*Acta Sci. Math.*, 51 (1987), 371—386

# $p$-algebras with Stone congruence lattices

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**1. Introduction.** In [12] we have described by means of subdirect factorization congruence distributive algebras $A$ whose congruence lattices $\mathrm{Con}(A)$ are atomic, Boolean, or Stonean. The purpose of this paper is to give an intrinsic characterization of those quasi-modular $p$-algebras whose congruence lattices are atomic, Stonean or relatively Stonean. To obtain this we use the representation of congruence relations of quasi-modular $p$-algebras in terms of congruence pairs. That means (see [11]), that every congruence relation $\alpha \in \mathrm{Con}(L)$ of a quasi-modular $p$-algebra $L$ can be uniquely represented by a congruence pair $(\alpha_B, \alpha_D)$, where $\alpha_B$ is a (Boolean) congruence relation of $B(L)$ and $\alpha_D$ a (lattice) congruence relation of $D(L)$.

We start with a description of congruence pairs corresponding to (relative) pseudocomplements in the lattice $\mathrm{Con}(L)$ (Theorem 1). By way of application, we characterize those quasi-modular $p$-algebras with atomic congruence lattices (Theorems 2, 3 and 4). As a second application we provide a characterization of (relative) Stone congruence lattices of quasi-modular $p$-algebras (Theorems 5, 6 and 11). Analogous, but deeper results, are obtained for distributive $p$-algebras (Theorems 7, 8 and 12).

**2. Preliminaries.** A (modular, distributive) *p-algebra* or *pseudocomplemented lattice* is an algebra $(L; \vee, \wedge, ^*, 0, 1)$ in which the deletion of the unary operation $^*$ yields a bounded (modular, distributive) lattice and $^*$ is the operation of pseudocomplementation, that is, $x \leqq a^*$ if and only if $a \wedge x = 0$. A $p$-algebra is said to be *quasi-modular* if it satisfies the identity

$$[(x\wedge y)\vee z^{**}]\wedge x = (x\wedge y)\vee(z^{**}\wedge x).$$

The variety of quasi-modular $p$-algebras properly contains the class of modular $p$-algebras and is properly contained in the class of $p$-algebras satisfying the identity

$$x = x^{**}\wedge(x\vee x^*).$$

Receivde May 25, 1984.

If, for any $p$-algebra $L$, we write

$$B(L)=\{x\in L\colon\ x=x^{**}\}\quad\text{and}\quad D(L)=\{x\in L\colon\ x^{*}=0\}$$

then $(B(L);\ \mathbf{V},\ \wedge,\ ^{*},0,1)$ is a Boolean algebra (of closed elements) when $a\mathbf{V}b$ is defined to be $(a^{*}\wedge b^{*})^{*}$, for any pair $a, b\in B(L)$, and $D(L)$ is a filter in $L$ (of dense elements). By a congruence relation of a $p$-algebra we mean a lattice congruence of $L$ preserving $^{*}$. The relation $\gamma$ of $L$ defined by $a\equiv b(\gamma)$ if and only if $a^{*}=b^{*}$ is a congruence relation of $L$, called the *Glivenko congruence* of $L$, and $L/\gamma\cong B(L)$. The lattice Con $(L)$ of all congruence relations of a $p$-algebra $L$ is algebraic and distributive, which implies that Con $(L)$ is a distributive $p$-algebra. The least and greatest elements of Con $(L)$ will be denoted by $\Delta$ and $\nabla$, respectively.

A distributive $p$-algebra $L$ in which the identity

$$x^{*}\vee x^{**}=1$$

holds is called a *Stone algebra (lattice)*. A *relative Stone algebra (lattice)* is a distributive lattice in which every interval $[a, b]$ is a Stone lattice.

A double $p$-algebra is an algebra $(L;\ \vee,\ \wedge,\ ^{*},\ ^{+},0,1)$ in which the deletion of $^{+}$ gives a $p$-algebra and the deletion of $^{*}$ gives a dual $p$-algebra, that is $a\vee x=1$ if and only if $x\geqq a^{+}$. The relation $\Phi$ of $L$ defined by

$$a\equiv b(\Phi)\ \text{ if and only if }\ a^{*}=b^{*}\ \text{ and }\ a^{+}=b^{+}$$

is a congruence relation of $L$, called the *determination congruence.* It is known that a double $p$-algebra is *regular* (that is, any two congruence relations of $L$ having a class in common are the same) if and only $\Phi=\Delta$ (see [16]).

A special class of distributive $p$-algebras is formed by the *Heyting algebras* $(L;\ \vee,\ \wedge,\ *,0,1)$, where $(L;\ \vee,\ \wedge,0,1)$ is a bounded lattice and $x\wedge y\leqq z$ if and only if $y\leqq x*z$. Then $x^{*}=x*0$ plays the role of a pseudocomplement of $x$. It is easy to verify that Con $(L)$ of a $p$-algebra $L$ is even a Heyting algebra.

A lattice with 0 is called *atomic,* if for every $a\neq 0$ there exists an atom $p\leqq a$.

We refer to [1], [8] or [10] for the standard results about $p$-algebras and to [1], [9] or [16] for the standard results about double $p$-algebras. For general lattice-theoretic terminology, notation and results we follow G. Grätzer [6].

**3. Congruence pairs.** Let $(L;\ \vee,\ \wedge,\ ^{*},0,1)$, henceforth simply $L$, be a quasi-modular $p$-algebra. Let Con $(L)$ denote the lattice of congruence relations of $L$. Since Con $(L)$ is a Heyting algebra, there exists a complete Boolean algebra $B(\text{Con}\,(L))$ of closed elements (congruences) and the filter of dense elements (congruences) $D(\text{Con}\,(L))$. We shall also consider Con $(B(L))$, the lattice of (Boolean) congruence relations of $B(L)$ and Con $(D(L))$, the lattice of (lattice) congruence relations of $D(L)$.

Having $\Theta \in \mathrm{Con}(L)$, the restrictions $\Theta_B = \Theta | B(L)$ and $\Theta_D = \Theta | D(L)$ are congruence relations of $B(L)$ and $D(L)$, respectively. Hence, there exists an isotone map $\Theta \mapsto (\Theta_B, \Theta_D)$ from $\mathrm{Con}(L)$ into $\mathrm{Con}(B(L)) \times \mathrm{Con}(D(L))$. The following definition is crucial (see also [11], [4]).

A pair $(\Theta_1, \Theta_2) \in \mathrm{Con}(B(L)) \times \mathrm{Con}(D(L))$ is said to be a *congruence pair* of $L$ if the following condition holds: $a \in B(L)$, $u \in D(L)$, $u \geqq a$ in $L$, and $a \equiv 1(\Theta_1)$ imply that $u \equiv 1(\Theta_2)$.

Theorem A (see [11, Theorem 1]). *Every congruence relation $\Theta$ of a quasi-modular p-algebra $L$ determines a congruence pair $(\Theta_B, \Theta_D)$ and, conversely, every congruence pair $(\Theta_1, \Theta_2)$ of $L$ determines a unique congruence relation $\Theta$ of $L$ having the property that $\Theta_B = \Theta_1$ and $\Theta_D = \Theta_2$. Moreover, $x \equiv y(\Theta)$ if and only if $x^* \equiv y^*(\Theta_1)$ and $x \vee x^* = y \vee y^*(\Theta_2)$.*

In what follows we shall often identify $\Theta \in \mathrm{Con}(L)$ with the corresponding congruence pair $(\Theta_B, \Theta_D)$. If there is no danger of confusion, we shall omit the subscripts in notation of some congruence pairs, e.g. $\Delta = (\Delta, \Delta)$, $\nabla = (\nabla, \nabla)$, $(\Delta, \alpha)$.

Clearly, having $\alpha \in \mathrm{Con}(B(L))$, there exists $\mathrm{Ker}\,\alpha = J \in I(B(L))$ (=the lattice of all ideals of $B(L)$) such that $\alpha = \Theta[J]$. Similarly, for $\beta \in \mathrm{Con}(D(L))$, $\mathrm{Ker}\,\beta = \{x \in D(L) \colon x \equiv 1(\beta)\}$ is a filter of $D(L)$, i.e. $\mathrm{Ker}\,\beta \in F(D(L))$.

