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On superalgebras of the polydisc algebra

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Let $\mathbf{T}$ be the unit circle and, for $n \geqq 1$, let A_n be the uniform closure in $C(\mathbf{T}^2)$ of the algebra of polynomials in $z^k w^l$, where k and l are integers, $l \geqq 0$, and $k \geqq 0$ whenever $0 \leqq l \leqq n-1$. Each A_n contains the polydisc algebra and the intersection of the A_n is the polydisc algebra. In this paper we give a characterization of the subspaces of $L^2(\mathbf{T}^2)$ which are invariant under multiplication by the functions in A_n. The characterization is somewhat complicated, as one would expect, since for $n>1$, A_n is not a Dirichlet algebra. In fact, for $n>1$, the point in the maximal ideal space of A_n represented by Lebesgue measure on $\mathbf{T}^2$ has an infinite dimensional set of representing measures. Nevertheless, as a result of our analysis, we find that each simply invariant subspace of $L^2(\mathbf{T}^2)$ for A_n is finitely generated and the number of generators required is $\leqq n$. Examples can be constructed where n generators are necessary. Our analysis enables us to extend results of the third author and to parametrize the weak-$*$ closed superalgebras of A_n.

1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the space of complex-valued continuous functions on X, and let A be a uniform algebra on X. For $\varphi \in M_A$, the maximal ideal space of A, set $A_0 = \{f \in A: \varphi(f) = 0\}$.

Definition 1.1. Let $\varphi \in M_A$, let σ be a representing measure (on X) for φ, and let $\mathfrak{M}$ be a (closed) subspace of $L^2(X, \sigma)$. Then $\mathfrak{M}$ is said to be *simply invariant* (for A) if $A\mathfrak{M} \subset \mathfrak{M}$, but $[A_0 \mathfrak{M}]_2 \neq \mathfrak{M}$ (where $[\ ]_2$ denotes L^2-closure).

Let ∂_A denote the Shilov boundary of A and N_φ denote the set of representing measures for $\varphi \in M_A$ whose support is contained in ∂_A. Note that N_φ is a weak-$*$

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compact convex set of probability measures on ∂_A. The general theory of simply invariant subspaces is known only in the case when $N_\varphi \cap L^1(X, \sigma)$ is finite dimensional. For instance, if A is a Dirichlet algebra then $N_\varphi \cap L^1(X, \sigma) = \{\sigma\}$, and the simply invariant subspaces of $L^2(X, \sigma)$ have been characterized (cf. [2, p. 132]). In particular, Beurling's theorem can be derived from that characterization (the disc algebra, after all, is a Dirichlet algebra on the unit circle $\mathbf{T}$).

In this note we focus our attention on the following class of function algebras, A_n, $n \geqq 1$, contained in $C(\mathbf{T}^2)$. The general theory of invariant subspaces does not apply to these algebras. Nevertheless, as we shall show, it is possible to give a fairly complete and concrete description of their invariant subspaces.

Definition 1.2. Let $\mathbf{T}^2$ be the 2-torus and let n be an integer, $n \geqq 1$. By A_n we shall denote the uniform algebra on $\mathbf{T}^2$ of all continuous functions on $\mathbf{T}^2$ that can be uniformly approximated by polynomials in $z^k w^l$, where $l \geqq 0$, and $k \geqq 0$ when $0 \leqq l \leqq n-1$.

Equivalently, A_n may be described as the set of all functions f in $C(\mathbf{T}^2)$ such that $\hat{f}$ is supported in the upper half-plane and, in the second quadrant, $\hat{f}$ is supported on or above the line $y = n$. We have $A_1 \supsetneqq A_2 \supsetneqq \dots$ and $\bigcap_{n=1}^{\infty} A_n = A_\infty$, the polidisc algebra. Observe that A_n is a Dirichlet algebra precisely when $n = 1$. Let σ be the Haar measure on $\mathbf{T}^2$ and define

$$\varphi_n(f) = \int_{\mathbf{T}^2} f \, d\sigma \quad (f \in A_n).$$

Clearly, $\varphi_n \in M_{A_n}$ and $\sigma \in N_{\varphi_n}$ for all n. Also note that $\partial A_n = \mathbf{T}^2$ for all n. However, $N_{\varphi_n} \cap L^1(\mathbf{T}^2, \sigma)$ is not finite dimensional for $n \geqq 2$, as may be seen quite easily.

