Acta Sci. Math., 51 (1987), 349—363

Concrete characterization of partial endomorphism semigroups of graphs

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Algebraic and elementary properties of partial endomorphism semigroups of graphs were studied by L. M. POPOVA [1, 2], JU. M. VAŽENIN [3] and A. M. KALMANOVIČ [4—6]. For these semigroups it is interesting to study the concrete characterization problem [7]: under which conditions is a partial transformation semigroup E equal to the partial endomorphism semigroup $E(G)$ of some graph G?

In the present paper we investigate this problem. The necessary and sufficient conditions for a partial transformation semigroup E to be equal to the partial endomorphism semigroup $E(G)$ of some graph G will be obtained in Theorem 2. We construct all kinds of such graphs in Theorem 1. At the end of the paper we apply our results to describe (in Theorem 3) graphs with equal partial endomorphisms and to investigate (in Theorem 4) the question: how are graphs determined by their partial endomorphism semigroups? Numerous other applications of Theorems 1 and 2 are briefly stated in [8].

1. Definitions, preliminary results

Let X be an arbitrary set with $|X|>1$, and let ϱ be a binary relation on X, $x\in X$, $A\subset X$. We put $X^2=X\times X$;

$$\varrho^{-1} = \{(x, y)\colon (y, x)\in\varrho\};\quad \operatorname{dom}\varrho = \{x\colon (\exists y)(x, y)\in\varrho\};$$

$$\varrho x = \{y\colon (x, y)\in\varrho\}\quad\text{and}\quad \varrho A = \{y\colon (\exists x\in A)(x, y)\in\varrho\}.$$

A one-valued binary relation $f\subset X^2$ is called a partial transformation of X (shortly p.transformation). If $x\in\operatorname{dom}f$, then fx denotes the image of x under f. A p.transformation f is called 3-bounded, if $|\operatorname{dom}f|<3$. We write $f=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, if $\operatorname{dom}f=\{a, b\}$ and $fa=c$, $fb=d$. The Cartesian power f^2 of f is the p.trans-

Received May 27, 1983.

formation of X^2 such that $f^2(x,y)=(fx,fy)$ for $x,y\in\operatorname{dom} f$. The identity transformation of a set A is denoted by Δ_A. We denote by $W(X)$ the symmetric semigroup of all p.transformations on a set X.

By a graph we mean a structure $G=(X,\varrho)$, where X is a non-empty set and $\varrho\subset X^2$. The elements of X and ϱ are called vertices and edges, respectively. The edge $(x,x)\in\varrho$ is called a loop. We denote $G^{-1}=(X,\varrho^{-1})$.

A p.transformation f of X is called a partial endomorphism (shortly p.endomorphism) of the graph G, if $f^2\varrho\subset\varrho$, i.e. $(x,y)\in\varrho$ implies $(fx,fy)\in\varrho$ for any $x,y\in\operatorname{dom} f$. The p.endomorphisms of G form a semigroup $E(G)$ (under the composition), which is called the p.endomorphism semigroup of the graph G. $E_3(G)$ denotes the 3-bounded p.endomorphism semigroup of G.

Let E be a p.transformation semigroup on a set X. The canonical relations of E are defined by the formulas:

$$\tau \overset{\mathrm{df}}{=} \bigcup\{f\colon f\in E\};\quad \beta \overset{\mathrm{df}}{=} \bigcup\{f^2\colon f\in E\};$$

$$A \overset{\mathrm{df}}{=} \{x\in X\colon X^2\subset\beta^{-1}(x,x)\};\quad B \overset{\mathrm{df}}{=} X\setminus A;$$

$$P \overset{\mathrm{df}}{=} \Delta_B\cup\{(x,y)\in X^2\setminus\Delta_X\colon \tau x\times\tau y\subset\beta(x,y)\};$$

$$R \overset{\mathrm{df}}{=} \Delta_A\cup\{(x,y)\in X^2\setminus\Delta_X\colon (\tau^{-1}x\times\tau^{-1}y)\setminus\Delta_X\subset\beta^{-1}(x,y)\};$$

$$Z \overset{\mathrm{df}}{=} P\cap R;\quad Q' \overset{\mathrm{df}}{=} X^2\setminus(P\cup R)$$

and

$$Q \overset{\mathrm{df}}{=} ((\beta Q')\setminus R)\cup((\beta^{-1}Q')\setminus P).$$

The intersections of any binary relation σ on X with the relations A^2, $(A\times B)\cup(B\times A)$ and B^2 are denoted by the same symbol but with indices 1, 2 and 3, respectively, i.e. $\sigma_1=\sigma\cap A^2$, $\sigma_2=\sigma\cap((A\times B)\cup(B\times A))$ and $\sigma_3=\sigma\cap B^2$.

We denote $\{f\in E\colon |\operatorname{dom} f|<3\}$ by E_3.

In the following lemmas the canonical relations of p.transformation semigroups will be investigated.

Lemma 1. *If E contains all 3-bounded identity p.transformations of X, then* (i) *τ is a quasi-order*[1]) *on X,* (ii) *β is a quasi-order on X^2 and* (iii) *the canonical relations of E and E_3 are equal.*

Lemma 2. *If $Q_1=Z_1=\emptyset$ ($Q_3=Z_3=\emptyset$), then P_1 (R_3) is non-empty.*

Lemma 3. *If $\tau a=X$ for all $a\in B$ then the following conditions hold:*

(i) *if $B\neq\emptyset$ and $Z_1\neq\emptyset$ then $R_1=A^2$, $P_1=A^2\setminus\Delta_A$ and $P_2\cup Q_2\neq\emptyset$;*

(ii) *if $Z_2\neq\emptyset$ then $P_2=R_2=(A\times B)\cup(B\times A)$;*

(iii) *if $A\neq\emptyset$ and $Z_3\neq\emptyset$ then $P_3=B^2$, $R_3=B^2\setminus\Delta_B$ and $R_2\cup Q_2\neq\emptyset$.*

[1]) A quasi-order on X is a reflexive and transitive binary relation on X.

The proofs of these lemmas easily follow from the definitions.

Now suppose that $E=E(G)$ for some graph $G=(X, \varrho)$. Then the canonical relations of E satisfy the following properties.

Lemma 4. *Let G be a graph such that $E(G)\neq W(X)$. Then*

(i) $X^2\setminus(\varrho\cup\varrho^{-1})\subset P$;

(ii) $\varrho\cap\varrho^{-1}\subset R$;

(iii) $\varrho\cap\Delta_X=\Delta_A$

and

(iv) $Q\subset(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$.

Proof. (i)—(iii) are obvious. It follows from the definition of Q' that $P\cap Q'=R\cap Q'=\emptyset$. Hence $Q'\subset(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$. Let $(x, y)\in Q$. By definition there exists a $(u, v)\in Q'$ such that either $\begin{pmatrix}x & y\\ u & v\end{pmatrix}\in E$ and $(x, y)\notin P$ or $\begin{pmatrix}u & v\\ x & y\end{pmatrix}\in E$ and $(x, y)\notin R$. Then either $\begin{pmatrix}x & y\\ u & v\end{pmatrix}$ is a p.endomorphism of G and $(x, y)\in\varrho\cup\varrho^{-1}$ or $\begin{pmatrix}u & v\\ x & y\end{pmatrix}$ is a p.endomorphism of G and $(x, y)\notin\varrho\cap\varrho^{-1}$. Since $(u, v)\in(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$, the vertices x, y are joined by one edge of G. Therefore $Q\subset(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$, i.e. (iv) holds.

