$. We have to prove that the pair L, I satisfies the properties given in Theorem B.

(1) First we prove that $\operatorname{Con} L \cong I(D)$ i.e. D is isomorphic to the semilattice of all compact congruences of L. Every congruence relation Θ of L is determined by its restrictions Θ_{M_1} and Θ_{M_2} to M_1 resp. M_2. By condition (iii) of Lemma 7 Θ_{M_1} is determined by its restriction to $(v_1]$ and similarly Θ_{M_2} is determined by its restriction to $(v_2]$. But the interval $[0_1, v_1]$ is a transpose of $[0_2, v_2]$, hence we get that Θ is determined by its restriction to the ideal $(v_2]$. This ideal is a Boolean lattice, thus every compact congruence relation of $(v_2]$ has the form $\Theta(0_2, x)$, $x \in (v_2]$. Let now, $\Theta(0_2, x)$ be a congruence relation of $(v_2]$. Under what conditions for x has this congruence relation an extension to L? Condition (iii) of Lemma 7 gives the following isomorphisms:

in M_1, $\mu_1\colon (u_1] \to D$, $\tau_1\colon (v_1] \to B(E \times D/\Theta')$,

in M_2, $\mu_2\colon (u_2] \to E \times B(D/\Theta')$, $\tau_2\colon (v_2] \to B(E \times D/\Theta')$.

If $\varepsilon_1\colon D \to B(E \times D/\Theta')$ and $\varepsilon_2\colon E \times B(D/\Theta')$ denote the identity maps, then $\tau_1 \pi_1 = \varepsilon_1 \mu_1$ and $\tau_2 \pi_2 = \varepsilon_2 \mu_2$ where $\pi_i(x) = (x \vee w_i) \wedge v_i$ $(i = 1, 2)$. By condition (iii) of

Lemma 7, the congruence relation $\Theta(0_2, x)$ can be extended to M_2 iff $\tau_2(x)\in \in\varepsilon_2(E\times B(D/\Theta'))$. Similarly, in M_1 we get that the congruence relation $\Theta(0_1, 0_1\vee x)$ of $(v_1]$ can be extended to M_1 iff $\tau_1(0_1\vee x)\in\varepsilon_1(D)$. Obviously the minimal extensions of $\Theta(0_2, x)$ and $\Theta(0_1, 0_1\vee x)$ to L are the same and $\varepsilon_1(D)$ is a sublattice of $\varepsilon_2(E\times B(D/\Theta'))$, so we obtain that $\Theta(0_2, x)$ has an extension to L iff $\tau_1(0_1\vee x)\in\varepsilon_1(D)$. This proves Con $L\cong I(D)$.

(2) Secondly we show Con $I\cong I(E)$. E is a direct factor of $(u_2]$ and $(e']$ is isomorphic to $B(E)$. Obviously $B(E\times B(D/\Theta'))=B(E)\times B(D/\Theta')$ hence the principal ideal $I=(e\vee e']$ is a direct factor of M_2. This means that I is again a lattice given by Lemma 7, namely if $K=E$ and $A=B(E)$. Thus by condition (iii) we have $K(\text{Con } I)\cong E$, i.e. Con $I\cong I(E)$.

(3) Finally, let Θ be a compact congruence relation of L. We have seen that Θ is the extension of some $\Theta(0_2, x)\in$Con $(u_2]$ where $\tau_2(x)\in\varepsilon_1(D)$, i.e. the restriction $\Theta\to\Theta_I$ is determined by the projection $D\to E\times D/\Theta'$. As we have seen this is exactly the given homomorphism φ.

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