Our hope is that an understanding of the A_n's will help us understand better the polydisc algebra A_∞. After all, in one obvious sense, A_∞ is the limit of the A_n. In another somewhat more vague sense, as we shall see, it appears that the lattice of invariant subspaces of A_∞ is approximated by the invariant subspace lattices of the A_n. The following proposition, however, shows that in still another sense all the A_n, $n < \infty$, are similar to A_1. Observe that for $n < \infty$, $w^{n-1} A_1 \subset A_n$ and therefore, $|A_n| = |A_1|$, where $|A_n| = \{|f| : f \in A_n\}$. However, $|A_\infty| \subsetneqq |A_1|$. Since $A_n \subset A_1$, there is a natural embedding ϱ_n of M_{A_1} into M_{A_n}, given by restriction. Similarly, $\varrho_\infty \colon M_{A_1} \to M_{A_\infty}$ is an embedding.

Proposition 1.3. *For each finite n, $\varrho_n \colon M_{A_1} \to M_{A_n}$ is surjective, while $\varrho_\infty \colon M_{A_1} \to M_{A_\infty}$ is not surjective.*

Proof. Let $\varphi \in M_{A_n}$. There are two possibilities: $|\varphi(z)| = 1$ or $|\varphi(z)| < 1$. In the first case, define $\tilde{\varphi}(\bar{z}^k w^l) = \overline{\varphi(z)}^k \varphi(w)^l$ $(k \geqq 0, l \geqq 1)$. Then $\tilde{\varphi} \in M_{A_1}$ and

$\tilde{\varphi}|_{A_n}=\varphi$. If $|\varphi(z)|<1$ then $\varphi(w^n)=\varphi(z^k)\varphi(\bar{z}^k w^n)$ for all $k\geqq 0$, so that $|\varphi(w)|^n\leqq\leqq|\varphi(z)|^k$ (all k), which implies that $\varphi(w)=0$. By [1, Theorem 5], φ has a unique extension $\tilde{\varphi}$ to A_1, and $\tilde{\varphi}\in M_{A_1}$. Therefore, $\varrho_n(\tilde{\varphi})=\varphi$. For the second assertion, observe that the proof just given shows that if $\varphi\in M_{A_1}$ and $|\varphi(z)|<1$ then $\varphi(w)=0$. It clearly follows that ϱ_∞ cannot be onto because M_{A_∞} can be identified with the bidisc $\mathbf{D}\times\mathbf{D}$.

Definition 1.4. We shall let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ denote the following subalgebras of $C(\mathbf{T}^2)$:

i) $\mathscr{A}$ is the uniform closure of the polynomials in the first variable z;

ii) $\mathscr{B}$ is the uniform closure of the polynomials in z, $\bar{z}$, and w;

and

iii) $\mathscr{C}$ is the uniform closure of the polynomials in z and $\bar{z}$.

Observe that:

i) $\mathscr{A}$ is isomorphic to the disc algebra;

ii) $\mathscr{C}$ is isomorphic to $C(\mathbf{T})$;

iii) $\mathscr{B}$ is isomorphic to the tensor product of the disc algebra and $C(\mathbf{T})$;

iv) $\mathscr{B}$ is also the uniform closure of $\bigcup_{k=0}^{\infty}\bar{z}^k A_n$ (all n);

v) $A_n\subsetneqq\mathscr{B}$ (all n);

vi) $\bigcap_{k=0}^{\infty} z^k A_n=w^n\mathscr{B}$ (all n);

vii) $\mathscr{B}=(\sum_{l=0}^{n-1}\oplus w^l\mathscr{C})\oplus w^n\mathscr{B}$ (all n); and

viii) $A_n=(\sum_{l=0}^{n-1}\oplus w^l\mathscr{A})\oplus w^n\mathscr{B}$ (all n).

