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The analytic behavior of the holiday numbers

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1. Introduction

Investigating Hilbert's fourth problem Z. I. Szabó [7] introduced the holiday numbers. In my previous paper [8] many combinatorial and algebraic properties of these numbers were treated. These properties are close to those of the Stirling numbers of the second kind. The aim of the present paper is to investigate the analytic behavior of the holiday numbers. We follow the main ideas of Harper [1], who investigated the analytic behavior of the Stirling numbers of the second kind.

We recall from [8] two possible definitions of the holiday numbers. The holiday numbers of the first kind are $\psi(m, i)$ (of the second kind $\varphi(m, i)$), where

$$(1) \qquad \sum_{m=0}^{\infty} (z^m/m!) \sum_{k=0}^{m} \psi(m, k) t^k = \left(1/\sqrt{1-2z}\right) \exp t\left(1/\sqrt{1-2z}-1\right),$$

$$(1') \qquad \sum_{m=0}^{\infty} (z^m/m!) \sum_{k=0}^{m} \varphi(m, k) t^k = \left(1/(1-2z)\right) \exp\left(t\left(1/\sqrt{1-2z}-1\right)\right).$$

The second definition is

$$(2) \qquad \psi(m, k) = (2m+k-1)\psi(m-1, k)+\psi(m-1, k-1),$$

$$\psi(0, 0) = 1, \quad \psi(0, t) = 0 \quad \text{for} \quad t \neq 0,$$

$$(2') \qquad \varphi(m, k) = (2m+k)\varphi(m-1, k)+\varphi(m-1, k-1),$$

$$\varphi(0, 0) = 1, \quad \varphi(0, t) = 0 \quad \text{for} \quad t \neq 0.$$

We use the notations $\psi_m = \sum_k \psi(m, k)$ and $\varphi_m = \sum_k \varphi(m, k)$.

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2. Results

Statement 1. *The holiday numbers are strongly logconcave in the following sense: for* $0 \leqq k \leqq n$,

$$\psi(n, k)^2 > \psi(n, k-1)\psi(n, k+1), \quad \varphi(n, k)^2 > \varphi(n, k-1)\varphi(n, k+1).$$

The statement is a special case of KURTZ's theorem [2]. It follows that the holiday numbers are of unimodal distribution, for any n their maximum value is attained at most two times. The statement is important to get the corollaries of our theorems.

Theorem 2. ψ_n *and* φ_n *admit asymptotic expansions in the powers of* $n^{1/3}$ *in the following way:*

$$\psi_n \sim \left(n!2^n/e\sqrt{3\pi}\right)e^{2^{-2/3}\cdot 3\cdot n^{1/3}}\left(n^{-1/2}+a_1 n^{-5/6}+a_2 n^{-7/6}+\ldots\right), \tag{3}$$

$$\varphi_n \sim \left(n!2^{n+1/3}/e\sqrt{3\pi}\right)e^{2^{-2/3}\cdot 3\cdot n^{1/3}}\left(n^{-1/6}+b_1 n^{-1/2}+b_2 n^{-5/6}+\ldots\right). \tag{3'}$$

We have also

$$\psi_{n+1}/\psi_n = 2n+(2n)^{1/3}+O(1), \quad \varphi_{n+1}/\varphi_n = 2n+(2n)^{1/3}+O(1),$$

$$\psi_n^{-2}\{-\psi_{n+1}^2+\psi_n\psi_{n+2}-2\psi_n\psi_{n+1}-\psi_n^2\} \sim (2/3)(2n)^{1/3}, \tag{4}$$

$$\varphi_n^{-2}\{-\varphi_{n+1}^2+\varphi_n\varphi_{n+2}-2\varphi_n\varphi_{n+1}-\varphi_n^2\} \sim (2/3)(2n)^{1/3}. \tag{4'}$$

Theorem 3. *The holiday numbers are asymptotically normal in the following sense:*

$$\lim_{n\to\infty}(1/\psi_n)\sum_{j=0}^{x_n}\psi(n, j) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{x} e^{-t^2/2}\,dt, \tag{5}$$

$$\lim_{n\to\infty}(1/\varphi_n)\sum_{j=0}^{y_n}\varphi(n, j) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{y} e^{-t^2/2}\,dt, \tag{5'}$$

where

$$x_n = \psi_{n+1}/\psi_n-(2n+2)+(x/\psi_n)\{-\psi_{n+1}^2+\psi_n\psi_{n+2}-2\psi_n\psi_{n+1}-\psi_n^2\}^{1/2}, \tag{6}$$

$$y_n = \varphi_{n+1}/\varphi_n-(2n+3)+(y/\varphi_n)\{-\varphi_{n+1}^2+\varphi_n\varphi_{n+2}-2\varphi_n\varphi_{n+1}-\varphi_n^2\}^{1/2}; \tag{6'}$$

or

$$x_n = (2n)^{1/3}+x((2/3)(2n)^{1/3})^{1/2}, \quad y_n = (2n)^{1/3}+y((2/3)(2n)^{1/3})^{1/2}. \tag{6''}$$