Given a quasi-modular $p$-algebra $L$, there is a map $\varphi(L) \colon B(L) \to F(D(L))$ defined as follows:

$$a\varphi(L) = \{x \in D(L) \colon x \geqq a^*\} = [a^*) \cap D(L).$$

This map proved instrumental in characterizing the quasi-modular $p$-algebras (see [13]). We shall need the following result.

Theorem B (see [13, Theorem 3]). *In a quasi-modular p-algebra $L$, the map $\varphi(L) \colon B(L) \to F(D(L))$ is a $\{0, 1, \vee\}$-homomorphism.*

Now, we can reformulate the definition of a congruence pair.

Lemma 1. *Let $L$ be a quasi-modular p-algebra and let $(\Theta_1, \Theta_2) \in \mathrm{Con}(B(L)) \times \mathrm{Con}(D(L))$. Then $(\Theta_1, \Theta_2)$ is a congruence pair if and only if $J\varphi(L) := \bigcup (a\varphi(L) \colon a \in J) \subseteqq \mathrm{Ker}\,\Theta_2$, where $J = \mathrm{Ker}\,\Theta_1$.*

Proof. Clearly, $a \in J = \mathrm{Ker}\,\Theta_1$ if and only if $a^* \equiv 1(\Theta_1)$. Therefore, $J\varphi(L) \subseteqq \mathrm{Ker}\,\Theta_2$ if and only if $(\Theta_1, \Theta_2)$ is a congruence pair.

From Lemma 1 we see that for every $\Theta_1 \in \mathrm{Con}(B(L))$ with $J = \mathrm{Ker}\,\Theta_1$ there exists a smallest $\delta(\Theta_1) \in \mathrm{Con}(D(L))$ such that $J\varphi(L) \subseteqq \mathrm{Ker}\,\delta(\Theta_1)$. That means, $(\Theta_1, \Theta_2)$ is a congruence pair of $L$ if and only if $\Theta_2 \geqq \delta(\Theta_1)$. Dually, for every $\Theta_2 \in \mathrm{Con}(D(L))$ there exists a largest ideal $J \in I(B(L))$ such that $J\varphi(L) \subseteqq \mathrm{Ker}\,\Theta_2$, i.e. $(\Theta[J], \Theta_2)$ is a congruence pair. Notation: $\tau(\Theta_2) = \Theta[J]$. Evidently, $(\Theta_1, \Theta_2)$ is a congruence pair of $L$ if and only if $\tau(\Theta_2) \geqq \Theta_1$.

An abstract description of the lattice of all congruence pairs of quasi-modular $p$-algebras can be found in [4]. In the next theorem we give a description of (relative) pseudocomplements in $\mathrm{Con}(L)$ by means of congruence pairs.

Theorem 1. *Let $L$ be a quasi-modular p-algebra and let $\alpha, \beta \in \mathrm{Con}(L)$. Then $(\alpha_B \vee \beta_B, \alpha_D \vee \beta_D)$, $(\alpha_B \wedge \beta_B, \alpha_D \wedge \beta_D)$ and $(\alpha_B * \beta_B \wedge \tau(\alpha_D * \beta_D), \alpha_D * \beta_D)$ are congruence pairs of $\alpha \vee \beta$, $\alpha \wedge \beta$ and $\alpha * \beta$, respectively. In particular,*

$$(\alpha_B^* \wedge \tau(\alpha_D), \alpha_D^*) \quad \text{and} \quad ((\alpha_B^* \wedge \tau(\alpha_D^*))^* \wedge \tau(\alpha_D^{**}), \alpha_D^{**})$$

*are congruence pairs of $\alpha^*$ and $\alpha^{**}$, respectively.*

Proof. Clearly, $(\alpha \vee \beta)_B \geqq \alpha_B \vee \beta_B$ and $(\alpha \vee \beta)_D \geqq \alpha_D \vee \beta_D$. Assume $a \equiv b(\alpha \vee \beta)$ for $a, b \in B(L)$. Then there exists a finite sequence $a = z_0, \ldots, z_n = b$ such that $z_{i-1} \equiv z_i(\alpha)$ or $z_{i-1} \equiv z_i(\beta)$ for every $i = 1, \ldots, n$. Therefore $z_{i-1}^{**} \equiv z_i^{**}(\alpha)$ or $z_{i-1}^{**} \equiv \equiv z_i^{**}(\beta)$, which implies $a \equiv b(\alpha_B \vee \beta_B)$. Hence $(\alpha \vee \beta)_B = \alpha_B \vee \beta_B$. A similar argument yields $(\alpha \vee \beta)_D = \alpha_D \vee \beta_D$, $(\alpha \wedge \beta)_B = \alpha_B \wedge \beta_B$ and $(\alpha \wedge \beta)_D = \alpha_D \wedge \beta_D$.

It is easy to verify that $(\alpha_B * \beta_B \wedge \tau(\alpha_D * \beta_D), \alpha_D * \beta_D)$ is a congruence pair of $L$. Clearly,

$$(\alpha_B, \alpha_D) \wedge (\alpha_B * \beta_B \wedge \tau(\alpha_D * \beta_D), \alpha_D * \beta_D) \leqq (\beta_B, \beta_D).$$

Assume $(\alpha_B, \alpha_D) \wedge (\eta_B, \eta_D) \leqq (\beta_B, \beta_D)$ in $\mathrm{Con}(L)$. Therefore, $\eta_B \leqq \alpha_B * \beta_B$ and $\eta_D \leqq \alpha_D * \beta_D$. Since $(\eta_B, \eta_D)$ is a congruence pair, we have $\eta_B \leqq \tau(\eta_D) \leqq \tau(\alpha_D * \beta_D)$. Hence $(\eta_B, \eta_D) \leqq (\alpha_B * \beta_B \wedge \tau(\alpha_D * \beta_D), \alpha_D * \beta_D)$. The last part of Theorem can be established in the same way because $(\alpha_B, \alpha_D)^* = (\alpha_B, \alpha_D) * (\Delta, \Delta)$.

Corollary 1 (see [1, Theorem 2]). *Let $L$ be a quasi-modular p-algebra. Then* $\mathrm{Con}(D(L)) \cong [\Delta, \gamma]$, *where $\gamma$ is the Glivenko congruence.*

Proof. Consider the map $\alpha_2 \mapsto (\Delta, \alpha_2)$ from $\mathrm{Con}(D(L))$ into $\mathrm{Con}(L)$. Since $\gamma = (\Delta, \nabla)$, we see that this map is an isomorphism between $\mathrm{Con}(D(L))$ and $[\Delta, \gamma]$.

Corollary 2. *Let $L$ be a quasi-modular p-algebra. Then* $\mathrm{Con}(B(L)) \cong [\gamma, \nabla]$.

Proof. Consider the map $\alpha_1 \mapsto (\alpha_1, \nabla)$ from $\mathrm{Con}(B(L))$ into $\mathrm{Con}(L)$. This map is an isomorphism between $\mathrm{Con}(B(L))$ and $[\gamma, \nabla]$.

**4. Atomic congruence lattices.** In [12] we have extended Tanaka's result [15, Theorem 1].

Theorem C. *Let $A$ be a congruence distributive algebra. Then the following conditions are equivalent:*

(i) $\mathrm{Con}(A)$ *is atomic;*

(ii) $D(\mathrm{Con}(A))$ *is a principal filter;*

(iii) $B(\mathrm{Con}\,(A))$ *is atomic and every dual atom of* $B(\mathrm{Con}\,(A))$ *is completely meet-irreducible in* $\mathrm{Con}\,(A)$;

(iv) $\mathrm{Con}\,(A)$ *satisfies the (infinite) identity*

$$\wedge(x_i^{**}\colon i\in I) = (\wedge(x_i\colon i\in I))^{**}.$$

Lemma 2. *Let $L$ be a quasi-modular p-algebra. Then $\alpha=(\alpha_B, \alpha_D)$ is an atom of* $\mathrm{Con}\,(L)$ *if and only if*

(i) $\alpha_B=\Delta$ *and $\alpha_D$ is an atom of* $\mathrm{Con}\,(D(L))$

*or*

(ii) $\alpha_D=\Delta$, $\alpha_B\leqq\tau(\Delta)$ *and $\alpha_B$ is an atom of* $\mathrm{Con}\,(B(L))$.