Definition 1.5. The closure in $L^2(\sigma)$ of A_n, $[A_n]_2$, will be denoted H_n^2 and the closure of $\mathscr{B}$ in $L^2(\sigma)$ will be denoted $\mathbf{H}^2$. Likewise, we define $H_n^\infty=[A_n]_*$ and $\mathbf{H}^\infty=[\mathscr{B}]_*$, where $[\ ]_*$ denotes weak-$*$ closure in $L^\infty(\sigma)$. For $p=2,\infty$, we set $H_{n,0}^p=\{f\in H_n^p|\int f\,d\sigma=0\}$. Finally, we define $\mathscr{L}^2=[\mathscr{C}]_2$, $\mathscr{L}^\infty=[\mathscr{C}]_*$, $\mathscr{H}^2=[\mathscr{A}]_2$, and $\mathscr{H}^\infty=[\mathscr{A}]_*$.

Observe that for $p=2,\infty$, $\mathscr{L}^p$ and $\mathscr{H}^p$ are spaces of functions in the first variable, z, only, while the splittings described above yield the decompositions

$$\mathbf{H}^p=(\sum_{l=0}^{n-1}\oplus w^l\mathscr{L}^p)\oplus w^n\mathbf{H}^p \quad (\text{all } n),$$

and

$$H_n^p=(\sum_{l=0}^{n-1}\oplus w^l\mathscr{H}^p)\oplus w^n\mathbf{H}^p.$$

These decompositions are crucial to our analysis. In Section 2 we use them to describe completely the non-simply invariant subspaces of $L^2(\sigma)$, and in Section 3 we use them to describe the simply invariant subspaces of $L^2(\sigma)$. Finally, in Section 4, we use them to determine the structure of the weak-$*$ closed superalgebras of H_n^∞.

2. Non-simply invariant subspaces

For $n<\infty$, H_n^2 is a simply invariant subspace (for A_n) while $\mathbf{H}^2$ is not. The following proposition gives an easy criterion to determine when an invariant subspace is simply invariant. First, we list some important properties of the algebras $A_{n,0}$:

(i) $A_{1,0}=zA_1$, and

(ii) $A_{n,0}=zA_n+[w, w^2, \ldots, w^{n-1}]$, where [ ] denotes linear span.

Proposition 2.1. *Let $\mathfrak{M}$ be an invariant subspace of $L^2(\sigma)$. Then $\mathfrak{M}$ is simply invariant for A_n if and only if* $z\mathfrak{M}\subsetneqq\mathfrak{M}$.

Proof. If $n=1$, $[A_{1,0}\mathfrak{M}]_2=[zA_1\mathfrak{M}]_2$, so that if $[A_{1,0}\mathfrak{M}]_2=\mathfrak{M}$, then $z\mathfrak{M}=\mathfrak{M}$. Conversely, if $z\mathfrak{M}=\mathfrak{M}$, then $[A_1z\mathfrak{M}]_2=[A_1\mathfrak{M}]_2=\mathfrak{M}$. If $n\neq 1$, $[A_{n,0}\mathfrak{M}]_2= =[z\mathfrak{M}+w\mathfrak{M}+\ldots+w^{n-1}\mathfrak{M}]_2$, by (ii) above, and therefore $[A_{n,0}\mathfrak{M}]_2=[z\mathfrak{M}+w\mathfrak{M}]_2$. Hence if $\mathfrak{M}$ is simply invariant, then $z\mathfrak{M}\subsetneqq\mathfrak{M}$. Assume now that $[A_{n,0}\mathfrak{M}]_2=\mathfrak{M}$. Then from what we have just seen, $[z\mathfrak{M}+w\mathfrak{M}]_2=\mathfrak{M}$. Consequently, $[z\mathfrak{M}+w^n\mathfrak{M}]_2= =[z(\mathfrak{M}+w^{n-1}\mathfrak{M})+w^n\mathfrak{M}]_2=[z\mathfrak{M}+w^{n-1}(z\mathfrak{M}+w\mathfrak{M})]_2=[z\mathfrak{M}+w^{n-1}\mathfrak{M}]_2$.

