Acta Sci. Math., 51 (1987), 403—411

A classification of the set of linear functions in prime-valued logic

IVAN STOJMENOVIĆ

1. Introduction

Let $P_k=\bigcup_{n\in\omega}\{f|f\colon E_k^n\to E_k\}$, where $E_k=\{0,1,\ldots,k-1\}$; i.e. P_k denotes the set of all k-valued logical functions. A subset G of P_k is said to be *closed* if it is closed under superposition (e.g. see [4]).

Let $H\subset P_k$ be a fixed closed set. If $F\subseteqq H$ then we say that

(i) F is *complete in* $H\Leftrightarrow$ every element of H is obtained from F by superposition;

(ii) F is *H-maximal* $\Leftrightarrow$ F is closed and no G exists such that $F\subset G\subset H$ (proper inclusion) and G is closed;

(iii) F is a *base in* $H\Leftrightarrow F$ is finite and complete in H and no complete subset of F exists;

(iv) F is a *pivotal set in* $H\Leftrightarrow F$ is finite and for every $f\in F$ there is an H-maximal F' such that $f\notin F'$ but $F-\{f\}\subseteqq F'$.

From these definitions it follows that a base is a complete pivotal set of functions.

The *rank* of a base (pivotal set) is the number of elements of the base (pivotal set).

Let m be the cardinality of the set of all H-maximal sets and suppose that this set is well-ordered. There exist closed sets H for which m is not finite ([5]). If m is finite then a subset F of H is complete in H iff F is not contained in any H-maximal set ([4]).

If $f\in H$, then the class a_f determined by f is an element of $\{0,1\}^m$ such that $a_i=0$ iff $f\in H_i$, where a_i is the i-th component of a_f and H_i is the i-th H-maximal set $(1\leqq i\leqq m)$ in the well-ordering mentioned above. For $F\subseteqq H$, one can define the class a_F determined by F as the union of classes determined by the elements of F. Therefore, if $F=\{f_1,\ldots,f_s\}$ then $a_F=\{a_{f_1},\ldots,a_{f_s}\}$. This set a_F can be represented as an element a'_F of $\{0,1\}^m$ such that $a'_F=\vee(a_{f_1},\ldots,a_{f_s})$, where bitwise OR operation $\vee$ is defined in the following way: the i-th component $a_F^{(i)}$ of a'_F is equal to 0 iff the i-th component of all classes a_{f_j} $(1\leqq j\leqq s)$ is equal to 0.

Received March 20, 1984.

From this definition it follows that the set F is complete iff $a'_F=1^m$. Also, we infer that F is a pivotal set if $a'_F \neq a'_{F\setminus\{f_j\}}$ for all j, $1 \leqq j \leqq s$. From these considerations one can remark that if F is complete (pivotal set, base), $f, f' \in F$ and $a_f = a_{f'}$ (i.e. f and f' are functions of the same class) then $F \cup \{f'\} \setminus \{f\}$ is complete (pivotal set, base) and $a_F = a_{F\cup\{f'\}\setminus\{f\}}$.

All P_2-maximal sets and maximal sets of P_2-maximal sets are described in [10]. P_3-maximal sets are determined in [4], and maximal sets of P_3-maximal sets are exhibited in [7] and other papers.

All different classes a_f for the set P_2 are investigated in [6], and for P_3 in [8], [9] and [11].

Let us recall some well-known closed sets in P_k.

The set L_k of *linear* functions is defined in the following way:

$$L_k = \left\{a_0 + \sum_{i=1}^{n} a_i x_i (\mathrm{mod}\, k) \mid a_0 \in E_k,\ a_i \in E'_k,\ 1 \leqq i \leqq n,\ n \in \omega, \text{ where } E'_k = E_k \setminus \{0\}\right\}.$$

Let $a = \sum_{i=1}^{n} a_i$. It is well-known that L_k is a P_k-maximal set iff k is a prime number ([4]).

