Acta Sci. Math., 51 (1987), 343—348

Square subgroup of an abelian group

A. M. AGHDAM

Given an abelian group G, we call R a ring over G if the additive group $R^+=G$. In this situation we write $R=(G, *)$, where $*$ denotes the ring multiplication. The multiplication is not assumed to be associative. Every group G can be provided with a ring structure in a trivial way, by defining all products to be 0; such a ring is called a zero-ring. In general, we call a group G a *nil group* if there is no ring on G other than the zero-ring.

Suppose that H is a subgroup of G. G is called *nil modulo H* if $G*G \leqq H$ for every ring $(G, *)$ on G. It is clear that if G is nil modulo both H_1 and H_2 then G is nil modulo $H_1 \cap H_2$, this suggests the following definition of the *square subgroup* $\Box G$ of G:

$$\Box G = \bigcap \{H \leqq G \mid G \text{ is nil modulo } H\}.$$

Clearly $\Box G$ is the smallest subgroup with the property that G is nil modulo $\Box G$. For the first time the square subgroup was studied in [1] by A. E. STRATTON and M. C. WEBB. The basic question about the square subgroup is whether $\frac{G}{\Box G}$ is a nil group? If this is not true in general then under what conditions it is true and why it fails?

In this note we are investigating the square subgroup of an abelian group. We will show that the square subgroup of a torsion reduced group is equal to itself and we will prove that

$$\frac{G}{\Box G} \cong \frac{D}{T} \oplus \frac{N}{\Box N}, \quad \Box D \leqq T \leqq D,$$

where D is the maximal divisible subgroup of G and N is the reduced part of G; also, if G is a non-torsion group then

$$\frac{G}{\Box G} \cong \frac{N}{\Box N}.$$

Received August 29, 1983, and in revised form January 29, 1985.

By an example we will show that the square subgroup of a torsion-free group, in general, is not a direct summand of the group.

All groups considered in this paper are abelian, with addition as the group operation.

Proposition 1. *If G is cyclic (finite or infinite) then* $\square G=G$.

Proof. Let $\langle x\rangle$ be a cyclic group, define a ring on $\langle x\rangle$ by $(mx)(nx)=mnx$. In this ring x is the neutral element of $\langle x\rangle$, so, $\langle x\rangle=\langle x\rangle^2$ and hence

$$\square\langle x\rangle = \langle x\rangle.$$

Proposition 2. $A=B\oplus H$ *implies that* $\square B\leqq\square A$.

Proof. Suppose that there is a ring $(B, *)$ over B. We can define a ring $(A, \circ)$ by putting

$$(b+h)\circ(b'+h') = b*b',$$

this implies that $A\circ A=B*B$, hence $\square B\leqq\square A$.

Theorem 3 ([2], page 288). *A p-group G is a nil group if and only if it is divisible.*

Theorem 4 ([2], page 287). *A multiplication μ on a p-group A is completely determined by the values $\mu(a_i, a_j)$ with a_i, a_j running over a p-basis of A. Moreover, any choice of $\mu(a_i, a_j)\in A$ with a_i, a_j from a p-basis of A subject to the sole condition that*

$$o(\mu(a_i, a_j)) \leqq \min(o(a_i), o(a_j))$$

extends to a multiplication on A.

Lemma 5. *The reduced part of a p-group G has unbounded order if and only if any p-basic subgroup of G has unbounded order.*

Proof. Let $G=D\oplus N$, D is the maximal divisible subgroup of G. Let B be a p-basic subgroup of N. If B has bounded order then B is a direct summand of G, hence $G=D\oplus B\oplus N'$ and $\frac{G}{B}\cong D\oplus N'$; by the definition of B, N' should be divisible, a contradiction, that is $N'=0$. Consequently $N=B$ and is of bounded order. This concludes that N has unbounded order if and only if B has unbounded order.

Lemma 6. *Let G be a p-group. If the reduced part of G has unbounded order then* $\square G=G$.