Proof. Suppose that $(\alpha_B, \alpha_D)$ is an atom of $\mathrm{Con}\,(L)$. Two cases can arise: $\alpha_D\neq\Delta$ or $\alpha_D=\Delta$. In the first event $(\Delta, \alpha_D)\leqq\alpha$, whence $\alpha=(\Delta, \alpha_D)$ and $\alpha_D$ is an atom of $\mathrm{Con}\,(D(L))$. In the second case we obtain (ii). The converse is trivial.

Theorem 2. *Let $L$ be a quasi-modular p-algebra. Then* $\mathrm{Con}\,(L)$ *is atomic if and only if*

(i) $\mathrm{Con}\,(D(L))$ *is atomic*

*and*

(ii) $\{a\in B(L)\colon a\varphi(L)=[1)\}$ *is an atomic ideal of $B(L)$, i.e. it is an atomic lattice.*

Proof. Assume that $\mathrm{Con}\,(L)$ is atomic. Therefore, $[\Delta, \gamma]$ is atomic as well. By Corollary 1 of Theorem 1 we obtain (i). Take $0\neq a\in B(L)$ with $a\varphi(L)=[1)$. By Lemma 1, $(\Theta[(a]], \Delta)\in\mathrm{Con}\,(L)$. There exists an atom $\alpha\in\mathrm{Con}\,(L)$ with $\alpha=(\alpha_B, \alpha_D)\leqq$ $\leqq(\Theta[(a]], \Delta)$. Hence $\alpha_D=\Delta$, $\alpha_B\leqq\tau(\Delta)$ and $\alpha_B$ is an atom of $\mathrm{Con}\,(B(L))$ (Lemma 2). Thus $\mathrm{Ker}\,\alpha_B=(b]$ and $b$ is an atom of $B(L)$ with $b\varphi(L)=[1)$.

Conversely, assume (i) and (ii). Take $\Delta\neq\alpha=(\alpha_B, \alpha_D)$ from $\mathrm{Con}\,(L)$. Two cases can occur: $\alpha_D\neq\Delta$ or $\alpha_D=\Delta$. In the first case, there is by (i) an atom $\beta\in\mathrm{Con}\,(D(L))$ with $\beta\leqq\alpha_D$. Hence $(\Delta, \beta)$ is by Corollary 1 to Theorem 1 an atom of $\mathrm{Con}\,(L)$ and $(\Delta, \beta)\leqq(\alpha_B, \alpha_D)$. In the second case, $\Delta\neq\alpha_B\leqq\tau(\Delta)$. There exists an atom $a\in J=\mathrm{Ker}\,\alpha_B$ by (ii). Hence $(\Theta[(a]], \Delta)$ is an atom of $\mathrm{Con}\,(L)$ (Lemma 2) and $(\Theta[(a]], \Delta)\leqq\alpha$.

Lemma 3. *Let $K$ be an ideal of a Boolean algebra $B$ and let $K$ be an atomic sublattice of $B$. Let $J$ be the ideal of $B$ generated by all atoms of $K$. Then $J^*=K^*$ in the lattice $I(B)$ of all ideals of $B$.*

Proof. Clearly $J\subseteqq K$. Therefore, $J^*\supseteqq K^*$. Take $b\in J^*$. If $(b]\cap K\neq(0]$, then there exists an atom $a\in K$ such that $a\leqq b$. Hence $a\in J\cap J^*=(0]$, a contradiction. Thus, $K\cap J^*=(0]$, which implies $J^*\subseteqq K^*$. So, $J^*=K^*$.

Theorem 3. *Let $L$ be a quasi-modular p-algebra. Then $(\beta_1, \beta_2)\in\mathrm{Con}\,(L)$ is the smallest element of $D(\mathrm{Con}\,(L))$ if and only if*

(i) *$\beta_2$ is the smallest element of* $D(\mathrm{Con}\,(D(L)))$

*and*

(ii) *the ideal* $K=\{a\in B(L)\colon a\varphi(L)=[1)\}$ *of* $B(L)$ *is atomic and* $\beta_1=\Theta[J]$, *where* $J$ *is the ideal of* $B(L)$ *generated by all atoms of* $K$.

Proof. Let $(\beta_1, \beta_2)\in \mathrm{Con}\,(L)$ be the smallest element of $D(\mathrm{Con}\,(L))$. It is easy to verify that $(\tau(\Delta), \alpha)\in \mathrm{Con}\,(L)$ for every $\alpha\in \mathrm{Con}\,(D(L))$. Moreover, $(\tau(\Delta), \alpha)^*=$ $=\Delta$ if and only if $\alpha\in D(\mathrm{Con}\,(D(L)))$. Therefore, $(\beta_1, \beta_2)\leqq(\tau(\Delta), \alpha)$ for every $\alpha\in D(\mathrm{Con}\,(D(L)))$. Thus $\beta_2$ is the smallest element of $D(\mathrm{Con}\,(D(L)))$ and $\beta_1\leqq\tau(\Delta)$. Since $\Delta=(\beta_1, \beta_2)^*=(\beta_1^*\wedge\tau(\Delta), \Delta)$, we see that $\beta_1^*\leqq\tau(\Delta)^*$. But $\beta_1\leqq\tau(\Delta)$ implies $\beta_1^*\geqq\tau(\Delta)^*$. Hence $\beta_1^*=\tau(\Delta)^*$. Clearly, $\beta_1=\Theta[M]$ and $\tau(\Delta)=\Theta[K]$, where $M$ is an ideal of $B(L)$ and $M\subseteq K$. According to Theorems C and 2, $K$ is atomic. Without difficulties one can check that $M$ contains all atoms of $K$, as $\beta_1^*=\tau(\Delta)^*$. Let $J$ denote the ideal of $B(L)$ generated by all atoms of $K$. Lemma 3 yields $\beta_1^*=\tau(\Delta)^*=$ $=\Theta[J]^*$. Now, $(\Theta[J], \beta_2)^*=(\beta_1^*\wedge\tau(\Delta), \beta_2^*)=\Delta$ implies $\beta_1\leqq\Theta[J]$. Eventually, $\beta_1=\Theta[J]$.

Conversely, let $L$ satisfy (i) and (ii). Take $\beta_1=\Theta[J]$ from $\mathrm{Con}\,(B(L))$ and $\beta_2\in\mathrm{Con}\,(D(L))$ as defined in (i) and (ii). Clearly, $(\beta_1, \beta_2)\in \mathrm{Con}\,(L)$, as $\beta_1\leqq\tau(\Delta)$. By Lemma 3, $\beta_1^*=\tau(\Delta)^*$. Therefore, $(\beta_1, \beta_2)^*=\Delta$; that means $(\beta_1, \beta_2)\in D(\mathrm{Con}\,(L))$. Consider $(\alpha_1, \alpha_2)\in D(\mathrm{Con}\,(L))$. Since $\alpha_2^*=\Delta$, we have $\beta_2\leqq\alpha_2$. In addition, $\alpha_1^*\wedge$ $\wedge\tau(\Delta)=\Delta$. Hence $\alpha_1^*\leqq\tau(\Delta)^*=\beta_1^*$. Clearly $\alpha_1=\Theta[M]$ for some ideal $M$ of $B(L)$. We claim that $M\supseteq J$. Really, if $J\nsubseteq M$, then there exists an atom $a\in J-M$ and $a\in M^*$. That means $\Theta[(a]]\leqq\alpha_1^*\wedge\beta_1=\Delta$, a contradiction. Therefore, $J\subseteq M$, as claimed. Hence $\beta_1\leqq\alpha_1$, and $(\beta_1, \beta_2)$ is the smallest dense congruence relation of $L$. The proof is complete.

Lemma 4. *Let $L$ be a quasi-modular p-algebra. Let* $a\in B(L)$ *with* $a\varphi(L)=[1)$. *Then* $(\Theta[(a]], \Delta)\in B(\mathrm{Con}\,(L))$.

Proof. Since $\Theta[(a]]\leqq\tau(\Delta)$, we see that $(\Theta[(a]], \Delta)\in \mathrm{Con}\,(L)$. By Theorem 1, $(\Theta[(a]], \Delta)^{**}=(\Theta[(a]]^{**}\wedge\tau(\Delta), \Delta)$. Since $\Theta[(a]]^{**}=\Theta[(a]]$, the proof is complete.

Theorem 4. *Let $L$ be a quasi-modular p-algebra. Then* $B(\mathrm{Con}\,(L))$ *is atomic if and only if*

(i) $B(\mathrm{Con}\,(D(L)))$ *is atomic*

*and*

(ii) $\{a\in B(L)\colon a\varphi(L)=[1)\}$ *is an atomic ideal of* $B(L)$.

