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On some generalizations of Boolean algebras

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0. We shall consider only lattices and algebras of the type $\tau_0=(2, 2, 1)$ with fundamental operations $+$, $\cdot$, $'$, where $+$ and $\cdot$ are binary and $'$ is unary. Algebras of type τ_0 are often studied mainly as generalizations of Boolean algebras, e.g. pseudocomplemented lattices, Stone algebras (see [1], [3]—[6]).

In [4] we introduced the notion of a locally Boolean algebra as follows. An algebra $(A; +, \cdot, ')$ is called a locally Boolean algebra if $(A; +, \cdot)$ is a distributive lattice and there exists a congruence R of $(A; +, \cdot, ')$ such that any congruence class of R is a Boolean algebra with respect to the operations $+$, $\cdot$, and $'$ restricted to this class.

We use a similar idea in this paper. In Section 1 we introduce a special congruence $\sim$ in a lattice $\mathfrak{A}=(A; +, \cdot)$ and by means of it we construct a new algebra $\mathfrak{A}_\sim$ of type (2, 2, 1). We show that all algebras $\mathfrak{A}_\sim$ form a variety (Theorem 1). In Section 2 we prove that if $\mathfrak{A}$ is distributive then it is isomorphic to a subdirect product of a Stone algebra and a distributive lattice with an additional constant operation $'$ whose value is the greatest element to this lattice.

1. Let $\mathfrak{A}=(A; +, \cdot)$ be a lattice. A congruence $\sim$ of $\mathfrak{A}$ will be called a b.u.-congruence of $\mathfrak{A}$ if it satisfies the following conditions (a)—(c):

(a) $\mathfrak{A}/\sim$ is a Boolean lattice;

(b) in any congruence class $[x]$ of $\sim$ there exists a greatest element $u([x])$;

(c) for any $x, y\in A$ we have:

$$u([x]+[y]) = u([x])+u([y]), \quad u([x]\cdot[y]) = u([x])\cdot u([y]).$$

Example 1. If $\mathfrak{A}$ is a finite chain then any congruence of it having two congruence classes is a b.u.-congruence. In fact a congruence class of a lattice must be convex.

If a lattice $\mathfrak{A}$ has a b.u.-congruence $\sim$ then we can define a new algebra $\mathfrak{A}_\sim$ of type τ_0 by putting $\mathfrak{A}_\sim=(A; +, \cdot, ')$ where the operations $+$ and $\cdot$ coincide in $\mathfrak{A}$

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and $\mathfrak{A}_\sim$ and the operation $'$ is defined by the formula $x'=u([x]^0)$ where $[x]^0$ is the complement of the congruence class $[x]$ in the lattice $\mathfrak{A}/\sim$.

We have

(i) any b.u.-congruence $\sim$ of a lattice $\mathfrak{A}$ is a congruence of $\mathfrak{A}_\sim$ such that $\mathfrak{A}_\sim/\sim$ is a Boolean algebra.

Lemma 1. *Any algebra* $\mathfrak{A}_\sim=(A; +, \cdot, ')$ *satisfies the following system of identities:*

(1) $$x+x=x, \quad x\cdot x=x,$$

(2) $$x+y=y+x, \quad x\cdot y=y\cdot x,$$

(3) $$(x+y)+z=x+(y+z), \quad (x\cdot y)\cdot z=x\cdot(y\cdot z),$$

(4) $$x\cdot(x+y)=x=x+(x\cdot y),$$

(5) $$((x')')'=x',$$

(6) $$(x+y)'=x'\cdot y', \quad (x\cdot y)'=x'+y',$$

(7) $$x+(x')'=(x')',$$

(8) $$x'+(x')'=y'+(y')',$$

(9) $$x'+(y'\cdot z')=(x'+y')\cdot(x'+z').$$

Proof. The proof follows easily from (a)—(c). We prove for example (8). Let us denote $x''=(x')'$. Let $x\in A$. Then

$$x'+x''=u([x]^0)+u([x']^0)=u([x]^0+[x']^0)=u([x]^0+[u([x]^0)]^0)=$$
$$=u([x]^0+([x]^0)^0)=u([x]^0+[x]).$$

But the element $u([x]^0+[x])$ is the greatest element of the greatest class of $\mathfrak{A}$ so it is fixed and consequently (8) holds.