Lemma 5. *If the canonical relations of $E=E(G)$ satisfy $Z_1=\emptyset$ (resp. $Z_3=\emptyset$) then $R_1=(\varrho\cap\varrho^{-1})_1$ and $P=X^2\setminus(\varrho\cup\varrho^{-1})$ (resp. $R=\varrho\cap\varrho^{-1}$ and $P_3=(X^2\setminus(\varrho\cup\varrho^{-1}))_3$).*

Proof. Suppose that $Z_1=\emptyset$. By Lemma 2, $Q_1\neq\emptyset$ or $P_1\neq\emptyset$. If $(x, y)\in Q_1$ then, by Lemma 4, x and y are joined by one edge of G. If $(a, b)\in P_1$ then, by the definition, $\begin{pmatrix}a & b\\ b & a\end{pmatrix}\in E$. On the other hand, $(a, b)\notin R_1$, whence $\begin{pmatrix}u & v\\ a & b\end{pmatrix}\notin E$ for some $u, v\in X$. Then $(a, b)\notin\varrho\cup\varrho^{-1}$. Therefore $(x, y)\notin\varrho\cup\varrho^{-1}$, if $(x, y)\in P$, and $(x, y)\in\varrho\cap\varrho^{-1}$, if $(x, y)\in R_1$. Using Lemma 4, we obtain that $R_1=(\varrho\cap\varrho^{-1})_1$ and $P=X^2\setminus(\varrho\cup\varrho^{-1})$. By the analogy the second statement of the lemma is proved.

Lemma 6. *If the canonical relations of $E=E(G)$ satisfy $B\neq\emptyset$ and $Z_2=Q=\emptyset$ then the following conditions hold*:

(i) *if $Z_1\neq\emptyset$ then $R_1=(\varrho\cap\varrho^{-1})_1$ and either $P_2=(X^2\setminus(\varrho\cup\varrho^{-1}))_2$ or $P_2=((\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho))_2$;*

(ii) *if $Z_3\neq\emptyset$ then $P_3=(X^2\setminus(\varrho\cup\varrho^{-1}))_3$ and either $R_2=((\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho))_2$ or $R_2=(\varrho\cap\varrho^{-1})_2$.*

Proof. Suppose that $Q=Z_2=\emptyset$ and $B, Z_1\neq\emptyset$. Then, by Lemma 3, $P_2\neq\emptyset$. Let $(a, b)\in P\cap(A\times B)$. Hence $(a, b)\notin R$ and, by the definition, $\begin{pmatrix}x & y\\ a & b\end{pmatrix}\notin E$ for some $x\in\tau^{-1}a$ and $y\in\tau^{-1}b$. Then $(a,b)\notin\varrho\cap\varrho^{-1}$, whence $(a,b)\in(\varrho\setminus\varrho^{-1})\cup\cup(\varrho^{-1}\setminus\varrho)$ or $(a,b)\notin\varrho\cup\varrho^{-1}$. Since $\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \begin{pmatrix}c & d\\ a & b\end{pmatrix}\in E$ for any $(c,d)\in P\cap(A\times B)$, we obtain by Lemma 4 that either $P_2=((\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho))_2$ or $P_2=(X^2\setminus(\varrho\cup\varrho^{-1}))_2$. Moreover, $(x,y)\notin P$ and, by Lemma 4, $(x,y)\in\varrho\cup\varrho^{-1}$. Since $\begin{pmatrix}x & y\\ u & v\end{pmatrix}, \begin{pmatrix}x & y\\ v & u\end{pmatrix}\in E$ for any $(u,v)\in R_1$, we obtain that $R_1=(\varrho\cap\varrho^{-1})_1$. Thus (i) holds; (ii) can be proved in the same manner.

2. Main results

Let E be a p.transformation semigroup on a set X. Using the canonical relations of E we define the following conditions U_k and binary relations $D_{k,l}$ on E:

$$U_1 \overset{\text{df}}{=} (A=X \ \&\ Z\neq\emptyset);\quad U_2 \overset{\text{df}}{=} \Big(\bigwedge_{i=1}^{3} Z_i\neq\emptyset\Big);$$

$$U_3 \overset{\text{df}}{=} (Q_3\neq\emptyset);\quad U_4 \overset{\text{df}}{=} (Q_3=\emptyset \ \&\ Q_1\neq\emptyset);$$

$$U_5 \overset{\text{df}}{=} (Q=Q_2 \ \&\ Q\neq\emptyset);$$

$$U_6 \overset{\text{df}}{=} (Q=Z_3=\emptyset \ \&\ (Z_2\neq\emptyset \vee Z_1=\emptyset));$$

$$U_7 \overset{\text{df}}{=} (Q=Z_1=\emptyset \ \&\ Z_2\neq\emptyset);$$

$$U_8 \overset{\text{df}}{=} (Z_3\neq\emptyset \ \&\ Q=Z_1=Z_2=\emptyset);$$

$$U_9 \overset{\text{df}}{=} (Z_1\neq\emptyset \ \&\ A\neq X \ \&\ Q=Z_2=Z_3=\emptyset);$$

$$U_{10} \overset{\text{df}}{=} (Q=Z_2=\emptyset \ \&\ Z_1\neq\emptyset \ \&\ Z_3\neq\emptyset);$$

$$D_{1,1} \overset{\text{df}}{=} \emptyset,\quad D_{1,2} \overset{\text{df}}{=} \Delta_X,\quad D_{1,3} \overset{\text{df}}{=} X^2;$$

$$D_{2,1} \overset{\text{df}}{=} \Delta_A;\quad D_{2,2} \overset{\text{df}}{=} A^2,\quad D_{2,3} \overset{\text{df}}{=} A\times X,$$

$$D_{2,4} \overset{\text{df}}{=} D_{2,3}^{-1};\quad D_{2,5} \overset{\text{df}}{=} X^2\setminus B^2,\quad D_{2,6} \overset{\text{df}}{=} X^2\setminus\Delta_B;$$

$$D_{3,1} \overset{\text{df}}{=} \beta(a,b) \quad\text{and}\quad D_{3,2} \overset{\text{df}}{=} D_{3,1}^{-1} \quad\text{for}\quad (a,b)\in Q_3;$$

$$D_{4,1} \overset{\text{df}}{=} ((R\setminus Q)\cup\beta^{-1}(a,b))\setminus P \quad\text{and}\quad D_{4,2} \overset{\text{df}}{=} D_{4,1}^{-1} \quad\text{for}\quad (a,b)\in Q_1;$$

$$D_{5,1} \overset{\text{df}}{=} (R\setminus P)\cup\beta(a,b)\cup\sigma \quad\text{and}\quad D_{5,2} \overset{\text{df}}{=} D_{5,1}^{-1} \quad\text{for}\quad (a,b)\in Q\cap(A\times B)$$

and either $\sigma=\beta(d,c)$, if there exists $(c,d)\in(Q\cap(A\times B))\setminus\beta(a,b)$, or $\sigma=\emptyset$ otherwise;

$$D_{6,1} \overset{\text{df}}{=} R; \quad D_{7,1} \overset{\text{df}}{=} R_1; \quad D_{8,1} \overset{\text{df}}{=} R \setminus B^2, \quad D_{8,2} \overset{\text{df}}{=} R \setminus (B \times X),$$

$$D_{8,3} \overset{\text{df}}{=} D_{8,2}^{-1}; \quad D_{9,1} \overset{\text{df}}{=} R, \quad D_{9,2} \overset{\text{df}}{=} (A \times X) \cup R,$$

$$D_{9,3} \overset{\text{df}}{=} D_{9,2}^{-1}; \quad D_{10,1} \overset{\text{df}}{=} R_1 \cup R_2, \quad D_{10,2} \overset{\text{df}}{=} (A \times X) \cup R_2,$$

$$D_{10,3} \overset{\text{df}}{=} R_1 \cup (R_2 \cap (A \times B)), \quad D_{10,4} \overset{\text{df}}{=} D_{10,2}^{-1}, \quad D_{10,5} \overset{\text{df}}{=} D_{10,3}^{-1}.$$

The sufficient conditions for a 3-bounded p.transformation semigroup to be equal to the 3-bounded p.endomorphism semigroup of some graph will be given in the following

Theorem 1. *Let E be a 3-bounded p.transformation semigroup on a set X and let the canonical relations of E satisfy the following conditions:*

(T$_1$) $\begin{pmatrix} a \\ x \end{pmatrix} \in E$ *for any* $a \in B$ *and* $x \in X$;

(T$_2$) $\begin{pmatrix} x & y \\ u & v \end{pmatrix}, \begin{pmatrix} x & y \\ v & u \end{pmatrix} \in E$ *imply* $(x, y) \in P$ *or* $(u, v) \in R$;

(T$_3$) $\left(\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in E \Leftrightarrow \begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} \in E\right) \Rightarrow \begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \in E$

and

$$\left(\begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} \in E \Leftrightarrow \begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \in E\right) \Rightarrow \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in E$$

for any $(x_i, y_i) \in Q$ $(i = \overline{1, 3})$ *such that* $(x_k, x_{k+1}), (y_k, y_{k+1}) \in \tau$ $(k = 1, 2)$.

