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Mal'cev conditions for varieties of subregular algebras

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Although every congruence uniquely determines anyone of its blocks, the converse apparently does not hold in general. This trivial fact has given origin to various definitions of congruence "regularity" or "nice" congruences or "a good theory" of ideals etc. Recall from the literature that an algebra $\mathfrak{A}$ is called *regular* if every congruence on $\mathfrak{A}$ is uniquely determined by anyone of its blocks; an algebra $\mathfrak{A}$ with nullary operations $c_1, \ldots, c_n$ is called *weakly regular* (with respect to $c_1, \ldots, c_n$) whenever every congruence Ψ on $\mathfrak{A}$ is uniquely determined by its blocks $[c_1]\Psi, \ldots, [c_n]\Psi$. A natural continuation of these two concepts was introduced by J. TIMM, [12]. We write $A, B, \ldots$ for the universes of algebras $\mathfrak{A}, \mathfrak{B}, \ldots$.

Definition 1. An algebra $\mathfrak{A}$ is said to have *subregular congruences* (briefly: $\mathfrak{A}$ is *subregular*) if every congruence Ψ on $\mathfrak{A}$ is uniquely determined by its blocks $[b]\Psi$, $b\in B$, for any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$.

It is already known that varieties of regular algebras and varieties of weakly regular algebras are definable by Mal'cev conditions, see [1, 2], [15] and [7] for the details. The objective of this note is to prove that also varieties of subregular algebras form Mal'cev class. We give here the explicit Mal'cev condition, see Theorem 1, since the characterizing identities enable us to prove that any variety of subregular algebras is congruence modular and n-permutable for some $n>1$. In addition we discuss the relationship between subregularity of tolerances and subregularity of congruences on algebras from a given variety. As a result of these considerations a simple Mal'cev condition for permutable varieties of subregular algebras is obtained.

Two lemmas will be needed in the sequel.

Lemma 1. *Let $\mathfrak{B}$ be a subalgebra of an algebra $\mathfrak{A}$, Ψ a congruence on $\mathfrak{A}$. The following conditions are equivalent:*

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(i) *Ψ is uniquely determined by its blocks* $[b]\Psi$, $b\in B$;

(ii) $\Psi=\Theta(\bigcup_{b\in B_0}(\{b\}\times A_b))$ *for some subsets* $B_0\subseteq B$ *and* $A_b\subseteq A$, $b\in B_0$.

Proof. (i)$\Rightarrow$(ii): If (i) holds then $\Psi=\Theta(\bigcup_{b\in B}([b]\Psi\times[b]\Psi))$. Using an evident fact that $\Theta(\bigcup_{b\in B}([b]\Psi\times[b]\Psi))=\Theta(\bigcup_{b\in B}(\{b\}\times[b]\Psi))$ the desired conclusion (ii) readily follows.

(ii)$\Rightarrow$(i): Conversely, suppose (ii). Then $\Psi\supseteq\bigcup_{b\in B_0}(\{b\}\times A_b)$ which gives that $[b]\Psi\times[b]\Psi\supseteq\{b\}\times A_b$ for every $b\in B_0$. By forming suitable set unions we obtain $\Psi=\bigcup_{b\in A}([b]\Psi\times[b]\Psi)\supseteq\bigcup_{b\in B}([b]\Psi\times[b]\Psi)\supseteq\bigcup_{b\in B_0}([b]\Psi\times[b]\Psi)\supseteq\bigcup_{b\in B_0}(\{b\}\times A_b)$. Hence $\Psi\supseteq\Theta(\bigcup_{b\in B}([b]\Psi\times[b]\Psi))\supseteq\Theta(\bigcup_{b\in B_0}(\{b\}\times A_b))=\Psi$, i.e. $\Psi=\Theta(\bigcup_{b\in B}([b]\Psi\times[b]\Psi))$, as required.

Remark 1. Evidently, the subsets B_0 and A_b, $b\in B_0$, from the previous lemma can be taken finite whenever Ψ is compact (=finitely generated).

H. A. Thurston has given a useful criterion for varieties of regular algebras in [13]. Lemma 2 shows that an analogue result holds for varieties of subregular algebras.

Lemma 2. *For a variety* $\mathbf{V}$ *the following conditions are equivalent:*

(i) *Every* $\mathfrak{A}\in\mathbf{V}$ *has subregular congruences;*

(ii) *a congruence on* $\mathfrak{A}\in\mathbf{V}$ *is trivial whenever it is trivial on a subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$.