Lemma 2. *Let* $\mathfrak{A}=(A; +, \cdot, ')$ *be an algebra satisfying* (1)—(9). *Then there exists a b.u.-congruence relation* $\sim$ *in the lattice* $(A; +, \cdot)$ *such that* $\mathfrak{A}$ *is identical with the algebra* $(A; +, \cdot)_\sim$.

Proof. Let us put for $x, y\in A$

$$x\sim y \Leftrightarrow x'=y'.$$

Obviously $\sim$ is an equivalence. If $a_1\sim a_2$ and $b_1\sim b_2$ then by (6) we have $(a_1+b_1)'=$ $=a_1'\cdot b_1'=a_2'\cdot b_2'=(a_2+b_2)'$, so $\sim$ satisfies the substitution law for $+$. Analogously $\sim$ satisfies the substitution law for $\cdot$ and $'$, so $\sim$ is a congruence in $\mathfrak{A}$ and consequently in $(A; +, \cdot)$.

To prove (a) it is enough to show that $\mathfrak{A}/\sim$ is a Boolean algebra. However by (5) we have $x''\sim x$ for any $x\in A$, so the identity $x''=x$ holds in $\mathfrak{A}/\sim$. By

(6), (5) and (8) we have

$$(x+x')' = (x'+x'')' = (y'+y'')' = (y+y')'$$

for any $x, y \in A$. So the identity $x+x'=y+y'$ holds in $\mathfrak{A}/\sim$. By (6) and (9) the distributive law

$$x\cdot(y+z) = x\cdot y+x\cdot z \tag{10}$$

holds in $\mathfrak{A}/\sim$, so $\mathfrak{A}/\sim$ is a Boolean algebra.

To prove (b) we shall show that the element x'' is the greatest element in the class $[x]$. We have already shown above that $x''\sim x$ for any $x\in A$, so $x''\in[x]$. If $x\sim y$ then $x'=y'$ and $x''=y''$. Now by (7) $x''=u([x])$.

The condition (c) follows at once from (5) and (6).

Finally $u([x]^0)=u([x'])=(x')''=x'$, so the operations $'$ in $(A; +, \cdot)_\sim$ and $\mathfrak{A}$ coincide.

Let us denote by L^* the class of all algebras of the form $\mathfrak{A}_\sim$ for some lattice $\mathfrak{A}$ and a b.u.-congruence $\sim$ of $\mathfrak{A}$. By Lemmas 1 and 2 we have

Theorem 1. *The class L^* is a variety defined by the identities* (1)—(9).

Let us denote by D^* the class of all algebras $\mathfrak{A}_\sim$ where $\mathfrak{A}$ is a distributive lattice.

Corollary 1. *The class D^* is a variety defined by the identities* (1)—(8) *and* (10).

This follows from Lemmas 1 and 2.

2. Let us denote by L_1 the variety of algebras of type τ_0 satisfying (1)—(4) and the following two identities:

$$x+y' = x', \tag{11}$$

$$x' = y'. \tag{12}$$

We denote by D_1 the variety of algebras of type τ_0 defined by (1)—(4), (11), (12) and (10). Thus the algebras from L_1 and D_1 are lattices with unit defined by an additional operation $'$.

The construction of algebras $\mathfrak{A}_\sim$ can suggest that any algebra from L^* is isomorphic to a subdirect product of a Boolean algebra and an algebra from L_1. This however is not true even for the variety D^* as it is shown by the following example.