Proof. Suppose that G is a p-group and the reduced part of G has unbounded order. Let $B=\bigoplus_{i\in I}\langle a_i\rangle$ be a p-basic subgroup of G. Let g be an arbitrary element of G with $o(g)=p^n$. By Lemma 5, B has unbounded order, hence there is a_K such that $o(a_K)>p^n$. In accordance with Theorem 4, a multiplication μ on G is uniquely determined if we put

$$\mu(a_i, a_j) = \begin{cases} 0 & \text{if either } \ i \neq K \ \text{ or } \ j \neq K, \\ g & \text{if } \ j = i = K; \end{cases}$$

hence $g\in \square G$, that is, $\square G=G$.

Lemma 7. *Let G be a reduced p-group, then* $\square G=G$.

Proof. If G has bounded order then $G=\bigoplus_{i\in A}\langle x_i\rangle$, and by Propositions 1, 2, $\square G=G$. If G has unbounded order then, by Lemma 6, $\square G=G$.

Theorem 8. *Let G be a reduced-torsion group, then* $\square G=G$.

Proof. $G=\bigoplus_p G_p$, G_p is a p-group. If G is reduced then G_p is reduced for all prime p. By Lemma 7 $\square G_p=G_p$. Therefore $\square G=G$.

Remark 1. Let G be a group. Let $R=(G, \eta)$ be a ring on G, then $\eta\in\mathrm{Hom}\,(G\otimes G, G)$ and $\eta(g_1\otimes g_2)=g_1g_2$, that is, $G^2=\mathrm{Im}\,\eta$, therefore

$$\square G = \langle \mathrm{Im}\,\eta \mid \eta\in\mathrm{Hom}\,(G\otimes G, G)\rangle.$$

Note. $A\otimes B$ means the tensor product of A and B.

Proposition 9. *Let G be a non-torsion group, then*

$$\langle \mathrm{Im}\,\theta \mid \theta\in\mathrm{Hom}\,(G, Z(p^\infty))\rangle = Z(p^\infty).$$

Proof. $Z(p^\infty)=\langle c_1, c_2, \ldots, c_n, \ldots \mid pc_1=0,\ pc_2=c_1, \ldots, pc_n=c_{n-1}, \ldots\rangle$. Let x be in G and the order of x be infinity, then the map $f(n)=nx$ ($n\in Z$, the set of integer numbers) defines a short exact sequence:

$$0 \to Z \xrightarrow{f} G \to M \to 0$$

which induces the short exact sequence:

$$0 \to \mathrm{Hom}\,(M, Z(p^\infty)) \to \mathrm{Hom}\,(G, Z(p^\infty)) \xrightarrow{f^*} \mathrm{Hom}\,(Z, Z(p^\infty)) \to 0,$$

the sequence being right exact because $\mathrm{Ext}\,(M, Z(p^\infty))=0$.

The definition of the map f^* is given by $f^*(\theta)=\theta f$ for all $\theta\in\mathrm{Hom}\,(G, Z(p^\infty))$. Now given $y\in Z(p^\infty)$ there is $\eta\in\mathrm{Hom}\,(Z, Z(p^\infty))$ such that $\eta(1)=y$. Since f^* is epic there is $\theta\in\mathrm{Hom}\,(G, Z(p^\infty))$ such that $f^*(\theta)=\eta$, hence $y=\eta(1)=(f^*(\theta))(1)=$ $=\theta(f(1))$, yielding the result.

Theorem 10. *Let G be a group, $G=D\oplus N$ where D is the maximal divisible subgroup of G. Then* $\frac{G}{\Box G}\cong\frac{D}{T}\oplus\frac{N}{\Box N}$, *where $\Box D\leqq T\leqq D$. If G is a non-torsion group, then* $\frac{G}{\Box G}\cong\frac{N}{\Box N}$.