Proof. The implication (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i): Let $\mathfrak{B}$ be an arbitrary subalgebra of $\mathfrak{A}\in\mathbf{V}$. We have to prove that any congruence Ψ on $\mathfrak{A}$ is uniquely determined by blocks $[b]\Psi$, $b\in B$. To do this take the congruence $\Psi'=\Theta(\bigcup_{b\in B}([b]\Psi\times[b]\Psi))$ on $\mathfrak{A}$. Clearly, the subset $[B]\Psi=\bigcup_{b\in B}[b]\Psi$ is a subalgebra of $\mathfrak{A}$, moreover, the equality $[B]\mathfrak{A}=[B]\Psi'$ follows from the construction of Ψ'. Since $\Psi\supseteq\Psi'$ we can consider the congruence Ψ/Ψ' on the quotient algebra $\mathfrak{A}/\Psi'\in\mathbf{V}$. Apparently, Ψ/Ψ' is trivial on the subalgebra $[B]\Psi'/\Psi'\cap([B]\Psi'\times[B]\Psi')$ of $\mathfrak{A}/\Psi'$ hence, by hypothesis (ii), Ψ/Ψ' is trivial on the whole algebra $\mathfrak{A}/\Psi'$. In other words we have $\Psi=\Psi'$ which was to be proved.

Now we state the promised Mal'cev condition for varieties of subregular algebras (announced in [4] at first).

Theorem 1. *For a variety* $\mathbf{V}$ *the following conditions are equivalent:*

(1) *every* $\mathfrak{A}\in\mathbf{V}$ *has subregular congruences;*

(2) *there exist unary polynomials* $u_1, \dots, u_n$, *ternary polynomials* $p_1, \dots, p_n$ *and*

4-ary polynomials $s_1, \ldots, s_n$ *such that*

$$x = s_1(x, y, z, u_1(z))$$
$$s_i(x, y, z, p_i(x, y, z)) = s_{i+1}(x, y, z, u_{i+1}(z)), \quad 1 \leqq i < n,$$
$$y = s_n(x, y, z, p_n(x, y, z))$$
$$u_i(z) = p_i(x, x, z), \quad 1 \leqq i \leqq n,$$

hold in $\mathbf{V}$;

(3) *there exist unary polynomials* $u_1, \ldots, u_n$ *and ternary polynomials* $p_1, \ldots, p_n$ *such that*

$$(u_i(z) = p_i(x, y, z),\ 1 \leqq i \leqq n) \Leftrightarrow x = y$$

holds in $\mathbf{V}$.

Proof. (1)⇒(2): Take $\mathfrak{A} = \mathfrak{F}_{\mathbf{V}}(x, y, z)$, the free algebra in $\mathbf{V}$ on free generators x, y and z. Choose the subalgebra $\mathfrak{B} = \mathfrak{F}_{\mathbf{V}}(z)$ of $\mathfrak{A}$ and consider the principal congruence $\Theta(x, y)$ on $\mathfrak{A}$. Since $\mathfrak{A}$ is subregular, Lemma 1 (see also Remark 1) yields

$$(*) \qquad \Theta(x, y) = \Theta(\langle b_1, a_1\rangle, \ldots, \langle b_m, a_m\rangle)$$

for some elements $b_1, \ldots, b_m \in B$ and $a_1, \ldots, a_m \in A$. Applying the binary scheme, see [5, Thm. 1], to the congruence on the right hand side we get that

$$x = \sigma_1(u_1, p_1),$$
$$\sigma_i(p_i, u_i) = \sigma_{i+1}(u_{i+1}, p_{i+1}), \quad 1 \leqq i < n,$$
$$y = \sigma_n(p_n, u_n),$$

where $\sigma_1, \ldots, \sigma_n$ are binary algebraic functions over $\mathfrak{A}$ and

$$\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle \in \{\langle b_1, a_1\rangle, \ldots, \langle b_m, a_m\rangle\}.$$

Using the fact that $\mathfrak{A} = \mathfrak{F}_{\mathbf{V}}(x, y, z)$ and $\mathfrak{B} = \mathfrak{F}_{\mathbf{V}}(z)$, the above equalities can be rewritten in the form

$$x = s_1(x, y, z, u_1(z), p_1(x, y, z)),$$
$$s_i(x, y, z, p_i(x, y, z), u_i(z)) = s_{i+1}(x, y, z, u_{i+1}(z), p_{i+1}(x, y, z)), \quad 1 \leqq i < n,$$
$$y = s_n(x, y, z, p_n(x, y, z), u_n(z)),$$