The set S_k of *selfdual* functions is defined as follows:

$$S_k = \{f \mid f(x_1+1, \ldots, x_n+1) = f(x_1, \ldots, x_n)+1 \ (\mathrm{mod}\, k),\ n = 1, 2, \ldots\}.$$

$T_k^j = \{f \mid f(j, \ldots, j) = j\}$ is the set of functions preserving j $(0 \leqq j \leqq k-1)$.

Let $\overline{X} = L_k \setminus X$ for each $X \subset L_k$. The intersection of the sets $X_1, \ldots, X_i \subset L_k$ will be denoted by $X_1 \ldots X_i$.

From the results in papers [1], [2], [3] it follows

Theorem 1. *Let $p \in \omega$ be an arbitrary prime. Then there are $p+2$ L-maximal sets. These are:*

(i) $L^j = L_p T_p^j$, *for every* j, $j = 0, 1, \ldots, p-1$,

(ii) $L^p = L_p S_p = \left\{a_0 + \sum_{i=1}^{n} a_i x_i \mid a = 1 \ (\mathrm{mod}\, p)\right\}$, *the set of linear selfdual functions,*

(iii) $L^{(1)} = \{a_0 + a_1 x \mid a_0,\ a_1 \in E_p\}$, *the set of unary linear functions.*

Let 0^t denote the sequence $\underbrace{00\ldots0}_{t}$, and 1^t denote $\underbrace{11\ldots1}_{t}$.

In this paper we prove that there exist $2p+4$ different classes determined by functions of L_p. The number of different classes determined by bases in L_p is $4\binom{p+1}{2}$, and the number of different classes determined by pivotal noncomplete sets of L_p is $\binom{p+4}{2}-2$.

2. Classification of L_p

Theorem 2. *Let* $p\in\omega$ *be an arbitrary prime. Then there are* $2p+4$ *different classes (denoted by* $c_1, c_2, \dots, c_{2p+4}$*) of functions in* L_p*. These classes and the corresponding sets of functions are:*

$$L^0L^1\dots L^{p-1}L^pL^{(1)}, \quad c_1 = 0^{p+2};$$

$$L^0L^1\dots L^{p-1}L^p\bar{L}^{(1)}, \quad c_2 = 0^{p+1}1;$$

$$\bar{L}^0\bar{L}^1\dots\bar{L}^{i-4}L^{i-3}\bar{L}^{i-2}\dots\bar{L}^pL^{(1)}, \quad c_i = 1^{i-3}0^{p+3-i}0, \quad \text{where} \quad 3 \leqq i \leqq p+3;$$

$$\bar{L}^0\bar{L}^1\dots\bar{L}^{j-p-5}L^{j-p-4}\bar{L}^{j-p-3}\dots\bar{L}^p\bar{L}^{(1)}, \quad c_j = 1^{j-p-4}01^{2p+5-j},$$

$$\text{where} \quad p+4 \leqq j \leqq 2p+4.$$

Proof. Let $f(x_1, \dots, x_n) = a_0 + \sum_{i=1}^{n} a_i x_i \pmod p$ and $\sum_{i=1}^{n} a_i = a$. Consider the equation $a_0 + ay = y$.

Case a) Let $a_0=0$, $a=1$. Then the equation is $y=y$ which is satisfied by every y. This implies that $f\in L^0L^1\dots L^p$. The function $f(x)=x$ is in the set $L^{(1)}$, and it is a function of the class c_1. The function $a_1x_1+\dots+a_nx_n$ where $a=1$ and $n\geqq 2$ is in the set $\bar{L}^{(1)}$, and so it is a function of the class c_2.

Case b) $a_0\neq 0$, $a=1$. Then we obtain $a_0=0$, so it has no solution. Hence, the function f is in the set $\bar{L}^0\bar{L}^1\dots\bar{L}^{p-1}L^p$. The function a_0+x for $a_0\neq 0$ is in the set $L^{(1)}$ and it is a function of the class c_{p+3}. The function $a_0+a_1x_1+\dots+a_nx_n \pmod p$ for $a_0\neq 0$ and $a=1$, $n\geqq 2$ is in the set $\bar{L}^{(1)}$, and it is a function of the class c_{2p+4}.