Example 2. Let us consider an algebra $\mathfrak{B}=(\{a, b, c\}; +, \cdot, ')$ where $(\{a, b, c\}; +, \cdot)$ is a lattice in which $a<b<c$ and $a'=c$, $b'=c'=a$. Then the equivalence relation $\sim$ with two classes $\{a\}$ and $\{b, c\}$ is a b.u.-congruence in the lattice $(\{a, b, c\}; +, \cdot)$ such that $\mathfrak{B}=(\{a, b, c\}; +, \cdot)_\sim$ (see the definition of $\sim$ in Lemma 2). However $\mathfrak{B}$ neither is a Boolean algebra nor belongs to D_1, and it is subdirectly irreducible since $\mathfrak{B}$ is a subdirectly irreducible Stone algebra (see [3]).

This example is not accidental. In fact, the next theorem shows that for algebras from D^* we always have a subdirect decomposition.

Let B_1 denote the variety of Stone algebras of type τ_0 (see [1]). We have that (ii) the identities (1)—(8), (10) and

$$x \cdot x' = y \cdot y' \tag{13}$$

form an equational base for the variety of Stone algebras.

In fact the identity (13) together with the identities $x \cdot (x \cdot x')' = x$, $(x \cdot x')'' = x \cdot x'$, $x \cdot (x \cdot y)' = x \cdot y'$, $x' + x'' = (x \cdot x')'$ form an equational base for B_1. Using subdirectly irreducible algebras from B_1 (see [3]) it is easy to check that these two systems of identities are equivalent.

For a variety V of algebras of type τ_0 we denote by $\mathrm{Id}\,(V)$ the set of all identities of type τ_0 satisfied in V. For two varieties V_1 and V_2 we denote by $V_1 \vee V_2$ the join of V_1 and V_2, and by $V_1 \otimes V_2$ the class of all algebras isomorphic to a subdirect product of two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ where $\mathfrak{A}_1 \in V_1$ and $\mathfrak{A}_2 \in V_2$.

Let $\mathfrak{A} = (A; +, \cdot, ')$ be an algebra of type τ_0.

Theorem 2. *The following four conditions are equivalent:*

(1°) $\mathfrak{A} \in D^*$,

(2°) $\mathfrak{A} \in B_1 \otimes D_1$,

(3°) $\mathfrak{A} \in B_1 \vee D_1$,

(4°) $\mathfrak{A}$ *satisfies the identities* (1)—(10).

To prove Theorem 2 we need some lemmas. In the next six lemmas we assume that the algebra $\mathfrak{A} = (A; +, ')$ belongs to D^*, so it satisfies (1)—(10) by Corollary 1.

Lemma 3. $\mathfrak{A}$ *satisfies the following identities:*

$$x' \cdot x'' = y' \cdot y'', \tag{14}$$

$$x \cdot x' = x \cdot x' \cdot x'', \tag{15}$$

$$(x+y)(x+y)' = xx' + yy', \tag{16}$$

$$(x \cdot y)(x \cdot y)' = xx' \cdot yy'. \tag{17}$$

Proof. By (6), (5), (3) and (8) we have $x'x'' = (x'' + x')' = (y'' + y')' = y'y''$. By (7) and (2) we have $xx' = xx''x' = xx'x''$. By (14) we can denote by e the constant element of A with $e = x'x''$ for any $x \in A$. By (15) and (10) we have

$$(x+y)(x+y)' = (x+y)(x+y)'(x+y)'' = (x+y) \cdot e = x \cdot e + y \cdot e =$$
$$= xx'x'' + yy'y'' = xx' + yy'.$$

Finally

$$(x \cdot y)(x \cdot y)' = (x \cdot y)(x \cdot y)'(x \cdot y)'' = xye = xyee = xe \cdot ye = xx' \cdot yy'.$$

We define in $\mathfrak{A}$ two relations R_1 and R_2 by putting for $a, b \in A$

$$aR_1 b \Leftrightarrow a + a'a'' = b + b'b'', \quad aR_2 b \Leftrightarrow aa' = bb'.$$