Proof. Assume that $B(\mathrm{Con}\,(L))$ is atomic. Let $\Delta\neq\alpha\in B(\mathrm{Con}\,(D(L)))$. Therefore, $(\Delta, \alpha)\in \mathrm{Con}\,(L)$. Clearly, $(\Delta, \alpha)^{**}=(\tau(\alpha^*)^*\wedge\tau(\alpha), \alpha)\neq\Delta$, by Theorem 1. By assumption there exists an atom $(\beta_1, \beta_2)$ of $B(\mathrm{Con}\,(L))$ such that $(\beta_1, \beta_2)\leqq$ $\leqq(\tau(\alpha^*)^*\wedge\tau(\alpha), \alpha)$. Evidently, $\beta_2^{**}=\beta_2$ in $\mathrm{Con}\,(D(L))$. Hence $\beta_2\leqq\alpha$. We claim that $\beta_2$ is an atom of $B(\mathrm{Con}\,(D(L)))$. First we show that $\beta_2 \neq \Delta$. Assume to the contrary that $\beta_2 = \Delta$. Hence $\beta_1 \leqq \tau(\Delta)$. Since $\tau(\Delta) \leqq \tau(\alpha^*)$, we get $\beta_1 \leqq \tau(\alpha^*)^* \leqq \leqq \tau(\Delta)^*$. Therefore $\beta_1 = \Delta$, a contradiction. Thus $\beta_2 \neq \Delta$. Take $\Delta \neq \eta \in B(\mathrm{Con}\,(D(L)))$ with $\eta \leqq \beta_2$. Therefore $\Delta \neq (\Delta, \eta) \leqq (\beta_1, \beta_2)$ implies $(\Delta, \eta)^{**} = (\beta_1, \beta_2)$, as $(\beta_1, \beta_2)$ is an atom of $B(\mathrm{Con}\,(L))$. But $(\Delta, \eta)^{**} = (\tau(\eta^*)^* \wedge \tau(\eta), \eta)$. Hence $\eta = \beta_2$ and $\beta_2$ is an atom of $B(\mathrm{Con}\,(D(L)))$, as claimed. The second part of Theorem follows from Lemma 4.

Conversely, let $L$ satisfy (i) and (ii). Consider $\Delta \neq (\alpha_1, \alpha_2) \in B(\mathrm{Con}\,(L))$. Clearly $\alpha_2 = \alpha_2^{**}$ in $\mathrm{Con}\,(D(L))$. If $\alpha_2 = \Delta$ then $\alpha_1 \leqq \tau(\Delta)$, and $\alpha_1 = \Theta[J]$, where $J$ is an ideal of $\{a \in B(L)\colon a\varphi(L) = [1)\}$. By (ii) there exists an atom $a \in J$. Put $\beta_1 = \Theta[(a]]$ in $\mathrm{Con}\,(B(L))$. Clearly $(\beta_1, \Delta)^{**} = (\beta_1, \Delta) \leqq (\alpha_1, \Delta)$, using Lemma 4. Thus $(\beta_1, \Delta)$ is an atom of $B(\mathrm{Con}\,(L))$. Assume $\alpha_2 \neq \Delta$. Then there exists an atom $\beta_2 \leqq \alpha_2$ in $B(\mathrm{Con}\,(D(L)))$ by (i). Since $(\Delta, \beta_2) \leqq (\alpha_1, \alpha_2)$, we see that

$$(\Delta, \beta_2)^{**} = (\tau(\beta_2^*)^* \wedge \tau(\beta_2), \beta_2) \leqq (\alpha_1, \alpha_2) = (\alpha_1, \alpha_2)^{**}.$$

It remains to verify that $(\Delta, \beta_2)^{**}$ is an atom of $B(\mathrm{Con}\,(L))$. Really, suppose that there exists $\Delta \neq (\eta_1, \eta_2) \in B(\mathrm{Con}\,(L))$ with $(\eta_1, \eta_2) \leqq (\Delta, \beta_2)^{**}$. Two cases can arise: $\eta_2 \neq \Delta$ or $\eta_2 = \Delta$. But $\eta_2 \neq \Delta$ implies $\beta_2 = \eta_2$. Moreover, $(\Delta, \beta_2) \leqq (\eta_1, \eta_2) \leqq \leqq (\Delta, \beta_2)^{**}$ implies $(\eta_1, \eta_2)^{**} = (\eta_1, \eta_2) = (\Delta, \beta_2)^{**}$. Assume $\eta_2 = \Delta$. Therefore, $\eta_1 \leqq \leqq \tau(\Delta) \leqq \tau(\beta_2^*)$. Similarly as above, $\eta_1 \leqq \tau(\beta_2^*)^* \leqq \tau(\Delta)^*$, which implies $\eta_1 = \Delta$, a contradiction. Thus, $(\Delta, \beta_2)^{**}$ is an atom of $B(\mathrm{Con}\,(L))$ and the proof is complete.

## 5. Stonean congruence lattices.

Lemma 5. *Let $L$ be a Stone lattice and $a \in L$. Then $[0, a]$ is also a Stone lattice.*

The proof is straightforward (see [8, 2.11]).

Theorem 5. *Let $L$ be a quasi-modular p-algebra. Then* $\mathrm{Con}\,(L)$ *is a Stone lattice if and only if*

(i) $\mathrm{Con}\,(D(L))$ *is a Stone lattice,*

(ii) *if* $(\alpha_1, \alpha_2) \in \mathrm{Con}\,(L)$ *then* $\mathrm{Ker}\,(\alpha_1^* \wedge \tau(\alpha_2^*)) = (a]$ *for some* $a \in B(L)$,

(iii) *if* $\alpha \in \mathrm{Con}\,(D(L))$ *then* $\tau(\alpha^{**}) \geqq (\tau(\alpha)^* \wedge \tau(\alpha^*))^*$.

Proof. Suppose that $\mathrm{Con}\,(L)$ is a Stone lattice. The condition (i) follows directly from Lemma 5 and Corollary 1 to Theorem 1. Take now $(\alpha_1, \alpha_2) \in \mathrm{Con}\,(L)$. By Theorem 1 and the hypothesis,

$$\nabla = (\alpha_1, \alpha_2)^* \vee (\alpha_1, \alpha_2)^{**} = (\alpha_1^* \wedge \tau(\alpha_2^*), \alpha_2^*) \vee ((\alpha_1^* \wedge \tau(\alpha_2^*))^* \wedge \tau(\alpha_2^{**}), \alpha_2^{**}).$$

Therefore,

$$(\alpha_1^* \wedge \tau(\alpha_2^*)) \vee [(\alpha_1^* \wedge \tau(\alpha_2^*))^* \wedge \tau(\alpha_2^{**})] = \nabla.$$

Consequently, $(\alpha_1^*\wedge\tau(\alpha_2^*))\vee(\alpha_1^*\wedge\tau(\alpha_2^*))^*=\nabla$. Hence, $(\alpha_1^*\wedge\tau(\alpha_2^*))^*$ is a complement of $\alpha_1^*\wedge\tau(\alpha_2^*)$ in $\mathrm{Con}(B(L))$, and $(\alpha_1^*\wedge\tau(\alpha_2^*))^*=(\alpha_1^*\wedge\tau(\alpha_2^*))^*\wedge\tau(\alpha_2^{**})$. Thus $\tau(\alpha_2^{**})\geqq\geqq(\alpha_1^*\wedge\tau(\alpha_2^*))^*$. As $(\tau(\alpha),\alpha)\in\mathrm{Con}(L)$ for every $\alpha$ from $\mathrm{Con}(D(L))$, this yields (iii). The condition (ii) follows from the fact that $\alpha_1^*\wedge\tau(\alpha_2^*)=\Theta[J]$ and $(\alpha_1^*\wedge\tau(\alpha_2^*))^*==\Theta[J^*]$ for some $J\in I(B(L))$. By the hypothesis, $J^*$ is a complement of $J$ in $I(B(L))$. It follows that $J=(a]$ and $J=(a^*]$ for some $a\in B(L)$ (see [5] or [7]).