Case c) $a\neq 1$. $y_1\neq y_2$ implies $(a-1)y_1\neq(a-1)y_2$. From this it follows that $(a-1)y$ takes on p different values, when y ranges from 0 to $p-1$. It follows that there exists exactly one y_0 such that $(a-1)y_0=-a_0$, i.e. $a_0+ay_0=y_0$. This implies that the function f is in the set L^{y_0}, and it is not in the sets L^i for $i\neq y_0$, $1\leqq i\leqq p-1$. Since $a\neq 1$, f is not in the set L^p. The function $f=i$ (constant) is in the set $L^{(1)}$, and it is a function of the class c_{i+3}. The function $f=i+ax_1+(p-a)x_2$ $(a\neq 0)$ is in the set $\bar{L}^{(1)}$ and it is a function of the class c_{p+4+i}.

Theorem is proved, because all possible cases have been considered.

3. Classes determined by bases of L_p

Theorem 3. *Let $p \in \omega$ be an arbitrary prime. Then the number of different classes determined by bases in L_p and the number of different classes determined by pivotal noncomplete sets in L_p for each rank are shown in the following table:*

rank	*bases*	*pivotal noncomplete sets*
1	0	$2p+3$
2	$3\binom{p+1}{2}$	$\binom{p+1}{2}+p+1$
3	$\binom{p+1}{2}$	0
$\geqq 4$	0	0

Proof. From the definitions it is easy to see that the class $c_1 = 0^{p+2}$ is not included in any pivotal set, and there is no base of rank 1. The classes $c_2, c_3, \ldots, c_{2p+4}$ are different from 0^{p+2} and 1^{p+2}. Hence, these classes define the classes determined by pivotal noncomplete sets of rank 1 of L_p.

We begin the investigation of bases and pivotal noncomplete sets of rank $\geqq 2$ by the following remarks:

$\vee(c_i, c_j) = 1^{p+1}0$ for $3 \leqq i, j \leqq p+3$;

$\vee(c_i, c_j) = 1^{p+2}$ for $p+4 \leqq i, j \leqq 2p+4$;

$\vee(c_2, c_i) \notin \{c_2, c_i, 1^{p+2}\}$ for $3 \leqq i \leqq p+3$;

$\vee(c_2, c_i) = c_i$ for $p+4 \leqq i \leqq 2p+4$;

$\vee(c_i, c_j) = 1^{p+2}$ for $3 \leqq i \leqq p+3$, $p+4 \leqq j \leqq 2p+4$ and $j \neq i+p+1$;

$\vee(c_i, c_{i+p+1}) = c_{i+p+1}$ for $3 \leqq i \leqq p+3$;

$\vee(c_2, c_i, c_j) = 1^{p+2}$ for $3 \leqq i < j \leqq p+3$.

From these remarks it follows that bases of rank 2 may contain any two functions of classes c_i and c_j, where i and j satisfy the condition $p+4 \leqq i < j \leqq 2p+4$, or the conditions $3 \leqq i \leqq p+3$, $p+4 \leqq j \leqq 2p+4$ and $j \neq i+p+1$.

Also, one can infer that pivotal incomplete sets of rank 2 consist either of two functions of classes c_i and c_j, $3 \leqq i < j \leqq p+3$, or a function of class c_2 and a function of class c_i, $3 \leqq i \leqq p+3$.