Lemma 4. *The relation R_1 is a congruence in $\mathfrak{A}$.*

Proof. Obviously R_1 is an equivalence. If aR_1a_1 and bR_1b_1 then $(a+b)+(a+b)'(a+b)''=(a+b)+e=a+e+b+e=a_1+e+b_1+e=(a_1+b_1)+e=(a_1+b_1)+(a_1+b_1)' \cdot (a_1+b_1)''$. So R_1 satisfies the substitution law for $+$. To show the substitution law for $\cdot$ we use the distributivity of $+$ with respect of $\cdot$. If aR_1b then by (6) and (4) we have $(a+a'a'')'=(b+b'b'')'$, hence $a'(a'+a'')=b'(b'+b'')$, so $a'=b'$, and consequently $a'R_1b'$.

Lemma 5. *R_2 is a congruence of $\mathfrak{A}$.*

Proof. Obviously R_2 is an equivalence. The substitution law for $+$, for $\cdot$ and for $'$ follows at once from (16), (17) and (14), respectively.

Lemma 6. *$R_1 \cap R_2 = \omega$ where ω is the diagonal.*

Proof. If aR_1b and aR_2b then

$$a = a + aa' = a + (a \cdot a'') \cdot a' = a + (aa') \cdot (a'a'') = a + (bb')(b'b'') =$$
$$= a + bb'b'' = a + be = (a+b)(a+e) = (a+b) \cdot (a+e)(a+e) =$$
$$= (a+b)(a+e) \cdot (b+e).$$

Analogously, we can prove that $b=(b+a) \cdot (b+e) \cdot (a+e)$. So by (2) $a=b$.

Lemma 7. *$\mathfrak{A}/R_1$ is a Stone algebra.*

Proof. We shall show that for any $x, y \in A$ we have $(x \cdot x')R_1(y \cdot y')$. In fact

$$(xx') + (xx')'(xx')'' = xx' + (x'+x'')x'x'' = xx' + x'x'' =$$
$$= x'(x+x'') = x'x'' = e.$$

Analogously $(yy')+(yy')'(yy')''=e$. So $xx'R_1yy'$. Thus the algebra $\mathfrak{A}/R_1$ satisfies (13) and by (ii) it is a Stone algebra.

Lemma 8. *$\mathfrak{A}/R_2$ belongs to D_1.*

Proof. By (11) and (12) we have to prove that for any $x \in A$ we have $(x+x')R_2x'$ and for any $x, y \in A$ we have $x'R_2y'$. In fact

$$(x+x')(x+x')' = (x+x') \cdot x'x'' = xx'x'' + x'x'' = x'x'' = x'(x')'.$$

Further $x'(x')'=e=y'(y')'$.

Proof of Theorem 2. By Corollary 1 condition (1°) is equivalent to (4°). Obviously $B_1 \otimes D_1 \subset B_1 \vee D_1$, so (2°)⇒(3°). Further $B_1 \vee D_1 \subset D^*$ since each of the identities (1)—(10) belongs to $\mathrm{Id}(B_1)$ by (ii), and each of the identities (1)—(10) belongs to $\mathrm{Id}(D_1)$. So any of (1)—(10) belongs to $\mathrm{Id}(B_1) \cap \mathrm{Id}(D_1)$. Thus (3°)⇒(4°). To complete the proof it is enough to show that (1°)⇒(2°) i.e. any algebra $\mathfrak{A} \in D^*$ is isomorphic to a subdirect product of a Stone algebra and an algebra from D_1. However, this follows from Lemmas 4—8 and the decomposition theorem (see [2], Theorem 2, p. 123).

Remark 1. The distributive law (10) in Theorem 2 is an essential assumption, i.e. we cannot omit this identity in condition (4°) and substitute D^* by L^* and D_1 by L_1 in conditions (1°)—(3°).

In fact, we have the following:

(iii) the variety L^* is essentially larger than the variety $B_1 \vee L_1$.