Conversely, suppose that $L$ satisfies (i)—(iii). Take $(\alpha_1,\alpha_2)\in\mathrm{Con}(L)$. Clearly $(\alpha_1,\alpha_2)\leqq(\tau(\alpha_2),\alpha_2)$. By Theorem 1 and the hypothesis,

$$(\alpha_1,\alpha_2)^*\vee(\alpha_1,\alpha_2)^{**}=(\alpha_1^*\wedge\tau(\alpha_2^*),\alpha_2^*)\vee(((\alpha_1^*\wedge\tau(\alpha_2^*))^*\wedge\tau(\alpha_2^{**}),\alpha_2^{**})=$$
$$=(\Theta[(a]],\alpha_2^*)\vee(\Theta[(a^*]],\alpha_2^{**})=\nabla,$$

because $(\alpha_1^*\wedge\tau(\alpha_2^*))^*\leqq(\tau(\alpha_2)^*\wedge\tau(\alpha_2^*))^*$. The proof is complete.

Corollary. *Let $L$ be a quasi-modular p-algebra and let* $\mathrm{Con}(L)$ *be a Stone lattice. Then for* $(\alpha_1,\alpha_2)\in\mathrm{Con}(L)$ *we have*

(i) $(\alpha_1^{**},\alpha_2^{**})\in\mathrm{Con}(L)$,

(ii) $(\alpha_1,\alpha_2)^{**}=(a_1^{**},\alpha_2^{**})^{**}=((\alpha_1^*\wedge\tau(\alpha_2^*))^*,\alpha_2^{**})$

*and*

(iii) $\alpha\in\mathrm{Con}(D(L))$ *implies* $\tau(\alpha^{**})\geqq\tau(\alpha)^{**}\vee\tau(\alpha^*)^*$.

Proof. (i) Since, by Theorem 5, $\alpha_1^{**}\leqq\tau(\alpha_2^{**})$, we have $(\alpha_1^{**},\alpha_2^{**})\in\mathrm{Con}(L)$. (ii) and (iii) follows from Theorems 1 and 5.

In [12] we have also investigated algebras whose congruence lattices satisfy the (infinite) identity

$$\vee(x_i^{**}\colon i\in I)=(\vee(x_i\colon i\in I))^{**}. \tag{1}$$

Theorem 6. *Let $L$ be a quasi-modular p-algebra. Then* $\mathrm{Con}(L)$ *satisfies the identity* (1) *if and only if*

(i) $\mathrm{Con}(L)$ *is a Stone lattice*

*and*

(ii) $B(\mathrm{Con}(L))$ *is finite.*

Proof. Assume that $\mathrm{Con}(L)$ satisfies the identity (1). Then by [12, Lemma 2], $\mathrm{Con}(L)$ is a Stone lattice and $B(\mathrm{Con}(L))$ is atomic. Moreover, [12, Theorem 9] says that $L$ has an irredundant discrete subdirect factorization with finitely subdirectly irreducible factors. Let $\{\alpha_i\colon i\in I\}$ denote the set of all dual atoms of $B(\mathrm{Con}(L))$. Then by [12, Theorem 2], $(L/\alpha_i\colon i\in I)$ is the subdirect factorization of $L$ in question. Therefore, every element $x\in L$ can be represented as $(x_i)_{i\in I}$, where $x_i\in L/\alpha_i$ for every $i\in I$. Take now the elements $v=0$ and $u=1$ from $L$, i.e. the smallest and the largest elements of $L$, respectively. Since the factorization of $L$ is discrete, there exists a finite subset $I_1 \subseteq I$ such that $\{i \in I: u_i \neq v_i\} = I_1$. Moreover, $0 \leqq x \leqq 1$ implies $u_i = x_i = v_i$ for every $i \in I - I_1$. But the factorization $(L/\alpha_i: i \in I)$ is irredundant that means $\wedge(\alpha_j: j \in I, i \neq j) \neq \Delta$ for every $i \in I$. Hence $I = I_1$ is finite. $B(\mathrm{Con}(L))$ is an atomic and complete Boolean algebra. Therefore, $B(\mathrm{Con}(L))$ is finite.

Conversely, assume that $L$ satisfies (i) and (ii). Therefore, $\mathrm{Con}(L)$ satisfies the identity $\vee(x_i^{**}: i \in I) = (\vee(x_i: i \in I))^{**}$ for every finite $I$. According to (ii), $\mathrm{Con}(L)$ enjoys the identity (1) for arbitrary $I$. The proof is complete.

Deeper results can be obtained for distributive $p$-algebras. First we recall two results.

Theorem D. *Let $L$ be a distributive lattice with* 0. *Then $L$ can be embedded in a generalized Boolean lattice $B$ such that every congruence relation of $L$ has one and only one extension to $B$, that means* $\mathrm{Con}(L) \cong \mathrm{Con}(B)$.

For the proof see [6, Lemma II.4.5].

Theorem E ([10, Theorem 2]). *Every distributive $p$-algebra can be embedded in a Heyting algebra $H$ of order* 3 *(i.e. $D(H)$ is relatively complemented) such that*

(i) *every congruence relation of $L$ has one and only one extension to $H$, i.e.* $\mathrm{Con}(L) \cong \mathrm{Con}(H)$,

(ii) $B(L) = B(H)$

*and*

(iii) *$D(H)$ is an extension of $D(L)$ such that* $\mathrm{Con}(D(L)) \cong \mathrm{Con}(D(H))$.

For the proof of (ii) and (iii) see the proof of [10, Theorem 2].

Theorem 7. *Let $L$ be a distributive $p$-algebra. Then* $\mathrm{Con}(L)$ *is a Stone lattice if and only if*

(i) *$D(L)$ is relatively complemented,*

(ii) *the dual lattice $\breve{L}$ is a Stone lattice*

*and*

(iii) *$B(\breve{L})$ is a complete Boolean algebra.*

Proof. Let $\mathrm{Con}(L)$ be a Stone lattice. Then there exists a Heyting algebra $H$ of order 3 such that $L$ is a subalgebra of the $p$-algebra $H$ and $\mathrm{Con}(L) \cong \mathrm{Con}(H)$ (Theorem E). It is well known that $\mathrm{Con}(H) \cong F(H)$, that means, every congruence relation of $H$ is uniquely determined by a filter of $H$. Hence $F(H)$ is a Stone lattice. Now we can apply [7, Satz 9]. Therefore, (a) the dual lattice $\breve{H}$ is a Stone lattice and (b) $B(\breve{H})$ is a complete Boolean algebra. Evidently, $B(\breve{H}) \subseteq B(H) = B(L)$ (Theorem E). Take $a \in L$. There exists a dual pseudocomplement $a^+ \in B(\breve{H})$ of $a$ in $H$, i.e. $a \vee x = 1$ if and only if $x \geqq a^+$. As $B(\breve{H}) \subseteq B(L)$, $a^+ \in L$ and $L$ is dual pseudocomplemented, that means $\check{L}$ is pseudocomplemented. Moreover, $\check{L}$ is a Stone lattice by (a). Again, $B(\check{L})\subseteq B(\check{H})\subseteq B(L)$ implies $B(\check{L})=B(\check{H})$. Hence, $B(\check{L})$ is a complete Boolean algebra by (b). We have established (ii) and (iii).

Now we shall prove (i). Using (a) we see that $H$ is a double $p$-algebra. Moreover, by the hypothesis, $D(H)$ is relatively complemented. Therefore, $H$ is a regular double $p$-algebra (see [9, Theorem 2]). Above we have shown that $L$ is a subalgebra of the double $p$-algebra $H$ (see also Theorem E). But the regular double $p$-algebras form a variety (see [9, Theorem 2] or [16]). Hence, again by [9, Theorem 2], $L$ is also regular and this implies that $D(L)$ is relatively complemented.

Conversely, let $L$ satisfy (i)—(iii). Then $L$ is a distributive double $p$-algebra. By [9, Theorem 2], $L$ is a regular double $p$-algebra, because $D(L)$ is relatively complemented. According to [9, Theorem 1], $L$ forms a (double) Heyting algebra $H$. But every congruence relation of $L$ is also a Heyting algebra congruence relation, that means $\mathrm{Con}(L)=\mathrm{Con}(H)$ (see [10, Lemma 1]). Therefore, $\mathrm{Con}(L)\cong F(H)=F(L)$. Now, conditions (ii) and (iii) imply by [7, Satz 9] that $F(L)$ is a Stone lattice. Thus, $\mathrm{Con}(L)$ is a Stone lattice and the proof is complete.