From the remarks above it follows that no pivotal set of rank $\geqq 3$ exists which contains a function of class c_i for $p+4 \leqq i \leqq 2p+4$. Hence, pivotal sets of rank $\geqq 3$ may contain only functions of the class c_2 and classes c_i for $3 \leqq i \leqq p+3$. But, from the first remark we conclude that $\vee(c_{i_1}, c_{i_2}, \ldots, c_{i_s}) = \vee(c_{i_1}, c_{i_2}) = 1^{p+1}0$ for

$3 \leqq i_1, \ldots, i_s \leqq p+3$. Hence, a pivotal set cannot contain functions from more than two classes c_i for $3 \leqq i \leqq p+3$. Therefore, no base or pivotal set of rank $\geqq 4$ exists. From $\vee(c_2, c_i, c_j) = 1^{p+2}$ $(3 \leqq i < j \leqq p+3)$ we conclude that pivotal sets of rank 3 are complete. Thus, no pivotal noncomplete set of rank 3 exists and a base of rank 3 consists of a function of class c_2 and two functions of the classes c_i and c_j, where $3 \leqq i < j \leqq p+3$.

From the above considerations the theorem follows.

Corollary 1. *The maximal rank of a base of the set L_p is 3, and the maximal rank of a pivotal noncomplete set is 2.*

Corollary 2. *There is no base of rank 1 (i.e. Sheffer function) in the set L_p.*

Corollary 3. *The number of different classes determined by bases in L_p (p prime) is* $4\binom{p+1}{2}$.

Corollary 4. *The number of different classes determined by pivotal noncomplete sets of L_p (p prime) is* $\binom{p+1}{2} + 3p + 4 = \binom{p+4}{2} - 2$.

The number of n-ary linear functions of class c_i $(1 \leqq i \leqq 2p+4)$ will be denoted by $t_n(i)$.

Theorem 4. $t_0(i) = 1$ *for* $3 \leqq i \leqq p+2$, $t_0(i) = 0$ *otherwise;* $t_1(1) = 1$, $t_1(p+3) = p-1$, $t_1(i) = p-2$ *for* $3 \leqq i \leqq p+2$, $t_1(i) = 0$ *otherwise;* $t_n(2) = ((p-1)^n - (-1)^n)/p$, $t_n(2p+4) = (p-1)t_n(2)$; $t_n(i) = ((p-1)^{n+1} + (-1)^n)/p$ *for* $p+4 \leqq i \leqq 2p+3$, $t_n(i) = 0$ *otherwise* $(n \geqq 2)$.

Proof. The statement follows easily from considerations in the proof of Theorem 2. For $n=0$ and $n=1$ the assertion is obvious. For $n>1$ $t_n(2)$ is equal to the number of sequences $a_1, \ldots, a_n$ which satisfy the condition $a_1 + \ldots + a_n = 1 \pmod p$. If $a_1 + \ldots + a_{n-1} = 1 \pmod p$, then no solution of the equation $a_1 + \ldots + a_n = 1 \pmod p$ exists (since $a_i \neq 0$, $1 \leqq i \leqq n$). If $a_1 + \ldots + a_{n-1} \neq 1 \pmod p$, then there exists exactly one solution of the equation $a_1 + \ldots + a_n = 1 \pmod p$. It follows that $t_n(2) = (p-1)^{n-1} - t_{n-1}(2)$, $t_2(2) = p-2$. By induction on n it is easy to prove that $t_n(2) = ((p-1)^n - (-1)^n)/p$. If $p+4 \leqq i \leqq 2p+3$, then from $t_n(i) = (p-1)^n - t_n(2)$ we obtain $t_n(i) = ((p-1)^{n+1} + (-1)^n)/p$.

The number of functions of the class c_i which depend on at most n variables is denoted by $t_{\leqq n}(i)$.

From Theorem 4 the following theorem is easily derived.

Theorem 5. $t_{\leqq 0}(i)=t_0(i)$; $t_{\leqq 1}(i)=t_0(i)+t_1(i)$;

$$t_{\leqq n}(1)=1, \quad t_{\leqq n}(i)=p-1 \quad for \quad 3\leqq i\leqq p+3;$$

$$t_{\leqq n}(2)=((p-1)^{n+1}-(p-1)^2)/(p-2)-((-1)^n+1)/2)/p;$$

$$t_{\leqq n}(2p+4)=(p-1)t_{\leqq n}(2);$$

$$t_{\leqq n}(i)=((p-1)^2((p-1)^n-1)/(p-2)+(1+(-1)^n)/2)/p \quad for \quad p+4\leqq i\leqq 2p+3.$$

Let B_i^n and P_i^n denote the number of bases and the number of pivotal incomplete sets of rank i which consist of functions depending on at most n variables.