Proof. $G\otimes G\cong(D\otimes D)\oplus(D\otimes N)\oplus(N\otimes D)\oplus(N\otimes N)$. Since N is reduced and $D\otimes D$, $D\otimes N$, $N\otimes D$ are divisible, $\mathrm{Hom}\,(D\otimes N, N)=\mathrm{Hom}\,(N\otimes D, N)=$ $=\mathrm{Hom}\,(D\otimes D, N)=0$. Hence

$$
\begin{aligned}
\mathrm{Hom}\,(G\otimes G, G) \cong{} & \mathrm{Hom}\,(D\otimes D, D)\oplus\mathrm{Hom}\,(D\otimes N, D)\\
& \oplus\mathrm{Hom}\,(N\otimes D, D)\oplus\mathrm{Hom}\,(N\otimes N, N)\\
& \oplus\mathrm{Hom}\,(N\otimes N, D).
\end{aligned}
\tag{1}
$$

So, by remark (1), $\Box G=T\oplus\Box N$ where $\Box D\leqq T\leqq D$. This implies $\frac{G}{\Box G}\cong\frac{D}{T}\oplus$ $\oplus\frac{N}{\Box N}$.

Suppose that G is a non-torsion group. If the group of rational numbers is a subgroup of D, then $D=H\oplus K$, where H is a direct sum of the groups of rational numbers and K is a direct sum of quasicyclic groups. Hence $D\otimes D=H\otimes H$ is a direct sum of the groups of rational numbers.

$$\mathrm{Hom}\,(D\otimes D, D)\cong\mathrm{Hom}\,(H\otimes H, H)\oplus\mathrm{Hom}\,(H\otimes H, K),$$

because of $\langle\mathrm{Im}\,\theta|\theta\in\mathrm{Hom}\,(Q, Q)\rangle=Q$ (Q is the group of rational numbers), by Proposition 9 and Remark 1 $\Box D=D$ and $\Box G=D\oplus\Box N$.

If D is a torsion group, then D is a direct sum of quasicyclic groups and N is a non-torsion group, hence $N\otimes N$ is non-torsion, too. By Proposition 9 $\langle\mathrm{Im}\,\eta|\eta\in\mathrm{Hom}\,(N\otimes N, D)\rangle=D$.

Consequently by (1) $\Box G=D\oplus\Box N$, this concludes that

$$\frac{G}{\Box G}\cong\frac{N}{\Box N}. \tag{2}$$

Remark 2. Let $G=Z(p^\infty)\oplus Z(p)$, then we have $G\otimes G\cong Z(p)\otimes Z(p)\cong Z(p)$. $\mathrm{Hom}\,(G\otimes G, G)\cong\mathrm{Hom}\,(Z(p)\otimes Z(p), Z(p^\infty))\oplus\mathrm{Hom}\,(Z(p)\otimes Z(p), Z(p))$. By remark (1) $\Box G=\langle c_1\rangle\oplus Z(p)$. We deduce that $\Box G$ is not a pure subgroup of G, consequently $\Box G$ is not a direct summand of G. $\frac{G}{\Box G}\cong\frac{Z(p^\infty)}{\langle c_1\rangle}$, that is (2) is not true in general when G is a torsion group. But $\frac{G}{\Box G}$ is a nil group.

The following example shows that the square subgroup of a torsion-free group, in general, is not a direct summand.

Example. Let A be the subgroup of $Qx_1 \oplus Qx_2$ generated by the set

$$\left\{\frac{1}{p}x_1, \frac{1}{p^2}x_1+\frac{1}{p^5}x_2 \,\middle|\, p \text{ is running over } \pi\right\}$$

where π is the set of all prime numbers. $\frac{1}{p}x_1 \in A$ implies $x_1 = p\left(\frac{1}{p}x_1\right) \in A$, hence,

(3) $$h_p(x_1) \geqq 1 \quad \text{for all} \quad p \in \pi.$$

Suppose that $x_1 \in q^2 A$ for some prime q, then

$$x_1 = q^2 \sum_{p\in\psi} \left[\frac{\alpha_p}{p}x_1 + \beta_p\left(\frac{1}{p^2}x_1+\frac{1}{p^5}x_2\right)\right],$$

where ψ is a finite set of prime numbers;

$$x_1 = q^2\left[\sum_{p\in\psi}\left(\frac{\alpha_p}{p}+\frac{\beta_p}{p^2}\right)x_1 + \sum_{p\in\psi}\frac{\beta_p}{p^5}x_2\right].$$

Since $\{x_1, x_2\}$ is an independent set of A, $\sum_{p\in\psi}\frac{\beta_p}{p^5}=0$, this implies