for some unary polynomials $u_1, \ldots, u_n$, ternary polynomials $p_1, \ldots, p_n$ and 5-ary polynomials $s_1, \ldots, s_n$ of $\mathbf{V}$. Moreover $(*)$ implies the identities $u_i(z) = p_i(x, x, z)$, $1 \leqq i \leqq n$. Now one can easily verify that the ternary polynomials $q_1, \ldots, q_n$ defined by $q_i(x, y, z) = s_i(x, z, z, p_i(y, z, z), p_i(x, y, z))$, $1 \leqq i \leqq n$, satisfy the identities

$$x = q_1(x, z, z),$$
$$q_i(x, x, z) = q_{i+1}(x, z, z), \quad 1 \leqq i < n,$$
$$z = q_n(x, x, z),$$

ensuring the $(n+1)$-permutability of $\mathbf{V}$, see [9, 10] or [8, p. 353]. Then, by [5, Thm. 2], unary scheme can be used to describe the congruence $\Theta(\langle b_1, a_1\rangle, \ldots, \langle b_m, a_m\rangle)$. In this way we obtain the same identities as above with an additional information that the polynomials $s_1, \ldots, s_n$ do not depend on the last variable. Hence we have

$$x = s_1(x, y, z, u_1(z)),$$
$$s_i(x, y, z, p_i(x, y, z)) = s_{i+1}(x, y, z, u_{i+1}(z)), \quad 1 \leqq i < n,$$
$$y = s_n(x, y, z, p_n(x, y, z)),$$
$$u_i(z) = p_i(x, x, z), \quad 1 \leqq i \leqq n,$$

as desired in (2).

The implication (2)$\Rightarrow$(3) is clear.

(3)$\Rightarrow$(1): Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}\in\mathbf{V}$, Ψ a congruence on $\mathfrak{A}$. Assume that $[d]\Psi=\{d\}$ for every $d\in B$. Apparently $\langle u_i(c), p_i(a, b, c)\rangle=\langle p_i(a, a, c), p_i(a, b, c)\rangle\in\Psi$, $i=1, \ldots, n$, hold for any $\langle a, b\rangle\in\Psi$ and $c\in B$. Since $u_i(c)\in B$, $i=1, \ldots, n$, we have further $u_i(c)=p_i(a, b, c)$, $i=1, \ldots, n$. Then the hypothesis (3) gives $a=b$ proving the triviality of Ψ. Lemma 2 completes the proof.

Remark 2. Putting $u_1(z)=\ldots=u_n(z)=z$ $(u_1(z)=c_1, \ldots, u_n(z)=c_n$ for nullary operations $c_1, \ldots, c_n$ of $\mathbf{V})$ in Theorem 1 (2), (3) we immediately get the well-known Mal'cev conditions for varieties of regular (resp. weakly regular) algebras.

We have already proved

Corollary 1. *Any variety of subregular algebras is n-permutable for some* $n>1$.

Furthermore, the identities from Theorem 1 (2) yield

Corollary 2. *Any variety of subregular algebras is congruence modular.*

Proof. Define 4-ary polynomials $m_0, \ldots, m_{2n+1}$ by $m_0(x, y, z, w)=x$, $m_{2i-1}(x, y, z, w)=s_i(x, w, w, u_i(w))$ $(1\leqq i\leqq n)$, $m_{2i}(x, y, z, w)=s_i(x, w, w, p_i(y, z, w))$ $(1\leqq i\leqq n)$ and $m_{2n+1}(x, y, z, w)=w$. Then $m_{2i-1}(x, y, y, w)=s_i(x, w, w, u_i(w))=$ $=s_i(x, w, w, p_i(y, y, w))=m_{2i}(x, y, y, w)$ $(1\leqq i\leqq n)$, $m_0(x, x, w, w)=x=$ $=s_1(x, w, w, u_1(w))=m_1(x, x, w, w)$, $m_{2i}(x, x, w, w)=s_i(x, w, w, p_i(x, w, w))=$ $=s_{i+1}(x, w, w, u_{i+1}(w))=m_{2i+1}(x, x, w, w)$ $(1\leqq i<n)$, $m_{2n}(x, x, w, w)=$ $=s_n(x, w, w, p_n(x, w, w))=w=m_{2n+1}(x, x, w, w)$ and $m_j(x, y, y, x)=x$, $0\leqq j\leqq 2n+1$, since $x=s_1(x, x, x, u_1(x))=\ldots=s_n(x, x, x, u_n(x))$. Thus $m_0, \ldots, m_{2n+1}$ are the Day polynomials and the desired result follows from [3] (see also [8, p. 355]).