From Theorems 2, 3 and 5 it is easy to prove the following

Theorem 6.

$$B_2^n=pt_{\leqq n}(2p+4)t_{\leqq n}(p+4)+\binom{p}{2}t_{\leqq n}^2(p+4)+pt_{\leqq n}(p+3)t_{\leqq n}(p+4)+$$

$$+pt_{\leqq n}(2p+4)t_{\leqq n}(3)+t_{\leqq n}(p+3)t_{\leqq n}(2p+4)+p^2t_{\leqq n}(3)t_{\leqq n}(p+4);$$

$$B_3^n=t_{\leqq n}(2)\left(pt_{\leqq n}(p+3)t_{\leqq n}(3)+\binom{p}{2}t_{\leqq n}^2(3)\right);$$

$$P_1^n=t_{\leqq n}(2)+t_{\leqq n}(p+3)+t_{\leqq n}(2p+4)+pt_{\leqq n}(3)+pt_{\leqq n}(p+4);$$

$$P_2^n=t_{\leqq n}(2)(t_{\leqq n}(p+3)+pt_{\leqq n}(3))+pt_{\leqq n}(p+3)t_{\leqq n}(3)+\binom{p}{2}t_{\leqq n}(3);$$

$$B_1^n=B_4^n=B_5^n=\ldots=P_3^n=P_4^n=\ldots=0.$$

Analogously one can obtain the numbers of bases and pivotal noncomplete sets which contain functions depending on exactly n variables.

4. Classification of L_p-maximal sets

We may assume further that $p\geqq 3$ (prime number). The properties of L_2-maximal sets follow immediately from Post's lattice ([10]).

Let us define some familiar closed sets in L_p: $L^{(0)}=\{0, 1, \ldots, p-1\}$, $L_{s0}=L^pL^0=\{a_1x_1+\ldots+a_nx_n|a=1,\ n=1, 2, \ldots\}$, $L_i^{(1)}=L^{(1)}L^i=\{a_0+a_1x\,|\,a_0+a_1i=i,\ a_0, a_1\in E_p\}$ for $0\leqq i\leqq p-1$, $L_p^{(1)}=L^{(1)}L^p=\{x, x+1, \ldots, x+k-1\}$.

We shall mean by the multiplicative order of $a\in E_p'$ the least integer $r(a)=r\geqq 1$ for which $a^r=1$ holds. If $p-1$ is divisible by j, then E_p' has $\varphi(j)$ elements of order j ($\varphi(j)$ denotes Euler's φ-function). Let $a_{i0}+a_ix\in L^{(1)}\setminus L^{(0)}$, $i\geqq 1$, $r(a_i)=r_i$. Let us denote by lcm $(r_1, r_2, \ldots)$ the least common multiple of the numbers $r_1, r_2, \ldots$.

Let the number $p-1$ have the decomposition to powers of primes $p-1=q_1^{\alpha_1}q_2^{\alpha_2}\ldots q_u^{\alpha_u}$ with all $q_1=2<q_2<\ldots<q_u$ primes, $\alpha_i\geqq 1, p_i=(p-1)/q_i$ and $L^{(1,i)}=\{a_0+ax|r(a)(\geqq 1)$ divides $p_i\}$, $i=1, 2, \ldots, u$.

The maximal sets for all L_p-maximal sets are determined by Demetrovics and Bagyinszki ([2]).