Then the following conditions hold:

(i) *E satisfies the one of the conditions* U_k $(1 \leqq k \leqq 10)$;

(ii) *if E satisfies U_k, then the equality*

$$E = E_3(G) \tag{1}$$

holds if and only if $G = (X, D_{k,l})$ *for some number l.*

Using the canonical relations of p.transformation semigroups we obtain the following concrete characterization of the p.endomorphism semigroup of a graph.

Theorem 2. *Let E be a p.transformation semigroup on a set X. Then E is equal to the p.endomorphism semigroup of some graph if and only if E is the left idealizer*[2]) *of its subsemigroup E_3 in the symmetric semigroup $W(X)$ and the canonical relations of E satisfy the conditions* (T$_1$)—(T$_3$).

[2]) If S is a subsemigroup of a semigroup T, then the left idealizer of S in T is the largest subsemigroup L of T such that S is a left ideal of L.

Before drawing up our main results we verify five lemmas which make the proof of the theorems easier. Let E be a p.transformation semigroup on a set X satisfying (T_1)—(T_3). In the following propositions we collect some properties of the canonical relations of E.

Lemma 7. *E contains all 3-bounded identity p.transformations of X and the canonical relations of E satisfy $(x, y)\in\beta(x, y)$, $\tau a=A$, $\tau^{-1}b=B$ and $\tau^{-1}a=\tau b=X$ for any $x, y\in X$, $a\in A$, $b\in B$.*

Proof. By the definitions of A and (T_1), $\begin{pmatrix}x\\x\end{pmatrix}\in E$ for any $x\in X$. Consider distinct elements $x, y\in X$. Clearly, $(x, x), (y, y)\in\tau$. If $(x, y)\in P\cup R$, then, by the definition, $\begin{pmatrix}x & y\\x & y\end{pmatrix}\in E$. If $(x, y)\in Q$ then $\begin{pmatrix}x & y\\x & y\end{pmatrix}\in E$ by (T_3). We show that $\emptyset\in E$. Since $|X|>1$, there exist two distinct elements $x, y\in X$. Then the p.transformations $\begin{pmatrix}x\\x\end{pmatrix}$, $\begin{pmatrix}y\\y\end{pmatrix}$ and $\emptyset=\begin{pmatrix}x\\x\end{pmatrix}\circ\begin{pmatrix}y\\y\end{pmatrix}$ belong to E. So E contains all 3-bounded identity p.transformations of X. It follows from the definitions of β and A that $(x, y)\in\beta(x, y)$ and $\tau a=A$, $\tau^{-1}b=B$, $\tau^{-1}a=\tau b=X$ for any $x, y\in X$, $a\in A$, $b\in B$.

Lemma 8. *The canonical relations of E satisfy the following conditions:*

(i) $Q_1=Q_1'$,

(ii) $Q_3=Q_3'$,

(iii) $(\beta Q')\setminus R=\begin{cases}Q', & \text{if } Q_3'=\emptyset,\\ (\beta\{(a, b), (b, a)\})\setminus R, & \text{if there is } (a, b)\in Q_3';\end{cases}$

(iv) $(\beta^{-1}Q')\setminus P=\begin{cases}Q', & \text{if } Q_1'=\emptyset;\\ (\beta^{-1}\{(a, b), (b, a)\})\setminus P, & \text{if there is } (a, b)\in Q_1'.\end{cases}$

Proof. Suppose that $(x, y)\in Q_1\cup Q_3$. Then, by Lemma 7, $x, y\in\tau x=\tau y$ and, by the definition of Q, there exists a $(u, v)\in Q'$ such that either $(x, y)\notin R$ and $\begin{pmatrix}u & v\\x & y\end{pmatrix}\in E$ or $\begin{pmatrix}x & y\\u & v\end{pmatrix}\in E$ and $(x, y)\notin P$. We prove that $(x, y)\in Q'$. Suppose the contrary. Let $(x, y)\in P\cup R$. It follows that $\begin{pmatrix}x & y\\y & x\end{pmatrix}\in E$, by the definitions of P and R. Then, by (T_2), either $(u, v)\in P$ or $(u, v)\in R$, which both contradict the assumption. Consequently, $(x, y)\notin P\cup R$, $(x, y)\in Q'$ and $Q_1'\cup Q_3'=Q_1\cup Q_3$. Hence (i) and (ii) are satisfied.

Suppose now that $Q_3'=\emptyset$. By Lemma 7, $Q'\subset(\beta Q')\setminus R$. Conversely, let $(x, y)\in(\beta Q')\setminus R$. Clearly, $(x, y)\in Q$. If $(x, y)\in Q_1$ then $(x, y)\in Q_1'\subset Q'$. Suppose that $(x, y)\notin Q_1$. Then $(x, y)\in Q_2$, since $Q_3=Q_3'=\emptyset$. Let, for example, $(x, y)\in A\times B$. According to $(x, y)\in(\beta Q')\setminus R$, there exists an $(a, b)\in Q'$ such that $(x, y)\notin R$ and

$\begin{pmatrix} a & b \\ x & y \end{pmatrix} \in E$. Then $(a, b) \in Q_1' \cup Q_2'$ since $Q_3' = \emptyset$. By Lemma 7, $(a, b) \in A \times B$. If it were $(x, y) \in P$, then, by definitions, $(a, b) \in P$, which would contradict $P \cap Q' = \emptyset$. Hence $(x, y) \in X^2 \setminus (P \cup R) = Q'$. Thus $(\beta Q') \setminus R = Q'$, if $Q_3' = \emptyset$. Suppose now that there exists an $(a, b) \in Q_3'$. By Lemma 7, $\tau a = \tau b = X$. If $(x, y) \in (\beta Q') \setminus R \subset Q$ then, by (T_3), one of the p.transformations $\begin{pmatrix} a & b \\ x & y \end{pmatrix}$ and $\begin{pmatrix} a & b \\ y & x \end{pmatrix}$ belongs to E, i.e. $(x, y) \in (\beta \{(a, b), (b, a)\}) \setminus R$. On the other hand, it is easy to see that $(a, b) \in Q_3'$ implies $(\beta \{(a, b), (b, a)\}) \setminus R \subset (\beta Q') \setminus R$. Therefore Condition (iii) is satisfied. The latter condition can be proved in a similar way.