For the next Theorem we need the following

Lemma 6. *Let $L$ be a distributive lattice with* 0. *Then* $B(\mathrm{Con}(L))$ *is finite if and only if $L$ is finite.*

Proof. By Theorem $D$ there is an extension $K$ of $L$ such that $K$ is a generalized Boolean lattice and $\mathrm{Con}(L)\cong\mathrm{Con}(K)$. Every congruence relation of $K$ is uniquely determined by its kernel, that means $\mathrm{Con}(K)\cong I(K)$. By assumption, $B(I(K))$ is finite. Take $a\in K$. We claim that $(a]\in B(I(K))$. Really, if $J\in I(K)$, then $J^*=\{x\in K\colon x\wedge y=0 \text{ for every } y\in J\}$. Consider $(a]^*$ and $(a]^{**}$. It suffices to show that $(a]^{**}=(a]$. Clearly $(a]\subseteq(a]^{**}$. Choose $b\in(a]^{**}$. Take $c=a\vee b$ and observe $(c]\in I(K)$. $(c]$ is a Boolean lattice. Since $(a]\vee((a]^*\wedge(c])=(c]$, we see that $b\leqq a$ and $(a]=(a]^{**}\in B(I(K))$, as claimed. Hence $K$ is finite, as $B(I(K))$ is finite. Consequently, $L$ is finite. The converse implication is trivial.

Theorem 8. *Let $L$ be a distributive $p$-algebra. Then* $\mathrm{Con}(L)$ *satisfies the identity* (1) *if and only if*

(i) $\mathrm{Con}(L)$ *is a Stone lattice,*

(ii) $D(L)$ *is finite*

*and*

(iii) $\{a\in B(L)\colon a\varphi(L)=[1)\}$ *is finite.*

Proof. Let $\mathrm{Con}(L)$ satisfy the identity (1). The condition (i) follows from Theorem 6. Again from Theorem 6 we know that $B(\mathrm{Con}(L))$ is finite. Hence, $B(\mathrm{Con}(L))$ is atomic. With regard to Theorem 4, $B(\mathrm{Con}(D(L)))$ is also atomic and the set of all atoms of $B(\mathrm{Con}(L))$ comprises

$$\{(\Delta, \alpha)\in \mathrm{Con}(L)\colon \alpha \text{ is an atom of } B(\mathrm{Con}(D(L)))\}$$

and

$$\{(\Theta[(a]], \Delta)\in \mathrm{Con}(L)\colon a\varphi(L) = [1) \text{ and } a \text{ is an atom of } B(L)\}.$$

This and Lemma 6 imply (ii), because $B(\mathrm{Con}(L))$ is finite. Now we shall establish (iii). Again by the hypothesis the set of atoms $a\in B(L)$ such that $a\varphi(L)=[1)$ is finite. Observe $b\in B(L)$ with $b\varphi(L)=[1)$. We claim that $b=a_1\vee\ldots\vee a_n$, where $a_i\varphi(L)=[1)$ and $a_i$ is an atom of $B(L)$ for every $i=1, \ldots, n$. Let $a\in B(L)$ be a join of atoms $a_i$ of $B(L)$ with $a_i\leqq b$, i.e. $a=a_1\vee\ldots\vee a_n$. Therefore, $a\leqq b$. Then there exists $c\in B(L)$ such that $a\wedge c=0$ and $b=a\vee c$. Hence $b\varphi(L)=a\varphi(L)\vee\vee c\varphi(L)=[1)$ (see Theorem B). Consequently, $c\varphi(L)=[1)$. If $t\leqq c$ and $t$ is an atom of $B(L)$ then by assumption $t\leqq a$. This implies $c=0$. Thus $b=a$, as claimed. Now it is easy to show that $\{a\in B(L)\colon a\varphi(L)=[1)\}$ is finite.

Conversely, let $L$ satisfy (i)—(iii). By Theorem 4, $B(\mathrm{Con}(L))$ is atomic and the set of all atoms of $B(\mathrm{Con}(L))$ is finite. Therefore, the Boolean algebra $B(\mathrm{Con}(L))$ is finite. The rest follows from Theorem 6.

Before closing this section we shall generalize Beazer's [1, Theorem 6] (see also [2]). We shall characterize those finite $p$-algebras, which have the same congruence lattices as the finite distributive $p$-algebras.

Having an (arbitrary) finite $p$-algebra $L$, then $\mathrm{Con}(L)$ is a finite distributive lattice, and thus, $\mathrm{Con}(L)$ can be considered as a finite double $p$-algebra. In this case we introduce the ideal $\bar{D}(\mathrm{Con}(L))$ of dual dense elements from $\mathrm{Con}(L)$, that means, $\alpha\in\bar{D}(\mathrm{Con}(L))$ if and only if $\alpha^+=\nabla$.

Theorem 9. *Let $L$ be a finite $p$-algebra. Then the following statements are equivalent:*

(i) *there exists a finite distributive $p$-algebra $L'$ such that* $\mathrm{Con}(L)\cong\mathrm{Con}(L')$;
(ii) $D(\mathrm{Con}(L))$ *is a Boolean lattice;*
(iii) $\bar{D}(\mathrm{Con}(L))$ *is a Boolean lattice;*
(iv) $\mathrm{Con}(L)$ *is a regular double $p$-algebra.*

Proof. By assumption, $\mathrm{Con}(L)$ is finite and distributive. Now the equivalence between (ii)—(iv) follows from [9, Theorem 2]. Assume (i). Then there exists a finite Heyting algebra $H$ of order 3 with $\mathrm{Con}(H)\cong\mathrm{Con}(L)$. Since $H$ is finite, we see that $H$ is a double $p$-algebra. Eventually, $H$ is regular, because $H$ is of order 3. The same is also true for the dual lattice $\breve{H}$. But $\mathrm{Con}(H)\cong F(H)\cong\breve{H}$. Hence $\breve{H}\cong\mathrm{Con}(L)$, and (iv) is true. Conversely, assume (iv). Let $H$ denote the dual lattice of $\mathrm{Con}(L)$. Clearly, $H$ is also a regular double $p$-algebra. By [9, Theorem 2] $H$ is in fact a Heyting algebra of order 3. Let $L'$ be $H$ considered as a $p$-algebra. Then

$\mathrm{Con}(L') \cong F(H)$, by [10, Lemma 1]. Since $H$ is finite, we see that $F(H) \cong \check{H} \cong$ $\cong \mathrm{Con}(L)$, and (i) is established.

Lemma 7. *Let $L$ be a finite quasi-modular p-algebra. Then* $\bar{D}(\mathrm{Con}(L)) =$ $=[\Delta, \gamma]$ *(that means that the Glivenko congruence is the largest dual dense element of* $\mathrm{Con}(L)$*).*

Proof. We know that $\gamma = (\Delta, \nabla)$. Take $\alpha = (\alpha_1, \alpha_2) \in \mathrm{Con}(L)$ with $\nabla = \gamma \vee \alpha$. Therefore, $\alpha_1 = \nabla$. As $(\alpha_1, \alpha_2)$ is a congruence pair, we see that $\alpha_2 = \nabla$. Now, assume $\alpha \geqq \gamma$ for some $\alpha \in \bar{D}(\mathrm{Con}(L))$. Corollary 2 to Theorem 1 says that $\nabla \neq \alpha \in [\gamma, \nabla] \cong \mathrm{Con}(B(L))$. But $\mathrm{Con}(B(L)) \cong B(L)$, as $L$ is finite. Take the complement $\alpha'$ of $\alpha$ in $[\gamma, \nabla]$. But $\alpha' \neq \nabla$ is impossible, because $\alpha \in \bar{D}(\mathrm{Con}(L))$. Hence $\alpha' = \nabla$, which implies $\alpha = \gamma$.

Theorem 10. *Let $L$ be a finite quasi-modular p-algebra. Then there exists a finite distributive p-algebra $L'$ such that* $\mathrm{Con}(L) \cong \mathrm{Con}(L')$ *if and only if* $\mathrm{Con}(D(L))$ *is a Boolean lattice.*

Proof. Corollary 1 to Theorem 1 and Lemma 7 imply that $\bar{D}(\mathrm{Con}(L)) =$ $=[\Delta, \gamma] \cong \mathrm{Con}(D(L))$. Hence, by Theorem 9, $\mathrm{Con}(D(L))$ is a Boolean lattice if and only if there exists a finite distributive lattice $L'$ such that $\mathrm{Con}(L) \cong \mathrm{Con}(L')$.