(4) $$\beta_p \equiv 0 \pmod p \quad \text{for all} \quad p \in \psi.$$

We deduce $1 = q^2 \sum_{p\in\psi}\left(\frac{\alpha_p}{p}+\frac{\beta_p}{p^2}\right)$, this implies $q \in \psi$, so, by (4)

(5) $$\beta_q \equiv 0 \pmod q.$$

Let $\psi^0 = \psi - \{q\}$, then

$$1 = q^2\left(\frac{\alpha_q}{q}+\frac{\beta_q}{q^2}\right) + q^2\sum_{p\in\psi^0}\left(\frac{\alpha_p}{p}+\frac{\beta_p}{p^2}\right) = q\alpha_q + \beta_q + q^2\sum_{p\in\psi^0}\left(\frac{\alpha_p}{p}+\frac{\beta_p}{p^2}\right),$$

this implies $\sum_{p\in\psi^0}\left(\frac{\alpha_p}{p}+\frac{\beta_p}{p^2}\right)$ is an integer, therefore $\beta_q \equiv 1 \pmod q$ a contradiction by (5). Consequently $x_1 \notin q^2 A$. By (3) $h_p(x_1)=1$ for all $p \in \pi$. Hence, $t(x_1) = (1, 1, \ldots, 1, \ldots)$. Let $Z_p = \frac{1}{p^2}x_1 + \frac{1}{p^5}x_2$ then $p^5 Z_p = p^3 x_1 + x_2$, and since $h_p(x_1)=1$, $h_p(x_2)=4$ for all $p \in \pi$. Hence $t(x_2) = (4, 4, \ldots, 4, \ldots)$ and $t(x_2) > t(x_1)$. $t(x_1)$ and $t(x_2)$ are not idempotent, that is any ring $R=(A, *)$ over A satisfies $x_1^2 = \alpha x_2$, $x_1 x_2 = x_2 x_1 = x_2^2 = 0$, α is a rational number.

Let $y=ux_1+vx_2$, $w=\beta x_1+\gamma x_2$ then, $yw=\alpha u\beta x_2$, this implies $A^2\leqq\langle x_2\rangle^*$ ($\langle x_2\rangle^*$ is a pure subgroup of A generated by x_2). Since $(A, *)$ was arbitrary,

$$\Box A \leqq \langle x_2\rangle^*. \tag{6}$$

Let $\alpha=1$, then $yw=\beta ux_2$, and by the structure of A, $\beta ux_2\in A$. Hence A is not a nil group. We claim that $A^2=\langle x_2\rangle^*$. For the proof it is enough to show that $\frac{1}{p^4}x_2\in A^2$.

Let $Z_p=\frac{1}{p^2}x_1+\frac{1}{p^5}x_2$, then $Z_p^2=\frac{1}{p^4}x_2$, so $\frac{1}{p^4}x_2\in A^2$ for all $p\in\pi$. By (6) $\Box A=\langle x_2\rangle^*$. Let $U=\{u\in Q\,|\,ux_1+vx_2\in A$ for some $v\in Q\}$. If $\Box A$ is a direct summand of A, then

$$A=\langle x_2\rangle^*\oplus B,\quad \frac{A}{\langle x_2\rangle^*}\cong B,\quad t(B)=t\left(\frac{A}{\langle x_2\rangle^*}\right)=t(U);$$

by the structure of A, $t(U)=(2, 2, \ldots, 2, \ldots)$. This implies $t(x_1)<t(B)<t(x_2)$ but this is impossible, since $r(A)=2$ ([2], page 112, Ex. 10).

Consequently $\Box A$ is not a direct summand of A.

Note. Since $\frac{A}{\Box A}$ is of rank one and its type is not idempotent, it follows that $\frac{A}{\Box A}$ is a nil group.

References

[1] A. E. Stratton, M. C. Webb, Abelian group, nil modulo a subgroup, need not have nil quotient group, *Publ. Math. Debrecen*, **27** (1980), 127—130.

[2] L. Fuchs, *Infinite Abelian Groups*, vol. 2, Academic Press (1973).

[3] A. E. Stratton, The type set of torsion-free rings of finite rank, *Comment. Math. Univ. St. Paul.*, **27** (1978), 199—211.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TABRIZ
TABRIZ, IRAN