Corollary 4. *Using the definitions of* x_n, y_n *in* (6), (6′) *or* (6″) *we have*

$$\psi_n^{-2}\{-\psi_{n+1}^2+\psi_n\psi_{n+2}-2\psi_n\psi_{n+1}-\psi_n^2\}^{1/2}\psi(n, [x_n]) \to \left(1/\sqrt{2\pi}\right)e^{-x^2/2},$$

$$\varphi_n^{-2}\{-\varphi_{n+1}^2+\varphi_n\varphi_{n+2}-2\varphi_n\varphi_{n+1}-\varphi_n^2\}^{1/2}\varphi(n, [y_n]) \to \left(1/\sqrt{2\pi}\right)e^{-y^2/2},$$

i.e.

$$\psi(n,[x_n])\sim(n!\,2^{n-7/6}/e\pi)\,e^{2^{-2/3}\cdot 3\cdot n^{1/3}}n^{-2/3}e^{-x^2/2},$$

$$\varphi(n,[y_n])\sim(n!\,2^{n-5/6}/e\pi)\,e^{2^{-2/3}\cdot 3\cdot n^{1/3}}n^{-1/3}e^{-y^2/2}.$$

Corollary 5. *Suppose, for $i=I_n$ the maximum value of $\psi(n,i)$ (for $i=J_n$ the maximum value of $\varphi(n,i)$) is attained. Then for every $\varepsilon>0$ there exists N such that for $n>N$*

$$|I_n-(2n)^{1/3}|<\varepsilon n^{1/6},\quad |J_n-(2n)^{1/3}|<\varepsilon n^{1/6}.$$

Corollary 6.

$$\max_j \psi(n,j)\sim(2\pi)^{-1/2}\psi_n^2\{-\psi_{n+1}^2+\psi_n\psi_{n+2}-2\psi_n\psi_{n+1}-\psi_n^2\}^{-1/2},$$

$$\max_j \varphi(n,j)\sim(2\pi)^{-1/2}\varphi_n^2\{-\varphi_{n+1}^2+\varphi_n\varphi_{n+2}-2\varphi_n\varphi_{n+1}-\varphi_n^2\}^{-1/2},$$

i.e.

$$\max_j \psi(n,j)\sim(n!\,2^{n-7/6}/e\pi)\,e^{2^{-2/3}\cdot 3\cdot n^{1/3}}n^{-2/3},$$

$$\max_j \varphi(n,j)\sim(n!\,2^{n-5/6}/e\pi)\,e^{2^{-2/3}\cdot 3\cdot n^{1/3}}n^{-1/3}.$$

The corollaries follow from the fact, that the convergence in Theorem 3 is uniform behind the integrals. Its reason is Statement 1, and the proof goes on the same way as in Harper's paper.

3. The proof of the theorems

In order to prove (3) and (3′) we have to give the asymptotic expansion of the coefficients of

$$(1/\sqrt{1-2z})\exp(1/\sqrt{1-2z}-1)\quad\text{and}\quad(1/(1-2z))\exp(1/\sqrt{1-2z}-1)$$

(cf. (1), (1′)). It is given in [6], in 25.3, in formula 25.35, in terms of Bessel—Wright functions. The asymptotic expansion of Bessel—Wright functions is given in [5], in 21.5, in formula 21.107. Comparing them we get (3) and (3′). By the theorem concerning the ratio of functions expanded in asymptotic power series ([4], 4.4, Thm. 5—6) we have the following expansions in the powers of $n^{1/3}$:

$$\psi_{n+1}/\psi_n\sim 2n+2^{1/3}n^{1/3}+c_1+\dots,\quad \varphi_{n+1}/\varphi_n\sim 2n+2^{1/3}n^{1/3}+d_1+\dots.$$

Now (4) and (4′) follow easily.

In order to prove Theorem 3 we recall a well-known theorem from probability theory and prove an easy lemma.

Lemma. *The polynomials*

$$P_m(t) = \sum_{k=0}^{m} \psi(m, k)t^k \quad \text{and} \quad Q_m(t) = \sum_{k=0}^{m} \varphi(m, k)t^k$$

have m distinct, real, negative roots.