Lemma 9. *The canonical relations of E satisfy the following conditions:*

(i) *if* $(x, y) \in A^2 \cup B^2$, *then* $(x, y) \in Q$ *iff* $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \notin E$;

(ii) *if* (x, y), $(u, v) \in Q$ *and* $x, y \in B$ *or* $u, v \in A$, *then one and only one of the p.transformations* $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ *and* $\begin{pmatrix} x & y \\ v & u \end{pmatrix}$ *belongs to E.*

Proof. Let $(x, y) \in A^2 \cup B^2$. By Lemma 7, $\begin{pmatrix} x & y \\ x & y \end{pmatrix} \in E$. Using ($T_2$) and the definitions of P and R, we obtain that the condition $(x, y) \in P \cup R$ is equivalent to $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \in E$. Since $Q' = X^2 \setminus (P \cup R)$ and, by Lemma 8, $Q_1 \cup Q_3 = Q_1' \cup Q_3'$, the first statement of the lemma follows. Suppose now that $(x, y), (u, v) \in Q$ and $x, y \in B$ or $u, v \in A$. We may assume that $x, y \in B$. Then, by Lemma 7, $\tau x = \tau y = X$ and, by Lemma 8, $(x. y) \in Q_3 = Q_3'$, $(x, y) \notin P$. By (T_3), one of the p.transformations $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ and $\begin{pmatrix} x & y \\ v & u \end{pmatrix}$ belongs to E. Suppose that $\begin{pmatrix} x & y \\ u & v \end{pmatrix}, \begin{pmatrix} x & y \\ v & u \end{pmatrix} \in E$. Then, by ($T_2$), $(u, v) \in R$ since $(x, y) \notin P$. Hence, by the definition of Q, $(u, v) \in (\beta^{-1} Q') \setminus P$, i.e., by Lemma 8, there exists an $(a, b) \in Q_1'$ such that $(u, v) \notin P$ and $\begin{pmatrix} u & v \\ a & b \end{pmatrix} \in E$. This implies $\begin{pmatrix} x & y \\ a & b \end{pmatrix}, \begin{pmatrix} x & y \\ b & a \end{pmatrix} \in E$ and, by (T_2), $(a, b) \in R$ or $(x, y) \in P$, a contradiction. Consequently, one and only one of the p.transformations $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ and $\begin{pmatrix} x & y \\ v & u \end{pmatrix}$ belongs to E. For $u, v \in A$ the proof is analogous, therefore it is omitted.

Lemma 10. *If* $Q = \emptyset$ *and* $Z_2 \neq \emptyset$ *then one of the relations* Z_1, Z_3 *is nonempty.*

Proof. If $Q = \emptyset$, $Z_2 \neq \emptyset$ and $Z_1 = \emptyset$ then, by Lemmas 2 and 3, $P_2 = R_2 = (A \times B) \cup (B \times A)$, $P_1 \neq \emptyset$ and $R_3 \neq \emptyset$. Let $(x, y) \in R_3$ and $(u, v) \in P_1$. Then for any $(a, b) \in A \times B$ we have $\begin{pmatrix} x & y \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ u & v \end{pmatrix} \in E$ and $\begin{pmatrix} x & y \\ u & v \end{pmatrix} \in E$. Since, by the definition of P, $\begin{pmatrix} u & v \\ v & u \end{pmatrix} \in E$, we obtain $\begin{pmatrix} x & y \\ v & u \end{pmatrix} \in E$ and, by (T_2), $(x, y \in P$. Consequently, $Z_3 \neq \emptyset$.

Later we will use the following

Lemma 11. *Let ϱ and σ be binary relations on X. Then* $\varrho\subset\sigma$, $\varrho\cap\varrho^{-1}=\sigma\cap\sigma^{-1}$ *and* $\varrho\cup\varrho^{-1}=\sigma\cup\sigma^{-1}$ *imply* $\varrho=\sigma$.

The proof follows from the relations:

$$\sigma = ((\sigma\cup\sigma^{-1})\setminus\sigma^{-1})\cup(\sigma\cap\sigma^{-1}) \subset ((\varrho\cup\varrho^{-1})\setminus\varrho^{-1})\cup(\varrho\cap\varrho^{-1}) = \varrho.$$

Now we turn to the main theorems of the paper.

Proof of Theorem 1. It is easy to see that (i) follows from Lemmas 2, 3 and 10.

Suppose now that E satisfies U_k $(1\leqq k\leqq 10)$. We prove that (1) holds for the graphs $G=(X, D_{k,l})$ and only for them. Note that it is sufficient to verify (1) for one of these graphs with mutually converse relations, since they have equal p.endomorphisms. Clearly, $\emptyset$ is a p.endomorphism of any graph and, by Lemma 7, $\emptyset\in E$. Thus for the proof of (1) we must show that a non-empty 3-bounded p.transformation $f=\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ of X is a p.endomorphism of a graph G iff $f\in E$. We investigate the following ten cases concerning E.

Case 1. Let E satisfy U_1, i.e. $A=X$ and $Z\neq\emptyset$. Then, by Lemma 3, $R=X^2$ and E consists of all 3-bounded p.transformations of X. One can easily see that E is equal to $E_3(G)$ for all graphs $G=(X, D_{1,l})$ $(l=\overline{1,3})$. On the other hand, if a graph $G=(X, \varrho)$ satisfies (1), then either $\varrho=X^2=D_{1,3}$ (if $\varrho\setminus\Delta_X\neq\emptyset$) or $\varrho=\Delta_X=D_{1,2}$ (if $\emptyset\neq\varrho\subset\Delta_X$) or $\varrho=\emptyset=D_{1,1}$.

Case 2. Let E satisfy U_2, i.e. $Z_i\neq\emptyset$ $(i=\overline{1,3})$. Then, by Lemma 3, E consists of all 3-bounded p.transformations of X which map no elements of A into B. One can easily see that E is equal to $E_3(G)$ for all graphs $G=(X, D_{2,l})$ $(l=\overline{1,6})$. On the other hand, let a graph $G=(X, \varrho)$ satisfy (1). Then for any distinct elements $a, b\in B$, by Lemma 3 $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\in E(G)$. Therefore either $(a, b)\in\varrho\cap\varrho^{-1}$ (and in this case $\varrho=X^2\setminus\Delta_B=D_{2,6}$) or $(a, b)\notin\varrho\cup\varrho^{-1}$ (and hence $\varrho\subset X^2\setminus B^2$). In the latter case $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\notin E(G)$ for any $(a, b)\in A\times B$. Then either $(a, b)\in\varrho\cap\varrho^{-1}$ (and in this case $\varrho=X^2\setminus B^2=D_{2,5}$) or $(a, b)\in\varrho\setminus\varrho^{-1}$ (and hence $\varrho=A\times X=D_{2,3}$) or $(a, b)\in\varrho^{-1}\setminus\varrho$ (and hence $\varrho=X\times A=D_{2,4}$) or $(a, b)\notin\varrho\cup\varrho^{-1}$ (and hence $\varrho\subset A^2$). In the latter case, by Lemma 3, $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\in E(G)$ for any distinct elements $a, b\in A$. Then either $(a, b)\in\varrho\cap\varrho^{-1}$ (and in this case $\varrho=A^2=D_{2,2}$) or $(a, b)\notin\varrho\cup\varrho^{-1}$ (and hence $\varrho=\Delta_A=D_{2,1}$).

Case 3. Let E satisfy U_3, i.e. $(a, b)\in Q_3$ for some $a, b\in B$. We prove (1) for the graph $G=(X, \varrho)$ with $\varrho=D_{3,1}=\beta(a, b)$. By Lemma 7, $(a, b)\in\varrho\setminus\varrho^{-1}$. If $L=\beta(a, b)\cup\beta(b, a)$ then, by the definitions of R and Lemma 8, $R\cup Q\subset L$ and $X^2=P\cup L$. Hence any pair $(x, y)\in X^2$ belongs to one of the relations $P\setminus L$ and L. If $(x, y)\in P\setminus L$ then, by the definition, $(x, y)\notin\varrho\cup\varrho^{-1}$ and, by Lemma 7 f belongs to both E and $E_3(G)$. Let $(x, y)\in L$, and for example, $(x, y)\in\beta(a, b)$. If $(x, y)\in\beta(b, a)$, then $(x, y)\in\varrho\cap\varrho^{-1}$ and $\begin{pmatrix}a & b\\ x & y\end{pmatrix}, \begin{pmatrix}a & b\\ y & x\end{pmatrix}\in E$ by the definition of β. Hence, from $f\in E$, it follows that $\begin{pmatrix}a & b\\ u & v\end{pmatrix}, \begin{pmatrix}a & b\\ v & u\end{pmatrix}\in E$ and $(u, v)\in\varrho\cap\varrho^{-1}$, whence f is a p.endomorphism of G. Conversely, if $f\in E_3(G)$, then $(u, v)\in\varrho\cap\varrho^{-1}$ and by the definition of G, $\begin{pmatrix}a & b\\ u & v\end{pmatrix}, \begin{pmatrix}a & b\\ v & u\end{pmatrix}\in E$. By Lemma 8, from the equality $Q_3=Q_3'$ and (T_2) it follows that $(u, v)\in R, f\in E$. Suppose now that $(x, y)\notin\beta(b, a)$. In this case $(x, y)\in Q$, $\begin{pmatrix}a & b\\ x & y\end{pmatrix}\in E$ and $(x, y)\in\varrho\setminus\varrho^{-1}$. If $f\in E$ then $\begin{pmatrix}a & b\\ u & v\end{pmatrix}\in E$ and $(u, v)\in\varrho$. Hence f is a p.endomorphism of G. Conversely, let $f\in E_3(G)$. Then $(u, v)\in\varrho=\beta(a, b)$. If $\begin{pmatrix}a & b\\ v & u\end{pmatrix}\in E$ then $(u, v)\in R$ and $f\in E$ by Lemma 8 and (T_2). If $\begin{pmatrix}a & b\\ v & u\end{pmatrix}\notin E$ then $(u, v)\notin R$ and $(u, v)\in Q$ by the definition of Q. In this case $\begin{pmatrix}x & y\\ v & u\end{pmatrix}\notin E$ and, by Lemma 9, $f\in E$. So (1) holds for the graph G.