Theorem 7 ([2]). *There are exactly two L^i-maximal sets for $0 \leqq i \leqq p$ (p is a prime number): L_{s0} and $L_i^{(1)}$.*

Theorem 8. *If $1 \leqq i \leqq p$ then there are exactly four different classes determined by functions in L^i, two different classes determined by bases of L^i (one for both of ranks 1 and 2) and two different classes determined by pivotal noncomplete sets in L^i (both of them are of rank 1).*

Proof. The function x belongs to the set $L_{s0} L_i^{(1)}$ and the function $2x+(p-1)y$ is in the set $L_{s0} \bar{L}_i^{(1)}$. The function $x+1$ is an element of the set $\bar{L}_{s0} L_p^{(1)}$ and the function $2x-i$ is in the set $\bar{L}_{s0} L_i^{(1)}$ for $1 \leqq i \leqq p-1$. Base functions $x+y+(p-i)$ and $2x+(p-1)y+1$ ([2]) belong to the sets $\bar{L}_{s0} \bar{L}_i^{(1)}$ for $0 \leqq i \leqq p-1$ and $\bar{L}_{s0} \bar{L}_p^{(1)}$ respectively. Thus all four possible classes determined by the functions in L^i are nonempty. The other parts of the theorem follow immediately.

We are going to investigate classes determined by functions in $L^{(1)}$.

Theorem 9 ([2]). *The following $u+p+1$ sets are $L^{(1)}$-maximal:*

$$L^{(1,i)} \cup L^{(0)}, \quad i = 1, 2, \ldots, u,$$

$$L_i^{(1)} \cup L^{(0)}, \quad i = 0, 1, \ldots, p-1,$$

$$L^{(1)} \setminus L^{(0)}.$$

The next three lemmas are useful for the classification of $L^{(1)}$.

Lemma 1. *For the elements of $L^{(1)}$ we have:*

(a) *$a_0 + x \in L_i^{(1)}$ iff $a_0 = 0$, for $i = 0, 1, \ldots, p-1$;*

(b) *If $a > 1$ then for each i $(0 \leqq i \leqq p-1)$ there exists exactly one a_0 for which $a_0 + ax \in L_i^{(1)}$;*

(c) *$a_0 \in L_i^{(1)}$ iff $a_0 = i$.*

The proof is omitted.

Lemma 2. *$L_i^{(1)} L_j^{(1)} = \{x\}$ for $0 \leqq i < j \leqq p-1$.*

Proof. From $a_0 + a_1 i = i$ and $a_0 + a_1 j = j$ it follows that $a_1 = 1$ and $a_0 = 0$.

Lemma 3. *Let t_i be a sequence such that $t_i = q_i$ or $t_i = 1$ for each $i = 1, 2, \ldots, u$, $t = (p-1)/(t_1 \ldots t_u)$ and a is a number for which $r(a) = t$. If we define the sets A_i $(1 \leqq i \leqq u)$ such that $A_i = \bar{L}^{(1,i)}$ for $t_i = 1$ and $A_i = L^{(1,i)}$ for $t_i = q_i$ then the function $f = a_0 + ax$ is in the set $A_1 A_2 \ldots A_u$.*

Proof. If $t_i = 1$ then p_i is not divisible by $r(a)$. Hence $a_0 + ax \in \bar{L}^{(1,i)} = A_i$. If $t_i = q_i$ then $r(a)$ divides p_i. Thus $a_0 + ax \in L^{(1,i)} = A_i$.

Theorem 10. *The number of different classes determined by functions in* $L^{(1)}$ *is* $p2^u+3$ *if* $p-1\neq q_1q_2\dots q_u$ *and* $p(2^u-1)+3$ *otherwise.*

Proof. Suppose that the $L^{(1)}$-maximal sets are ordered as in Theorem 9. $0^{u+p}1$ is the class determined by the functions in the set $L^{(1)}$. The $(u+p+1)$-component of all other classes is 0. From Lemmas 1—3 we infer that the class 0^{u+p+1} is determined only by the function x and the class 0^u1^p0 is determined by functions a_0+x for $a_0\neq 0$. We may assume further that $f=a_0+ax$ and $a>1$. From Lemma 2 it follows that exactly one component among the components $u+1, u+2, \dots, u+p$ is equal to 0. We derive from Lemma 3 that all the 2^u possible classes with respect to the first u $L^{(1)}$-maximal sets are nonempty. But, if $p-1=q_1\dots q_u$ for $t_i=q_i$ $(1\leqq i\leqq u)$ we get $t=a=1$ in Lemma 3. It follows from Lemma 1 (b) and Lemma 2 that each of these classes with respect to the first u $L^{(1)}$-maximal sets can be supplemented to a class determined by functions in $L^{(1)}$ in p different ways.