Corollary (see [1, Theorem 6]). *Let $L$ be a finite modular p-algebra. Then there exists a finite distributive p-algebra $L'$ such that* $\mathrm{Con}(L) \cong \mathrm{Con}(L')$.

Proof. $D(L)$ is a finite modular lattice. It is well known that the congruence lattice of a finite modular lattice is Boolean. Hence $\mathrm{Con}(D(L))$ is a Boolean lattice. The rest follows from Theorem 10.

**6. Relative Stone congruence lattices.** We start with general results.

Lemma 8. *Let $L$ be a distributive lattice with 1. The following statements are equivalent:*

(i) *$L$ is relative Stone;*

(ii) *for every $a \in L$, $[a, 1]$ is a Stone lattice;*

(iii) *for every $a \leqq b$ in $L$, $[a, b]$ is a relative Stone lattice;*

(iv) *$L$ is a Brouwerian lattice (i.e. relatively pseudocomplemented) satisfying the identity $x * y \vee y * x = 1$.*

Proof. The equivalences between (i), (ii) and (iii) follow from Lemma 5. The equivalence between (i) and (iv) can be found in [8, 2.10].

Lemma 9. *Let $L$ be a Heyting algebra. Then $L$ is a relative Stone lattice if and only if*

(i) *$L$ is a Stone lattice*

*and*

(ii) $D(L)$ *is relative Stone.*

For the proof see [8, 2.13].

Lemma 10. *Let $L$ be a quasi-modular p-algebra and let $B(L)$ be finite. Then* $(\alpha_1, \alpha_2)\in D(\text{Con}(L))$ *if and only if*

(i) $\alpha_2^* = \Delta$, *i.e.* $\alpha_2 \in D(\text{Con}(D(L)))$

*and*

(ii) $\alpha_1 \geqq \tau(\Delta)$.

Proof. Assume $(\alpha_1, \alpha_2)\in D(\text{Con}(L))$. Then, by Theorem 1, $\Delta = (\alpha_1, \alpha_2)^* = (\alpha_1^* \wedge \tau(\Delta), \Delta)$. So, $\alpha_2^* = \Delta$. Moreover, $\Delta = \alpha_1^* \wedge \tau(\Delta)$ in $\text{Con}(B(L))$. Since $B(L)$ is finite, we have $B(L) \cong \text{Con}(B(L))$. Hence $\alpha = \alpha^{**}$ for every $\alpha \in \text{Con}(B(L))$. Now, $\Delta = \alpha_1^* \wedge \tau(\Delta)$ implies $\alpha_1^{**} = \alpha_1 \geqq \tau(\Delta)$ proving (ii). Conversely, (i) and (ii) imply $(\alpha_1, \alpha_2)^* = (\alpha_1^* \wedge \tau(\Delta), \Delta) = \Delta$, as $\alpha_1^* \leqq \tau(\Delta)^*$.

Lemma 11. *Let $B$ be a Boolean algebra. Then* $\text{Con}(B)$ *is a relative Stone lattice if and only if $B$ is finite.*

Proof. Let $\text{Con}(B)$ be a relative Stone lattice. This is equivalent to the fact that $I(B/J)$ is a Stone lattice for every $J \in I(B)$ (Lemma 8). But $I(B)$ is a Stone lattice if and only if $B$ is complete (see [5] or [7, Satz 9]). By [3, Theorem 4.3] every infinite complete Boolean algebra contains an ideal $J$ such that $B/J$ is not complete. That means $I(B/J)$ is not a Stone lattice. Hence $B$ is finite. The converse is trivially true.

Theorem 11. *Let $L$ be a quasi-modular p-algebra. Then* $\text{Con}(L)$ *is a relative Stone lattice if and only if*

(i) $\text{Con}(L)$ *is a Stone lattice,*

(ii) $B(L)$ *is finite,*

(iii) $\text{Con}(D(L))$ *is a relative Stone lattice*

*and*

(iv) *for any* $\alpha, \beta \in \text{Con}(D(L))$ *with* $\alpha \geqq \beta$, $\beta \in D(\text{Con}(D(L)))$ *and* $\tau(\beta) \geqq \tau(\Delta)$ *it is true that* $\tau(\alpha * \beta)^* \leqq \tau((\alpha * \beta) * \beta)$.

Proof. Let $\text{Con}(L)$ be relative Stone. (i) follows from Lemma 9. Corollary 2 to Theorem 1 says that $\text{Con}(B(L)) \cong [\gamma, \nabla]$. Using Lemma 8 we see that $[\gamma, \nabla]$ is also relative Stone. Hence $\text{Con}(B(L))$ is relative Stone. By Lemma 11, $B(L)$ is finite and (ii) is established. The condition (iii) follows from the hypothesis and Corollary 1 to Theorem 1. Eventually we shall prove (iv). Lemma 9 and the hypothesis imply that $D(\text{Con}(L))$ is relative Stone. Take $\alpha_2 = \alpha$ and $\beta_2 = \beta$ from $\text{Con}(D(L))$ with $\alpha \geqq \beta$, $\beta \in D(\text{Con}(D(L)))$ and $\tau(\beta) \geqq \tau(\Delta)$. Since $(\tau(\Delta), \alpha_2), (\tau(\Delta), \beta_2) \in D(\text{Con}(L))$ (see Lemma 10), there exist $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in D(\text{Con}(L))$ with $(\alpha_1, \alpha_2) \geqq (\beta_1, \beta_2)$.

By the hypothesis, $[(\beta_1, \beta_2), \nabla]$ is a Stone lattice (Lemma 8). The pseudocomplements of elements in this interval can be calculated using Theorem 1. Therefore,

$$(\alpha_1, \alpha_2) * (\beta_1, \beta_2) = (\alpha_1 * \beta_1 \wedge \tau(\alpha_2 * \beta_2), \alpha_2 * \beta_2)$$

and

$$((\alpha_1, \alpha_2) * (\beta_1, \beta_2)) * (\beta_1, \beta_2) = (\alpha_1 * \beta_1 \wedge \tau(\alpha_2 * \beta_2), \alpha_2 * \beta_2) * (\beta_1, \beta_2) =$$
$$= (((\alpha_1 * \beta_1) \wedge \tau(\alpha_2 * \beta_2)) * \beta_1 \wedge \tau((\alpha_2 * \beta_2) * \beta_2), (\beta_1 * \beta_2) * \beta_2).$$

By the hypothesis

$$(\alpha_1, \alpha_2) * (\beta_1, \beta_2) \vee ((\alpha_1, \alpha_2) * (\beta_1, \beta_2)) * (\beta_1, \beta_2) = \nabla.$$

Since $B(L)$ is finite, we have $\mathrm{Con}\,(B(L)) \cong B(L)$. This implies that in $\mathrm{Con}\,(B(L))$ pseudocomplements are complements (i.e. $\alpha^* = \alpha'$) and $\alpha * \beta = \alpha' \vee \beta$. Bearing this in mind we see that $((\alpha_1 * \beta_1) \wedge \tau(\alpha_2 * \beta_2)) * \beta_1$ is the complement of $\alpha_1 * \beta_1 \wedge \tau(\alpha_2 * \beta_2)$ in $[\beta_1, \nabla]$. Therefore,

$$((\alpha_1 * \beta_1) \wedge \tau(\alpha_2 * \beta_2)) * \beta_1 = (\alpha_1 * \beta_1)^* \vee \tau(\alpha_2 * \beta_2)^* \vee \beta_1 \leqq \tau((\alpha_2 * \beta_2) * \beta_2)$$

and consequently, $\tau(\alpha_2 * \beta_2)^* \leqq \tau((\alpha_2 * \beta_2) * \beta_2)$.