Conversely, let a graph $G=(X, \varrho)$ satisfy (1). Then, by Lemma 4, $(a, b)\in(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$. If $(a, b)\in\varrho$ then $\beta(a, b)\subset\varrho$, and $\beta(a, b)\subset\varrho^{-1}$ otherwise. By Lemma 5, $R=\varrho\cap\varrho^{-1}$. If $(u, v)\in P\setminus(R\cup Q)$ then $\begin{pmatrix}a & b\\ u & v\end{pmatrix}, \begin{pmatrix}a & b\\ v & u\end{pmatrix}\notin E(G)$ and $(u, v)\notin\varrho\cup\varrho^{-1}$. Using Lemma 4 we obtain that $\varrho\cup\varrho^{-1}=R\cup Q$. From Lemmas 8, 9 it follows that the relation $\sigma=\beta(a, b)$ satisfies $\sigma\cap\sigma^{-1}=R$ and $\sigma\cup\sigma^{-1}=R\cup Q$. Then, by Lemma 11, either $\varrho=\beta(a, b)=D_{3,1}$ or $\varrho=\beta(b, a)=D_{3,2}$.

Case 4. Let E satisfy U_4, i.e. $Q_3=\emptyset$ and $(a, b)\in Q_1$ for some $a, b\in A$. We prove (1) for the graph $G=(X, \varrho)$ with $\varrho=D_{4,1}=((R\setminus Q)\cup\beta^{-1}(a, b))\setminus P$. Since $X^2=(R\setminus(P\cup Q))\cup P\cup Q$, any pair $(u, v)\in X^2$ belongs to one of the relations $R\setminus(P\cup Q)$, P and $Q\setminus P$. If $(u, v)\in R\setminus(P\cup Q)$ then $(u, v)\in\varrho\cap\varrho^{-1}$ and f belongs to both E and $E_3(G)$. Suppose that $(u, v)\in P$. Then $(u, v)\notin\varrho\cup\varrho^{-1}$ and, by the definition of P, either $u\in B$ (if $u=v$) or $\begin{pmatrix}u & v\\ a & b\end{pmatrix}, \begin{pmatrix}u & v\\ b & a\end{pmatrix}\in E$ (if $u\neq v$). Let $f\in E$. Then, by (T_2), $(x, y)\in P$ since $R\cap\Delta_B=R\cap Q'=\emptyset$. Consequently, $(x, y)\notin\varrho\cup\varrho^{-1}$ and f is a p.endomorphism of G. Conversely, if $f\in E_3(G)$, then $(x, y)\notin\varrho\cup\varrho^{-1}$, $(x, y)\in P$ and $f\in E$. Further, suppose that $(u, v)\in Q\setminus P$. By Lemma 9, there is a unique mapping of $\{u, v\}$ onto $\{a, b\}$. Let, for example, $\begin{pmatrix}u & v\\ a & b\end{pmatrix}\in E$. It follows that $(u, v)\in\varrho$. If $f\in E$, then $\begin{pmatrix}x & y\\ a & b\end{pmatrix}\in E$ and either $(x, y)\in P$ or $(x, y)\in Q$. Consequently, either

$(x, y)\notin \varrho\cup\varrho^{-1}$ or $(x, y)\in \varrho\setminus\varrho^{-1}$. Hence f is a p.endomorphism of G. Conversely, let $f\in E_3(G)$. Then either $(x, y)\notin\varrho\cup\varrho^{-1}$ or $(x, y)\in\varrho\setminus\varrho^{-1}$. From the definition of G it follows that either $(x, y)\in P$ or $(x, y)\in Q\setminus P$. In the former case $f\in E$ by the definition of P. In the latter case, by (T_2), $\begin{pmatrix}x & y\\ v & u\end{pmatrix}\notin E$ since $\begin{pmatrix}x & y\\ a & b\end{pmatrix}, \begin{pmatrix}u & v\\ a & b\end{pmatrix}\in E$. Then, by Lemma 9, $f\in E$. Thus (1) holds for the graph G.

Conversely, let a graph $G=(X, \varrho)$ satisfy (1). Then, by Lemma 4, $(a, b)\in(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$. It follows that $(\beta^{-1}(a, b))\setminus P\subset\varrho$, if $(a, b)\in\varrho$, and $(\beta^{-1}(a, b))\setminus P\subset\varrho^{-1}$ otherwise. By Lemma 5, $P=X^2\setminus(\varrho\cup\varrho^{-1})$. If $(x, y)\in R\setminus(P\cup Q)$ then $\begin{pmatrix}x & y\\ a & b\end{pmatrix}, \begin{pmatrix}x & y\\ b & a\end{pmatrix}\notin E$ and, by (1), $(x, y)\in\varrho\cap\varrho^{-1}$. Thus, by Lemma 4, $\varrho\cap\varrho^{-1}= =R\setminus(P\cup Q)$. From Lemmas 8 and 9 it follows that the relation $\sigma=((R\setminus Q)\cup \cup\beta^{-1}(a, b))\setminus P$ satisfies $\sigma\cap\sigma^{-1}=R\setminus(P\cup Q)$ and $\sigma\cup\sigma^{-1}=X^2\setminus P$. Hence, by Lemma 11, either $\varrho=\sigma=D_{4,1}$ or $\varrho=\sigma^{-1}=D_{4,2}$.