As we have already seen in Corollary 1, the subregularity of congruences implies the n-permutability of a given variety. Considering further the subregularity of tolerances or the subregularity of compatible reflexive relations something more can be stated. Simultaneously, an application of general compatible relations enable us to derive Mal'cev condition for permutable varieties of subregular algebras in a very simple form. First it will be convenient to make

Definition 2. An algebra $\mathfrak{A}$ is said to have *subregular tolerances (subregular compatible reflexive relations)* if every tolerance (compatible reflexive relation, respectively) T on $\mathfrak{A}$ is uniquely determined by subsets $\{x \in A: \langle b, x\rangle \in T\}$, $b \in B$, for any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$.

It is a routine to paraphrase Lemma 1 for tolerance and for compatible reflexive relations. On the other hand Lemma 2 is redundant for the proof of our Theorem 2.

Theorem 2. *For a variety* $\mathbf{V}$ *the following conditions are equivalent:*

(1) *every* $\mathfrak{A} \in \mathbf{V}$ *has permutable and subregular congruences;*

(2) *every* $\mathfrak{A} \in \mathbf{V}$ *has subregular compatible reflexive relations;*

(3) *every* $\mathfrak{A} \in \mathbf{V}$ *has subregular tolerances;*

(4) *there exist unary polynomials* $u_1, \ldots, u_n$, *ternary polynomials* $p_1, \ldots, p_n$ *and* $(3+n)$*-ary polynomial* s *such that*

$$x = s(x, y, z, u_1(z), \ldots, u_n(z))$$

$$y = s(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z))$$

$$u_i(z) = p_i(x, x, z), \quad 1 \leqq i \leqq n,$$

hold in $\mathbf{V}$.

Proof. The implication (1)$\Rightarrow$(2) is a direct consequence of Werner's theorem [14]; (2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(4): Analogously as in the proof of Theorem 1 we take $\mathfrak{A} = \mathfrak{F}_{\mathbf{V}}(x, y, z)$, $\mathfrak{B} = \mathfrak{F}_{\mathbf{V}}(z)$ and $T(x, y)$ the smallest tolerance on $\mathfrak{A}$ containing the pair $\langle x, y\rangle$. Then the hypothesis (3) (here a modified version of Lemma 1 is used) gives

$$(**) \qquad T(x, y) = T(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle)$$

for some elements $u_1, \ldots, u_n \in B$ and $p_1, \ldots, p_n \in A$. Consequently there is a $2n$-ary algebraic function σ over $\mathfrak{A}$ such that

$$x = \sigma(u_1, \ldots, u_n, p_1, \ldots, p_n),$$

$$y = \sigma(p_1, \ldots, p_n, u_1, \ldots, u_n).$$

Since $\mathfrak{A} = \mathfrak{F}_{\mathbf{V}}(x, y, z)$ and $\mathfrak{B} = \mathfrak{F}_{\mathbf{V}}(z)$, the above two equalities can be expressed as

$$x = s(x, y, z, u_1(z), \ldots, u_n(z), p_1(x, y, z), \ldots, p_n(x, y, z)),$$

$$y = s(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z), u_1(z), \ldots, u_n(z))$$

for some $(3+2n)$-ary polynomial s, unary polynomials $u_1, \ldots, u_n$ and ternary polynomials $p_1, \ldots, p_n$. Identities $u_i(z) = p_i(x, x, z)$, $1 \leqq i \leqq n$, follow directly from $(**)$. From all the above identities one gets Mal'cev polynomial p by $p(x, y, z) = s(x, z, z, p_1(y, z, z), \ldots, p_n(y, z, z), p_1(x, y, z), \ldots, p_n(x, y, z))$. Hence $\mathbf{V}$ is permutable and, again by [14], tolerances can be replaced by compatible reflexive relations in (* *). Then

$$x = s(x, y, z, u_1(z), \dots, u_n(z)),$$
$$y = s(x, y, z, p_1(x, y, z), \dots, p_n(x, y, z)),$$
$$u_i(z) = p_i(x, x, z), \quad 1 \leqq i \leqq n,$$

as required.

(4)⇒(1): Ternary polynomials $p_1, \dots, p_n$ satisfy condition (3) from Theorem 1, i.e. every algebra in **V** has subregular congruences. Permutability of **V** is ensured by Mal'cev polynomial $p(x, y, z) = s(x, z, z, p_1(y, z, z), \dots, p_n(y, z, z))$.

Remark 3. We have just proved that congruence permutability is implicit in subregularity of tolerances or in subregularity of compatible reflexive relations. The same phenomenon holds for regularity and weak regularity, see the earlier paper [6].

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