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A characterization of semimodularity in lattices of finite length

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Dedicated to the memory of András P. Huhn

1. Introduction. In this note we consider only lattices of finite length. By $x \prec y$ we mean that x is a lower cover of y. If L is a lattice of finite length, we denote by $J(L)$ the set of all join-irreducible elements ($\neq 0$) of L. Equivalently, an element u is in $J(L)$ if and only if it has precisely one lower cover which will be denoted by u'. A lattice L of finite length is called (upper) semimodular if the neighbourhood condition (N) holds in L:

$$\text{(N)} \qquad a \wedge b \prec a \Rightarrow b \prec a \vee b \quad (a, b \in L).$$

It is the aim of this paper to show that (N) can be replaced by a seemingly weaker condition ($\overline{\text{N}}$) which may be called restricted neighbourhood condition or neighbourhood condition for join-irreducible elements:

$$(\overline{\text{N}}) \qquad u \wedge b = u' \prec u \Rightarrow b \prec u \vee b \quad (u \in J(L),\ b \in L).$$

After a preliminary lemma in Section 2, we show the equivalence of (N) and ($\overline{\text{N}}$) in Section 3. Applying this result to the atomistic case (i.e., to the case in which each join-irreducible element ($\neq 0$) is an atom) we get the well-known result that in these lattices semimodularity is equivalent to the so-called covering property.

2. Preliminary remarks. In this section we prove the following

Lemma. *Let L be a lattice of finite length. If $c \prec d$ $(c, d \in L)$, then there exists a join-irreducible element $u \in J(L)$ such that $u \leqq d$, $u \not\leqq c$ and $u \wedge c = u'$.*

Proof. If $d \in J(L)$, then put $u = d$. Let now $d \notin J(L)$ and consider the set of all $v \in J(L)$ which have the property $v < d$ and $v \not\leqq c$. It is clear that this set is not empty. Choose an element $u \in J(L)$ which is minimal with respect to this property. Since L is of finite length, such a minimal element always exists. From $u < d$

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and $u \not\leq c$ it follows that $u \wedge c \leq u'$. We show that equality holds. From the assumption $u \wedge c < u'$ we get the existence of an element $u_* \in J(L)$ having the properties $u_* \leq u'$ and $u_* \not\leq u \wedge c$. This implies

$$u_* \leq u' \prec u < d \quad \text{and} \quad u_* \not\leq c.$$

(Note that $u_* \leq c$ yields together with $u_* < u$ that $u_* \leq u \wedge c$, a contradiction.) But this in turn contradicts the minimality of $u \in J(L)$. Thus our assumption was false, i.e., we have $u \wedge c = u'$, which was to be proved.

Remark. The preceding lemma was implicitly used in the proof of the main theorem of [2] and it was explicitly given in [3]. We have included the proof here in order to make the paper self-contained. This lemma is a generalization of a property which is trivially fulfilled in atomistic lattices of finite length.

3. Results. Using the lemma of the preceding section, we prove here the following

Theorem. *Let L be a lattice of finite length. Then the neighbourhood condition* (N) *holds if and only if the restricted neighbourhood condition* $(\overline{\text{N}})$ *holds.*

Proof. (N)$\Rightarrow(\overline{\text{N}})$: This implication is obviously true.

$(\overline{\text{N}})\Rightarrow$(N): Assuming $(\overline{\text{N}})$ we show that (N) also holds. In other words, in lattices of finite length the restricted neighbourhood condition already implies the (upper) semimodularity of the lattice. Without loss of generality we may assume that $a, b \in L$ are incomparable elements. In order to prove the assertion assume

$$(*) \qquad a \wedge b \prec a \quad (a, b \in L).$$

We show that then $b \prec a \vee b$ also holds. If $a = u \in J(L)$, it follows by $(\overline{\text{N}})$ that $b \prec u \vee b = a \vee b$ and nothing is to be proved. Assume now $a \notin J(L)$. By the lemma of Section 2 there exists a join-irreducible element $u \in J(L)$ having the properties $u < a$, $u \not\leq a \wedge b$ and $u' = u \wedge (a \wedge b) = u \wedge b$. From $u < a$, $u \not\leq a \wedge b$ and $a \wedge b \prec a$ we obtain $a = (a \wedge b) \vee u$. This means that

$$a \vee b = (a \wedge b) \vee u \vee b = b \vee u.$$

Now

$$u' = u \wedge b \prec u$$

implies by $(\overline{\text{N}})$ that

$$(**) \qquad b \prec u \vee b = a \vee b.$$

To sum it up: under the assumption $(\overline{\text{N}})$, the relation $(*)$ implies $(**)$ which means that the lattice is semimodular. This finishes the proof of the theorem.

Corollary. *Let L be an atomistic lattice of finite length. Then L is (upper) semimodular if and only if the covering property*

$$\text{(C)} \qquad p(\in L) \text{ atom}, \; b \in L, \; b \wedge p = 0 \Rightarrow b \prec b \vee p$$

holds.

Proof. In the atomistic case ($\bar{\text{N}}$) reduces to (C) implying semimodularity by the preceding theorem. The converse statement is obviously true.

We remark that the assertion of the corollary holds even for arbitrary atomistic lattices by [1, Theorem 7.10, p. 32].

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