Case 5. Let E satisfy U_5, i.e. $Q=Q_2$ and $(a, b)\in Q$ for some $a\in A$, $b\in B$. We prove (1) for the graph $G=(X, \varrho)$ with $\varrho=D_{5,1}=(R\setminus P)\cup\beta(a, b)\cup\sigma$ where $\sigma=\beta(d, c)$, if there exists a $(c, d)\in(Q\cap(A\times B))\setminus\beta(a, b)$, and $\sigma=\emptyset$ otherwise. By U_5 and Lemma 8, $Q=Q'$. Since $X^2=(P\setminus(R_1\cup R_2))\cup(R\setminus P_3)\cup Q$, any pair $(x, y)\in X^2$ belongs to one of the relations $P\setminus(R_1\cup R_2)$, $R\setminus P_3$ and Q. If $(x, y)\in P\setminus(R_1\cup R_2)$ then, by the definition of G, $(x, y)\notin\varrho\cup\varrho^{-1}$ and f belongs to both E and $E_3(G)$. Suppose that $(x, y)\in R\setminus P_3$. This implies $(x, y)\in\varrho\cap\varrho^{-1}$. If $f\in E$ then $(u, v)\notin P_3$ since $(x, y)\in P_3$ otherwise. We show that $(u, v)\in R$. If $x, y\in B$ then by $Q=Q_2$ and, Lemma 9, $\begin{pmatrix}x & y\\ v & u\end{pmatrix}\in E$ and, by (T_2), $(u, v)\in R$. Now suppose that one of the elements x and y belongs to A, for example, $x\in A$. Since $f\in E$, $u\in A$. It follows that either $u, v\in A$ or $(u, v)\in A\times B$. In the former case, by $Q_2=Q'\neq\emptyset$, (T_2) and Lemmas 3, 9, $(u, v)\in R$. In the latter case $(x, y)\in R_2$ and $f\in E$ imply $(u, v)\in R_2$. So $(u, v)\in\varrho\cap\varrho^{-1}$ and f is a p.endomorphism of G. Conversely, if $f\in E_3(G)$ then $(u, v)\in\varrho\cap\varrho^{-1}$ and, by the definition of G, $(u, v)\in R\setminus P_3$. Thus $f\in E$. Now suppose that $(x, y)\in Q\cap(A\times B)$. Then either $\begin{pmatrix}a & b\\ x & y\end{pmatrix}\in E$ or $\begin{pmatrix}a & b\\ x & y\end{pmatrix}\notin E$. In the latter case, by (T_3), $\begin{pmatrix}c & d\\ x & y\end{pmatrix}\in E$. We may suppose that $\begin{pmatrix}a & b\\ x & y\end{pmatrix}\in E$. Then $(x, y)\in\varrho\setminus\varrho^{-1}$. Assume that $f\in E$. It follows $u\in A$ and $\begin{pmatrix}a & b\\ u & v\end{pmatrix}\in E$. If $v\in A$ then, by $Q=Q_2$ and Lemma 9, $\begin{pmatrix}u & v\\ v & u\end{pmatrix}\in E$. Consequently, $\begin{pmatrix}a & b\\ v & u\end{pmatrix}\in E$ and, by (T_2), $(u, v)\in R_1$. If $v\in B$ then $(u, v)\notin P_2$ since $(a, b)\in P_2$ otherwise. Hence (u, v) belongs to R_2 or Q_2. Moreover, in the latter case $(u, v)\in\beta(a, b)$. So $(u, v)\in\varrho\cap\varrho^{-1}$ or $(u, v)\in\varrho\setminus\varrho^{-1}$, whence $f\in E_3(G)$. Conversely, let $f\in E_3(G)$. If $\begin{pmatrix}x & y\\ v & u\end{pmatrix}\in E_3(G)$ then $(u, v)\in\varrho\cap\varrho^{-1}$. By the definition of G,

$(u, v)\in R\setminus P_3$. Then by the definition of R, $f\in E$. If $\begin{pmatrix} x & y \\ v & u \end{pmatrix}\notin E_3(G)$ then $(u, v)\in$ $\in(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$. It follows from the definition of G that $(u, v)\in Q\cap(A\times B)$ and $\begin{pmatrix} a & b \\ u & v \end{pmatrix}\in E$, since $\begin{pmatrix} a & b \\ x & y \end{pmatrix}\in E$. Then, by Lemma 9, $f\in E$. Thus (1) holds for the graph G.

Conversely, let a graph $G=(X, \varrho)$ satisfy (1). Then, by Lemma 4, $(a, b)\in(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$. If $(x, y)\in P_3$ then $\begin{pmatrix} x & y \\ a & b \end{pmatrix}$ and $\begin{pmatrix} x & y \\ b & a \end{pmatrix}$ are p.endomorphisms of G, and $(x, y)\notin\varrho\cup\varrho^{-1}$. If $(x, y)\in P_2\cap(A\times B)$ then $\begin{pmatrix} x & y \\ a & b \end{pmatrix}\in E$ and $\begin{pmatrix} a & b \\ x & y \end{pmatrix}$ $\notin E$, whence $(x, y)\notin\varrho\cup\varrho^{-1}$. Moreover $\begin{pmatrix} a & b \\ x & y \end{pmatrix}, \begin{pmatrix} a & b \\ y & x \end{pmatrix}\in E_3(G)$ for $(x, y)\in R_1$ and $\begin{pmatrix} a & b \\ x & y \end{pmatrix}$ $\in E$, $\begin{pmatrix} x & y \\ a & b \end{pmatrix}\notin E$ for $(x, y)\in R_2\cap(A\times B)$. In these cases $(x, y)\in\varrho\cap\varrho^{-1}$. If $(x, y)\in P_1\setminus R_1$ then $\begin{pmatrix} a & b \\ x & y \end{pmatrix}\notin E$ and, by Lemmas 8 and 9, $\begin{pmatrix} x & y \\ y & x \end{pmatrix}\in E$, whence $(x, y)\notin\varrho\cup\varrho^{-1}$. If $(x, y)\in R_3\setminus P_3$ then $\begin{pmatrix} x & y \\ a & b \end{pmatrix}\notin E$ and, by Lemmas 8 and 9 $\begin{pmatrix} x & y \\ y & x \end{pmatrix}\in E$, whence $(x, y)\in\varrho\cap$ $\cap\varrho^{-1}$. So $X^2\setminus(\varrho\cup\varrho^{-1})=P\setminus R_1$ and $\varrho\cap\varrho^{-1}=R\setminus P_3$. Moreover, $\beta(a, b)\subset\varrho$, if $(a, b)\in\varrho$, and $\beta(a, b)\subset\varrho^{-1}$ otherwise. Let there exist a $(c, d)\in(Q\cap(A\times B))\setminus\beta(a, b)$. Then, by Lemma 4, (c, d) belongs to $(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$. It follows that either $(d, c)\in\varrho$, if $(a, b)\in\varrho$, or $(d, c)\in\varrho^{-1}$ otherwise. Consequently, $\beta(d, c)\subset\varrho$, if $(a, b)\in\varrho$, and $\beta(d, c)\subset\varrho^{-1}$ otherwise. So the relation $\sigma=(R\setminus P)\cup\beta(a, b)\cup\beta(d, c)$ satisfies the conditions: $\sigma\cap\sigma^{-1}=R\setminus P_3=\varrho\cap\varrho^{-1}$, $\sigma\cup\sigma^{-1}=X^2\setminus(P\setminus R_1)=\varrho\cup\varrho^{-1}$ and either $\sigma\subset\varrho$ or $\sigma\subset\varrho^{-1}$. By Lemma 11, $\varrho=\sigma=D_{5,1}$ or $\varrho=\sigma^{-1}=D_{5,2}$.

Cases 6 and 7. If E satisfies U_6 (or U_7), then it is easy to verify that (1) holds for the graph $G=(X, \varrho)$ with $\varrho=R=D_{6,1}$ (or $\varrho=R_1=D_{7,1}$). On the other hand, if a graph $G=(X, \varrho)$ satisfies (1), then by Lemmas 4 and 7 $\varrho=R=D_{6,1}$ (or $\varrho=R_1=$ $=D_{7,1}$).

Case 8. Let E satisfy U_8, i.e. $Q=Z_1=Z_2=\emptyset$ and $Z_3\neq\emptyset$. We prove (1) for the graph $G=(X, \varrho)$ with $\varrho=D_{8,1}$ (and $\varrho=D_{8,2}$). If $(x, y)\in P$ then $(x, y)\notin\varrho\cup\varrho^{-1}$ and f belongs to both E and $E_3(G)$. Suppose that $(x, y)\in R\setminus P$. Then $(x, y)\notin B^2$ since U_8 and Lemma 3 imply $B^2\subset P$. If $(x, y)\in A^2$ then $(x, y)\in\varrho\cap\varrho^{-1}$ and, by Lemma 9, $\begin{pmatrix} x & y \\ y & x \end{pmatrix}\in E$. Hence $f\in E$ implies $\begin{pmatrix} x & y \\ v & u \end{pmatrix}\in E$ and, by (T_2), $(u, v)\in R$. It follows that $(u, v)\in\varrho\cap\varrho^{-1}$ and f is a p.endomorphism of G. Conversely, if $f\in E_3(G)$, then $(u, v)\in\varrho\cap\varrho^{-1}$ and, by the definition of G, $(u, v)\in R_1$. Thus $f\in E$. Let $(x, y)\notin A^2$. Without loss of generality, we can assume that $(x, y)\in A\times B$. For the graph G with the relation $D_{8,1}$ (or $D_{8,2}$), it follows $(x, y)\in\varrho\cap\varrho^{-1}$ (or $(x, y)\in\varrho\setminus\varrho^{-1}$). If $f\in E$ then, by the definition of R and (T_2), $(u, v)\in R_1\cup R_2$. Hence $(u, v)\in\varrho\cap\varrho^{-1}$