Indeed, by Theorem 2 we have $B_1 \subset D^* \subset L^*$. Further $\mathrm{Id}(L^*) \subset \mathrm{Id}(L_1)$, as it is easy to check. So $L_1 \subset L^*$, and consequently $B_1 \vee L_1 \subset L^*$. Let us take the lattice $N_5 = (\{a, b, c, 0, 1\}; +, \cdot)$ where $0 < a < b < 1$, $0 < c < 1$, the elements a and c are incomparable and the elements b and c are incomparable. We consider in N_5 an equivalence $\sim$ with two equivalence classes $\{0, a, b\}$ and $\{c, 1\}$. Obviously $\sim$ is a b.u.-congruence in N_5 where $u(\{0, a, b\}) = b$ and $u(\{c, 1\}) = 1$. Hence the algebra $(N_5)_\sim$ belongs to L^*. However $(N_5)_\sim$ does not belong to $B_1 \vee L_1$, as the identity

$$\text{(d)} \qquad x + y \cdot y' = (x+y) \cdot (x+y')$$

belongs to $\mathrm{Id}(B_1) \cap \mathrm{Id}(L_1) = \mathrm{Id}(B_1 \vee L_1)$, while $(N_5)_\sim$ does not satisfy (d) since $a + c \cdot c' = a + c \cdot u(\{0, a, b\}) = a + c \cdot b = a$ and $(a+c) \cdot (a+c') = (a+c) \cdot (a+b) = b$.

Let B_0 denote the variety of Boolean algebras of type τ_0.

Corollary 2. *$B_0 \vee D_1 = B_0 \otimes D_1$ and $B_0 \vee D_1$ is defined by the identities* (1)—(10) *and*

$$\text{(18)} \qquad x + x' = y + y'.$$

Proof. Let us denote by K the variety of algebras of type τ_0 defined by (1)—(10) and (18). Obviously $B_0 \otimes D_1 \subset B_0 \vee D_1$ and $B_0 \vee D_1 \subset K$, since $\mathrm{Id}(K) \subset \subset (\mathrm{Id}(B_0) \cap \mathrm{Id}(D_1))$. If $\mathfrak{A} \in K$ then $\mathfrak{A} \in D^*$, since $\mathrm{Id}(D^*) \subset \mathrm{Id}(K)$. So by Theorem 2 $\mathfrak{A}$ is isomorphic to a subdirect product of two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with $\mathfrak{A}_1 \in B_1$ and $\mathfrak{A}_2 \in D_1$. But $\mathfrak{A}$ satisfies (18), so also $\mathfrak{A}_1$ does. Thus $\mathfrak{A}_1$ satisfies (1)—(10), (13) and (18), whence it is easy to show that $\mathfrak{A}_1$ is a Boolean algebra. This completes the proof.

Example 3. Let X be a set. Put $Y = \{\langle A, B \rangle : A, B \in 2^X, A \subset B\}$. We define an algebra $\mathfrak{B}_0$ of type τ_0 by putting $\mathfrak{B}_0 = (Y; +, \cdot, ')$ where $+ = \cup$, $\cdot = \cap$ and $(\langle A, B \rangle)' = \langle X \setminus A, X \rangle$.

By Corollary 2 and Theorem 2 $\mathfrak{A}$ belongs to D^* since it is a subdirect product of a Boolean algebra $\mathfrak{A}_1=(2^X; \cup, \cap, ')$ and an algebra $\mathfrak{A}_2=(2^X; \cup, \cap, ')$ where $Z'=X$ for any $Z\subset X$. If $|X|=1$ then $\mathfrak{B}_0$ has only 3 elements: $\langle\emptyset; \emptyset\rangle$, $\langle\emptyset, X\rangle$ and $\langle X, X\rangle$. But $\mathfrak{B}_0$ neither is a Stone algebra nor belongs to D_1. So $\mathfrak{B}_0$ is not a direct product of a Stone algebra and an algebra from D_1. This shows that Theorem 2 cannot be strengthed to direct product.

Remark 2. We can obtain results dual to those of this paper by assuming the existence of a least element $o([x])$ in (b), and by substituting u by o in (c). Then (7) must be substituted by $x+x''=x$, and so on.

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