Proof of the lemma. We prove the statement by mathematical induction. It holds for $P_0(t)=Q_0(t)=1$. By (2) and (2′) we have

$$P_m(x) = (2m-1+x)P_{m-1}(x)+xP'_{m-1}(x), \tag{7}$$

$$Q_m(x) = (2m+x)Q_{m-1}(x)+xQ'_{m-1}(x). \tag{7'}$$

Let the roots of $P_{m-1}(x)$ be $z_1<z_2<\ldots<z_{m-1}<0$ by hypothesis. There are $m-2$ roots of P_m by Rolle's theorem in (z_1, z_{m-1}). There are two other roots by

$$P_m(z_{m-1}) = z_{m-1}P'_m(z_{m-1}) < 0 \quad \text{and} \quad P_m(0) = (2m-1)!! \quad \text{(see (2))},$$

and

$$\operatorname{sign} P_m(z_1) = \operatorname{sign} z_1 P'_{m-1}(z_1) = -\operatorname{sign} P'_{m-1}(-\infty) = -\operatorname{sign} P_m(-\infty).$$

A similar method applies for Q_m.

We continue the proof of Theorem 3. Let the roots of $P_n(x)$ be $\{-y_{nk}: k=1, \ldots, n\}$, the roots of $Q_n(x)$ be $\{-x_{nk}: k=1, \ldots, n\}$. We define the independent random variables Y^*_{nk} and X^*_{nk} by

$$P(Y^*_{nk} = 0) = y_{nk}/(1+y_{nk}), \quad P(Y^*_{nk} = 1) = 1/(1+y_{nk}),$$

$$P(X^*_{nk} = 0) = x_{nk}/(1+x_{nk}), \quad P(X^*_{nk} = 1) = 1/(1+x_{nk}).$$

Let $Z^*_n=\sum_k Y^*_{nk}$, $S^*_n=\sum_k X^*_{nk}$, F_{nk} and E_{nk} the distribution function of X^*_{nk}, Y^*_{nk}. Using (7), (7′) we have

$$E(Z^*_n) = \sum_k \frac{1}{1+y_{nk}} = \frac{P'_n(x)}{P_n(x)}\bigg|_{x=1} = \frac{P_{n+1}(x)-(2n+1+x)P_n(x)}{xP_n(x)}\bigg|_{x=1} =$$

$$= \psi_{n+1}/\psi_n-(2n+2),$$

$$E(S^*_n) = \varphi_{n+1}/\varphi_n-(2n+3),$$

$$D^2(Z^*_n) = E\big(Z^*_n-E(Z^*_n)^2\big)^2 = \sum_k \left(\frac{1}{1+y_{nk}}-\frac{1}{(1+y_{nk})^2}\right) = \tag{8}$$

$$= \frac{P'_n(x)}{P_n(x)}+\left(\frac{P'_n(x)}{P_n(x)}\right)'\bigg|_{x=1} = \frac{-\psi^2_{n+1}+\psi_n\psi_{n+2}-2\psi_n\psi_{n+1}-\psi^2_n}{\psi^2_n},$$

(we used (7) twice),

$$D^2(S^*_n) = E\big(S^*_n-E(S^*_n)^2\big)^2 = (-\varphi^2_{n+1}+\varphi_n\varphi_{n+2}-2\varphi_n\varphi_{n+1}-\varphi^2_n)/\varphi^2_n. \tag{8'}$$

Let us define

$$(9) \qquad Z_n = \big(1/D(Z_n^*)\big)\big(Z_n^* - E(Z_n^*)\big) = \sum_k \big(1/D(Z_n^*)\big)\big(X_{nk}^* - E(X_{nk}^*)\big),$$

$$(9') \qquad S_n = \big(1/D(S_n^*)\big)\big(S_n^* - E(S_n^*)\big) = \sum_k \big(1/D(S_n^*)\big)\big(Y_{nk}^* - E(Y_{nk}^*)\big).$$

From (4), (4'), (8), (8') we get $D(Z_n^*) \to \infty$, $D(S_n^*) \to \infty$. We are in a position to apply the Lindeberg—Feller Theorem ([3], p. 295) for Z_n^* and S_n^*, since

$$|X_{nk}^* - E(X_{nk}^*)| \leqq 1, \quad |Y_{nk}^* - E(Y_{nk}^*)| \leqq 1,$$

and for a number n large enough

$$\sum_k \int_{|x| \geqq \varepsilon} x^2 \, dF_{nk}(x) = 0, \quad \sum_k \int_{|x| \geqq \varepsilon} x^2 \, dE_{nk}(x) = 0.$$

Since the generating function of a sum of independent random variables is the product of the generating functions,

$$\prod_{k=1}^{n} \frac{x + x_{nk}}{1 + x_{nk}} = \frac{P_n(x)}{P_n(1)}, \quad \prod_{k=1}^{n} \frac{x + y_{nk}}{1 + y_{nk}} = \frac{Q_n(x)}{Q_n(1)},$$

we have $P(Z_n^* = a) = \psi(n, a)/\psi_n$, $P(S_n^* = a) = \varphi(n, a)/\varphi_n$. Now the theorem is proved by (9), (9').

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