(resp. $(u, v)\in\varrho\setminus\varrho^{-1}$ or $(u, v)\in\varrho\cap\varrho^{-1}$) for the graph G with the relation $D_{8,1}$ (resp. $D_{8,2}$). It follows that f is a p.endomorphism of G. It is easy to verify that $f\in E_3(G)$ implies $f\in E$. So (1) holds for the graph G with any of the relations $D_{8,l}$ $(l=\overline{1,3})$. Conversely, let a graph $G=(X, \varrho)$ satisfy (1). Then, by Lemmas 5 and 6, $P=X^2\setminus(\varrho\cup\varrho^{-1})$, $R_1=(\varrho\cap\varrho^{-1})_1$ and either $R_2=(\varrho\cap\varrho^{-1})_2$ or $R_2=((\varrho\setminus\varrho^{-1})\cup\cup(\varrho^{-1}\setminus\varrho))_2$. Since $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in E$ for any $(a, b), (c, d)\in R_2\cap(A\times B)$, the relation ϱ equals one of the relations $D_{8,l}$ $(1\leqq l\leqq 3)$.

Case 9. Let E satisfy U_9, i.e. $A\neq X$, $Z_1\neq\emptyset$ and $Q=Z_2=Z_3=\emptyset$. We prove (1) for the graph $G=(X, \varrho)$ with $\varrho=D_{9,1}$ (and $\varrho=D_{9,2}$). If $(u, v)\in R\setminus(P_2\cup P_3)$ then $(u, v)\in\varrho\cap\varrho^{-1}$ and f belongs to both E and $E_3(G)$. If $(u, v)\in P_3$ then $(u, v)\notin\varrho\cup\cup\varrho^{-1}$ and $f\in E$ is equivalent to $(x, y)\in P_3$, i.e. $(x, y)\notin\varrho\cup\varrho^{-1}$. The latter is equivalent to $f\in E_3(G)$. Suppose that $(u, v)\in P_2\cap(A\times B)$. Then $(u, v)\in\varrho\setminus\varrho^{-1}$ (or $(u, v)\notin\varrho\cup\varrho^{-1}$) for the graph G with the relation $D_{9,2}$ (or $D_{9,1}$). Let $f\in E$. If $x, y\in B$ then, by $Q=\emptyset$ and Lemma 9, $\begin{pmatrix}x & y\\ y & x\end{pmatrix}\in E$. Hence $\begin{pmatrix}x & y\\ v & u\end{pmatrix}\in E$ and, by (T_2), $(x, y)\in P_3$. It follows that $(x, y)\notin\varrho\cup\varrho^{-1}$ and f is a p.endomorphism of G. If $(x, y)\notin B^2$ then $(u, v)\in P_2$ and $f\in E$ imply $(x, y)\in P_2\cap(A\times B)$. Thus $(x, y)\in\varrho\setminus\varrho^{-1}$ (or $(x, y)\notin\varrho\cup\cup\varrho^{-1}$) for the graph G with the relation $D_{9,2}$ (or $D_{9,1}$). Therefore f is a p.endomorphism of G. Conversely, if $f\in E_3(G)$ then $(x, y)\notin\varrho\cup\varrho^{-1}$. Hence, by the definition of G, either $(x, y)\in P_3$ or $(x, y)\in P_2$; moreover, in the latter case, $(x, y)\in\varrho\setminus\varrho^{-1}$ for the graph G with the relation $D_{9,1}$. It follows that (x, y) belongs to P_3 or $P_2\cap(A\times B)$, whence, by the definition of P, $f\in E$. Thus (1) holds for the graph G with each of the relations $D_{9,l}$ $(l=\overline{1,3})$.

Conversely, let a graph $G=(X, \varrho)$ satisfy (1). Then, by Lemmas 3 and 5, $R_1=A^2$, $R=\varrho\cap\varrho^{-1}$ and $P_3=(X^2\setminus(\varrho\cup\varrho^{-1}))_3$. Since $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in E$ for any $(a, b), (c, d)\in P_2\cap(A\times B)$, from Lemma 6 it follows that the relation ϱ equals one of the relations $D_{9,l}$ $(1\leqq l\leqq 3)$.

Case 10. Let E satisfy U_{10}, i.e. $Q=Z_2=\emptyset$ and $Z_1, Z_3\neq\emptyset$. We prove (1) for the graph $G=(X, \varrho)$ with $\varrho=D_{10,l}$ $(l=\overline{1,3})$. By U_{10} and Lemma 3, $P_3=B^2$ and $R_1=A^2$. If $(x, y)\in B^2$ or $(u, v)\in A^2$ then, by the definition of G, $(x, y)\notin\varrho\cup\varrho^{-1}$ or $(u, v)\in\varrho\cap\varrho^{-1}$, respectively. It follows that f belongs to both E and $E_3(G)$. If $(x, y)\in A^2$ and f belongs to E or $E_3(G)$, then by Lemmas 4 and 9, $(u, v)\in A^2$. Analogously, $(x, y)\in B^2$ if $(u, v)\in B^2$ and f belongs to E or $E_3(G)$. Suppose now that $(x, y)\notin B^2$ and $(u, v)\notin A^2$. Hence (x, y) and (u, v) belong to $(A\times B)\cup(B\times A)$. Let, for example, $(x, y)\in A\times B$. If $f\in E$ then $(u, v)\in A\times B$ and, by the definitions, the following condition holds:

$$\text{(2)}\qquad \begin{cases}\text{either } (x, y), (u, v)\in P_2, \text{ or } (x, y), (u, v)\in R_2,\\ \text{or } (x, y)\in P_2 \text{ and } (u, v)\in R_2.\end{cases}$$

It follows that f is a p.endomorphism of G for any $D_{10,l}$ $(l=\overline{1,3})$. Conversely, if $f\in E_3(G)$, then $(u,v)\in A\times B$ and, by the definition of G, (2) is satisfied. Hence, by the definition of P and R, $f\in E$. So (1) holds for the graph G with any of the relations $D_{10,l}$ $(l=\overline{1,5})$.

Conversely, let a graph $G=(X,\varrho)$ satisfy (1). Then, by Lemmas 3 and 6, $P_3=(X^2\setminus(\varrho\cup\varrho^{-1}))_3=B^2$ and $R_1=(\varrho\cap\varrho^{-1})_1=A^2$. Moreover, either $P_2=((\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho))_2$ and $R_2=(\varrho\cap\varrho^{-1})_2$, or $P_2=(X^2\setminus(\varrho\cup\varrho^{-1}))_2$ and one of the following equalities holds: $R_2=(\varrho\cap\varrho^{-1})_2$ or $R_2=((\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho))_2$. We show that other cases are impossible. By Lemma 3, there exist (a,b) and (c,d) in $A\times B$ such that $(a,b)\in P_2$ and $(c,d)\in R_2$. Using $Z_2=\emptyset$ and the definition of P and R, we obtain that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in E$ and $\begin{pmatrix} c & d \\ a & b \end{pmatrix}\notin E$. Clearly, $\begin{pmatrix} x & y \\ u & v \end{pmatrix}\in E$ if (x,y) and (u,v) belong to $P_2\cap(A\times B)$ (or $R_2\cap(A\times B)$). It follows that the relation ϱ equals one of the relations $D_{10,l}$ $(1\leqq l\leqq 5)$. The proof is complete.