Conversely, suppose that $L$ satisfies (i)—(iv). With regard to (i) and Lemma 9 it suffices to show that $D(\mathrm{Con}\,(L))$ is a relative Stone lattice. Take $(\beta_1, \beta_2) \in D(\mathrm{Con}\,(L))$. By Lemma 10, $\beta_2^* = \Delta$ and $\tau(\beta_2) \geqq \tau(\Delta)$. We want to show that $[(\beta_1, \beta_2), \nabla]$ is a Stone lattice (Lemma 8). Take $(\alpha_1, \alpha_2) \geqq (\beta_1, \beta_2)$ in $\mathrm{Con}\,(L)$. Evidently, $(\alpha_1, \alpha_2) * (\beta_1, \beta_2)$ and $((\alpha_1, \alpha_2) * (\beta_1, \beta_2)) * (\beta_1, \beta_2)$ is a pseudocomplement of $(\alpha_1, \alpha_2)$ and $(\alpha_1, \alpha_2) * (\beta_1, \beta_2)$, respectively, in $[(\beta_1, \beta_2), \nabla]$. By Theorem 1,

$$(\delta_1, \delta_2) = (\alpha_1, \alpha_2) * (\beta_1, \beta_2) \vee ((\alpha_1, \alpha_2) * (\beta_1, \beta_2)) * (\beta_1, \beta_2) =$$
$$= ((\alpha_1 * \beta_1 \wedge \tau(\alpha_2 * \beta_2))) \vee (((\alpha_1 * \beta_1 \wedge \tau(\alpha_2 * \beta_2)) * \beta_1 \wedge \tau((\alpha_2 * \beta_2) * \beta_2)), \alpha_2 * \beta_2 \vee$$
$$\vee (\alpha_2 * \beta_2) * \beta_2).$$

Condition (iii) implies $\alpha_2 * \beta_2 \vee (\alpha_2 * \beta_2) * \beta_2 = \nabla$. Clearly, $\alpha_2 \leqq (\alpha_2 * \beta_2) * \beta_2$ yields $\beta_1 \leqq \alpha_1 \leqq \tau((\alpha_2 * \beta_2) * \beta_2)$. The last condition, (ii) and (iv) imply

$$(\alpha_1 * \beta_1 \wedge \tau(\alpha_2 * \beta_2)) * \beta_1 = (\alpha_1 \wedge \beta_1^*) \vee \tau(\alpha_2 * \beta_2)^* \vee \beta_1 \leqq \tau((\alpha_2 * \beta_2) * \beta_2).$$

Now, it is easy to see that $(\delta_1, \delta_2) = \nabla$. Thus $\mathrm{Con}\,(L)$ is relative Stone and the proof is complete.

Before establishing the last theorem we need a concept. A lattice $L$ is said to be *locally finite* if all intervals in $L$ are finite.

Lemma 12. *Let $L$ be a Stone lattice. Assume that $B(L)$ is finite. Let $J \in I(L)$. Then $J \in D(I(L))$, i.e. $J^* = (0]$, if and only if $J \cap D(L) \neq \emptyset$.*

Proof. Assume that $J\in D(I(L))$. Assume to the contrary that $J\cap D(L)=\emptyset$. It is well known that there exists a prime ideal $P\in I(L)$ such that $J\subseteq P$ and $P\cap D(L)=\emptyset$. Note that $a\in P$ implies $a^{**}\in P$, as $a^*\wedge a^{**}=0$. Let $a\in B(L)$ be the join of all elements from $P\cap B(L)$. Since $B(L)$ is finite and $L$ is a Stone lattice, we have $(a]=P$. Evidently $a\neq 1$. Hence $(a^*]\leqq J^*$, a contradiction. Thus $J\cap D(L)\neq 0$. The converse statement is trivially true.

Theorem 12. *Let $L$ be a distributive $p$-algebra. Then* Con$(L)$ *is a relative Stone lattice if and only if*

(i) *$B(L)$ is finite,*

(ii) *$D(L)$ is locally finite and relatively complemented,*

(iii) *the dual lattice $\breve{L}$ is a Stone lattice*

*and*

(iv) *the ideal of dual dense elements $\bar{D}(L)$ (i.e. $\bar{D}(L)=D(\breve{L})$) is locally finite and relatively complemented.*

Proof. Suppose that Con$(L)$ is a relative Stone lattice. Combining Lemma 9, Theorem 7 and Theorem 11 we get (i), (iii) and that $D(L)$ is relatively complemented. In other words, $L$ is a regular double $p$-algebra (see [9, Theorem 2]). Again by this theorem we get that $\bar{D}(L)$ is also relatively complemented. By Theorem 11 Con$(D(L))$ is a relative Stone lattice. But Con$(D(L))\cong F(D(L))$. Take $a\in D(L)$. Then $[[1), [a)]$ is an interval in the lattice of all filters $F(D(L))$. Since $[a)$ is a Boolean lattice and $[[1), [a)]=F([a))$, we see that $[a)$ is finite, as $[[1), [a)]$ is a relative Stone lattice (see Lemma 11). Thus $D(L)$ is locally finite and (ii) is completely established. It remains to prove the locally finiteness of $\bar{D}(L)$. Since every congruence relation $\Theta\in$Con$(L)$ is also a Heyting algebra congruence relation of $L$ ([10, Lemma 1]), we see that Con$(L)\cong F(L)$. Take $b\in\bar{D}(L)$, i.e. $b^+=1$. Evidently, $(b]$ is a Boolean lattice and $F((b])\cong[[b), [0)]$. By assumption $[[b), [0)]$ is a relative Stone lattice. Therefore, by Lemma 11, $(b]$ is a finite Boolean lattice. Thus $\bar{D}(L)$ is locally finite, and of course, relatively complemented.

Conversely, suppose that $L$ satisfies (i)—(iv). Theorem 7 says that Con$(L)$ is a Stone lattice. According to Lemma 10 it suffices to prove that $D($Con$(L))$ is a relative Stone lattice. This follows from the fact (Lemma 9) that for every $\alpha\in D($Con$(L))$, $[\alpha, \nabla]$ is a Stone lattice. Again [9, Theorem 2] and [10, Lemma 1] imply that Con$(L)\cong F(L)$. Let Ker $\alpha=K\in F(L)$. With regard to Lemma 12, $K\cap\bar{D}(L)\neq\emptyset$. Take $b\in K\cap\bar{D}(L)$. By (iv), $(b]$ is a finite Boolean lattice. Thus Con$(L)$ is a relative Stone lattice and the proof is complete.

## References


[1] R. Beazer, On congruence lattices of some $p$-algebras and double $p$-algebras, *Algebra Universalis*, **13** (1981), 379—388.

[2] J. Berman, Congruence relations of pseudocomplemented distributive lattices, *Algebra Universalis*, 3 (1973), 288—293.

[3] Ph. Dwinger, On the completeness of the quotient algebras of a complete Boolean algebra. I, *Indag. Math.*, **20** (1958), 448—456.

[4] S. El-Assar, Two notes on the congruence lattice of the $p$-algebras, *Acta Math. Univ. Comenian*, XLVI—XLVII (1985), 13—20.

[5] O. Frink, Pseudo-complements in semi-lattices, *Duke Math. J.*, **29** (1962), 505—514.

[6] G. Grätzer, *General Lattice Theory*, Birkhäuser Verlag (Basel, 1978).

[7] T. Katriňák, Pseudokomplementäre Halbverbände, *Mat. Časop.*, **18** (1968), 121—143.

[8] T. Katriňák, Die Kennzeichnung der distributiven pseudokomplementären Halbverbände, *J. Reine Angew. Math.*, **241** (1970), 160—179.

[9] T. Katriňák, The structure of distributive double $p$-algebras. Regularity and congruences, *Algebra Universalis*, 3 (1973), 238—246.

[10] T. Katriňák, The congruence lattice of distributive $p$-algebras, *Algebra Universalis*, 7 (1977), 265—271.

[11] T. Katriňák, Essential and strong extensions of $p$-algebras, *Bull. Soc. Roy. Sci. Liège*, **49** (1980), 119—124.

[12] T. Katriňák and S. El-Assar, Algebras with Boolean and Stonean congruence lattices, *Acta Math. Hung.*, **48** (1986), 301—316.

[13] T. Katriňák and P. Mederly, Constructions of $p$-algebras, *Algebra Universalis*, **17** (1983), 288—316.

[14] W. C. Nemitz, Implicative semi-lattices, *Trans. Amer. Math. Soc.*, **117** (1965), 128—142.

[15] T. Tanaka, Canonical subdirect factorization of lattices, *J. Sci. Hiroshima Univ., Ser. A*, **16** (1952), 239—246.

[16] J. Varlet, A regular variety of type $\langle 2, 2, 1, 1, 0, 0\rangle$, *Algebra Universalis*, 2 (1972), 218—223.


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