The proof is complete.

Corollary 5. *Each base of* $L^{(1)}$ *contains a constant.*

Corollary 6. *For* $p=3$ *there are exactly* 6 *classes determined by functions in* $L^{(1)}$: $0^5, 0^41, 01^30, 10110, 11010, 1^300$.

Theorem 11 ([2]). *The cardinality of the bases of* $L^{(1)}$ *is* $\geqq 3$.

Theorem 12. *The maximal rank of classes determined by bases in* $L^{(1)}$ *is* $u+2$.

Proof. Each base of $L^{(1)}$ contains a function of the class $0^{u+p}1$. There is a subset of the base containing no more than u functions for which bitwise OR gives the value 1^u with respect to the first u components. From Lemmas 1 and 3 we obtain that no more than one component among components $u+1, \dots, u+p$ has the value 0. Hence, except the $u+1$ functions considered above, this base may contain at most one function. Thus, each base of $L^{(1)}$ consists of at most $u+2$ functions.

Theorem 13. *If* $p-1=q_1^{\alpha_1}$ *(for example, if* $p=3$ *or* $p=5$*) then each base in* $L^{(1)}$ *contains exactly three functions.*

Proof. In the case $p-1=q_1^{\alpha_1}$ we have $u=1$ and so this theorem is proved by using Theorems 11 and 12.

Acknowledgement. The author is thankful for the comments given by the referee which have certainly improved the readibility of the paper.

References

[1] J. Bagyinszki, J. Demetrovics, The structure of the maximal linear classes in prime-valued logics, *C.R. Math. Rep. Acad. Sci. Canada*, **2** (1980), 209—213.

[2] J. Bagyinszki, J. Demetrovics, The lattice of linear classes in prime-valued logics, *Banach Center Publ.*, PWN, 8/1979.

[3] J. Bagyinszki, J. Demetrovics, Lineáris osztályok szerkezete prímszám értékű logikában, *MTA SzTAKI Közlemények*, 16/1976, 25—52.

[4] S. V. Jablonskiĭ, Functional constructions in k-valued logics, *Trudy Mat. Inst. Steklov*, **51** (1958), 5—142. (Russian)

[5] Yu. I. Yanov, A. A. Mučnik, Existence of k-valued closed classes without a finite basis, *Dokl. Akad. Nauk. SSSR*, **127** (1959), 44—46. (Russian)

[6] L. Krnić, Classes of bases of propositional logic, *Glas. Mat.*, **20** (1965), 23—32. (Russian)

[7] D. Lau, Submaximale Klasses von P_3, *Elektron. Informationsverarb. Kybernet.*, **18** (1982), 227—243.

[8] M. Miyakawa, Functional completeness and structure of three-valued logic I. Classification of P_3, *Researches of Electrotechnical Laboratory*, 717, (Tokyo, 1971) 1—85.

[9] M. Miyakawa, Enumeration of bases of three-valued logical functions, in: *Finite Algebra and Multiple-Valued Logic* (Proc. Conf. Szeged, 1979), Colloq. Math. Soc. János Bolyai, 28, North-Holland (Amsterdam, 1981), 469—487.

[10] E. Post, *Two-valued Iterative Systems of Mathematical Logic*, Ann. of Math. Stud., 5, Princeton Univ. Press (1941).

[11] I. Stojmenović, Classification of P_3 and the enumeration of bases of P_3, *Rev. of Res. Fac. of Sci. Math. Ser.*, **14** (1984), 73—80.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
21000 NOVI SAD, DR. ILIJE DJURIČIĆA 4
YUGOSLAVIA