Proof of Theorem 2. Let $E=E(G)$ for some graph $G=(X,\varrho)$. From the definition of a p.endomorphism it follows that E is a left idealizer of its subsemigroup $E_3=E_3(G)$ in the symmetric semigroup $W(X)$. We prove (T_1)—(T_3). Clearly, these conditions hold for $E=W(X)$. Suppose that $E\neq W(X)$. Then, by Lemma 4, a vertex $a\in X$ has no loop iff $a\in B$. Thus $\begin{pmatrix} a \\ x \end{pmatrix}$ is a p.endomorphism of G for any $a\in B$ and $x\in X$, i.e. (T_1) holds. Consider now $(x_i,y_i)\in Q$ $(i=\overline{1,3})$ such that $(x_k,x_{k+1}),(y_k,y_{k+1})\in\tau$ $(k=1,2)$. Then, by Lemma 4, $(x_i,y_i)\in(\varrho\setminus\varrho^{-1})\cup(\varrho^{-1}\setminus\varrho)$ and (T_3) is satisfied. Suppose that $\begin{pmatrix} x & y \\ u & v \end{pmatrix},\begin{pmatrix} x & y \\ v & u \end{pmatrix}$ are p.endomorphisms of G. Then $(x,y)\notin\varrho\cup\varrho^{-1}$ or $(u,v)\in\varrho\cap\varrho^{-1}$, whence, by Lemma 4, $(x,y)\in P$ or $(u,v)\in R$. Thus (T_2) holds.

Conversely, let a p.transformation semigroup E satisfy the conditions of Theorem 2. Then the semigroups E and E_3 are determined by each other and, by Lemmas 1 and 7, their canonical relations are equal. Moreover, E_3 satisfies (T_1)—(T_3). By Theorem 1, $E_3=E_3(G)$ for some graph G. Then $E=E(G)$ since $E(G)$ is the left idealizer of $E_3(G)$ in $W(X)$. Theorem 2 is proved.

Remark. The conditions (T_1)—(T_3) are independent.

3. Applications

Two graphs, G and G' are called E-equivalent, if $E(G)=E(G')$.

As an application of Theorems 1 and 2 we describe E-equivalent graphs and graphs with isomorphic or elementarily equivalent [9] p.endomorphism semigroups.

Theorem 3. *The graphs $G=(X,\varrho)$ and $G'=(X,\varrho')$ are E-equivalent iff either*

$\varrho=\varrho'$ or $\varrho=\varrho'^{-1}$ or ϱ, ϱ' simultaneously belong to one of the following classes:

$$K_0 \overset{\mathrm{df}}{=} \{\emptyset, \Delta_X, X^2\};$$

$$K(A) \overset{\mathrm{df}}{=} \{\Delta_A, A^2, A\times X, X\times A, X^2\setminus B^2, X^2\setminus \Delta_B\};$$

$$K(A, \alpha, \gamma) \overset{\mathrm{df}}{=} \{\alpha\cup\gamma, \alpha\cup(\gamma\cap(A\times B)), \alpha\cup(\gamma\cap(B\times A))\};$$

$$K(A, \delta, \zeta) \overset{\mathrm{df}}{=} \{A^2\cup\delta\cup\zeta, (A\times X)\cup\delta\cup\zeta, (X\times A)\cup\delta\cup\zeta\};$$

$$K(A, \gamma) \overset{\mathrm{df}}{=} \{A^2\cup\gamma, A^2\cup(\gamma\cap(A\times B)), A^2\cup(\gamma\cap(B\times A)), (A\times X)\cup\gamma, (X\times A)\cup\gamma\},$$

for some proper subset $A\subset X$, $B=X\setminus A$ and symmetrical relations α, ζ, γ, δ such that $\Delta_A\subset\alpha\subsetneqq A^2$, $\emptyset\neq\zeta\subset B^2\setminus\Delta_B$, $\gamma, \delta\subset(A\times B)\cup(B\times A)$ and $\gamma\neq\emptyset$.

The proof follows from Theorems 1 and 2.

Denote by $\mathfrak{A}$ the class of all p.endomorphism semigroups of graphs. The signature Ω of $\mathfrak{A}$ consists of the single symbol $\cdot$ for the binary semigroup ooeration. A n-place predicate $\Phi(x_1, \ldots, x_n)$ is called formular in $\mathfrak{A}$ if there exists a formula $F(x_1, \ldots, x_n)$ of the signature Ω such that for every semigroup $S\in\mathfrak{A}$ and for every $x_1, \ldots, x_n\in S$, $F(x_1, \ldots, x_n)$ is true iff $\Phi(x_1, \ldots, x_n)$ is true. Consider the following formulas:

$$O(x) \overset{\mathrm{df}}{=} (\forall y)(x\cdot y = y\cdot x = x);\quad I(x) \overset{\mathrm{df}}{=} (\neg O(x)\ \&\ x\cdot x = x); \tag{3}$$

$$J(x) \overset{\mathrm{df}}{=} (I(x)\ \&\ (\forall y)(I(y)\ \&\ x\cdot y = y \Rightarrow y\cdot x = x));$$

$$M(x) \overset{\mathrm{df}}{=} (J(x)\ \&\ (\forall y)(J(y)\ \&\ y\cdot x \neq x \Rightarrow O(x\cdot y)));$$

$$T(x, y) \overset{\mathrm{df}}{=} (M(x)\ \&\ M(y)\ \&\ (\exists z)(\neg O(y\cdot z\cdot x))).$$

Let $G=(X, \varrho)$ be a graph, and let $E=E(G)$. We write $\bar{x}=\binom{x}{x}$ for $x\in X$. Clearly, $x\mapsto\bar{x}$ $(x\in X)$ is a one-to-one mapping of X into E. Then any relation σ on X is mapped onto the relation $\bar{\sigma}$ on E, and any condition U on the relations σ_i on X is transformed into the condition $\bar{U}$ on the relations $\bar{\sigma}_i$ on E. Using (3) we can prove that the relations X, ϱ, $D_{k,l}$ and the canonical relations of E are mapped onto the relations $\bar{X}$, $\bar{\varrho}$, $\bar{D}_{k,l}$ and so on, which are determined by formular predicates in $\mathfrak{A}$. For example, $\bar{X}$ and $\bar{\tau}$ are determined by the formulas $M(x)$ and $T(x, y)$, resp. Denote by F_k and $R_{k,l}$ the formulas that determine the conditions $\bar{U}_k$ and the relations $\bar{D}_{k,l}$, resp. Clearly, F_k is a proposition and $R_{k,l}$ a two-place predicate.

Theorem 4. *Let G and G′ be graphs, and let $E=E(G)$, $E'=E(G')$. Then the following holds:*

(i) *if E and E′ are elementarily equivalent then G is elementarily equivalent to a graph that is E-equivalent to G′;*

(ii) *E and E′ are isomorphic iff G is isomorphic to a graph that is E-equivalent to G′.*

Proof. If E and E' are elementarily equivalent and E satisfies U_k then F_k is true for E and E'. Therefore, by Theorem 1, the relations of G and G' are equal to $D_{k,l}$ and $D_{k,m}$ for some l and m. Consider the graph G_1 with the vertex-set of G' and the relation $D_{k,l}$ for the semigroup E'. By Theorem 1, $E(G')=E(G_1)$. Thus the formulas $M(x)$ and $P_{k,l}(x, y)$ determine the graphs $\bar{G}$ (on E) and $\bar{G}_1$ (on E') such that $G\cong\bar{G}$ and $G_1\cong\bar{G}_1$. On the analogy of [3] we obtain that G is elementarily equivalent to one of the graphs $\bar{G}_1$ and $\bar{G}_1^{-1}$. So G is elementarily equivalent to a graph that is E-equivalent to G', i.e. (i) holds.

Now suppose that $E\cong E'$. Then the semigroups are elementarily equivalent. Using the previous reasoning, we can prove that an isomorphism of E onto E' determines the isomorphism of G onto one of the graphs G_1 and G_1^{-1}. Hence G is isomorphic to a graph that is E-equivalent to G'. Theorem 3 implies the converse assertion. Thus (ii) holds. This completes the proof.

Consider a graph G with a reflexive (or antireflexive) relation ϱ such that $\varrho\neq\emptyset$, Δ_X, X^2. One can easily see that the E-equivalence class of G consists only of the graphs G and G^{-1}. Hence Theorem 4 yields the results of [2, 3] on p.endomorphism semigroups of reflexive graphs.

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