By repeating this argument, we find that $[z\mathfrak{M}+w^n\mathfrak{M}]_2=[z\mathfrak{M}+w\mathfrak{M}]_2=\mathfrak{M}$. But $\bar{z}w^n\in A_n$, so $\mathfrak{M}=[z\mathfrak{M}+w^n\mathfrak{M}]_2=z[\mathfrak{M}+\bar{z}w^n\mathfrak{M}]_2=z\mathfrak{M}$, as desired.

Corollary 2.2. *Let $\mathfrak{M}$ be an invariant subspace of $L^2(\sigma)$. Then $\mathfrak{M}$ is not simply invariant if and only if*

$$\mathfrak{M} = \chi_{E_1} q\mathbf{H}^2\oplus\chi_{E_2}L^2(\sigma),$$

where χ_{E_1} and χ_{E_2} denote the characteristic functions of two measurable sets E_1 and E_2, $\chi_{E_1}\in\mathcal{L}^\infty$, $\chi_{E_1}+\chi_{E_2}\leqq 1$, and $|q|=1$ a.e. (σ).

Proof. The sufficiency is clear. If $z\mathfrak{M}=\mathfrak{M}$, then $\mathfrak{M}$ is invariant under $\mathscr{B}$. Since $\mathscr{B}$ contains the Dirichlet algebra A_1 on which σ is multiplicative, we may apply [6, First example, p. 165] to conclude that $\mathfrak{M}$ is of the form $\mathfrak{M}=\chi_E q[D]_2$, where $D=\{f\in L^\infty\colon f\mathfrak{M}\subset\mathfrak{M}\}$, q is unimodular, and $\chi_E\in D$. By [5, Example 3.(1)], D has the form $D=\chi_F\mathbf{H}^\infty+(1-\chi_F)L^\infty$, where $\chi_F\in\mathbf{H}^\infty$. Letting $\chi_{E_1}=\chi_E\chi_F$ and $\chi_{E_2}=\chi_E(1-\chi_F)$, we see [5] that $\chi_{E_1}\in\mathcal{L}^\infty$ and $\mathfrak{M}$ has the desired representation.

An alternate proof of this result may be based on [4] as follows. Since $z\mathfrak{M}=\mathfrak{M}$, $\mathfrak{M}$ is invariant under $\mathbf{H}^\infty$. But $\mathbf{H}^\infty$ may be viewed as the non-self-adjoint crossed product determined by the identity automorphism of $L^\infty(\mathbf{T})$. Hence the result follows from the analysis in Section 3 of [4] (see in particular Theorem 3.3 and Proposition 3.4).

Before we proceed, we need a definition.

Definition 2.3. Let $\mathfrak{M}$ be an invariant subspace of $L^2(\sigma)$. Then we define $\mathfrak{M}_{-\infty}$ to be $[\bigcup_{k\geq 0} \bar{z}^k\mathfrak{M}]_2$ and $\mathfrak{M}_\infty$ to be $[\bigcap_{k\geq 0} z^k\mathfrak{M}]_2$.

Clearly $\mathfrak{M}_\infty\subset\mathfrak{M}\subset\mathfrak{M}_{-\infty}$. Moreover, both $\mathfrak{M}_\infty$ and $\mathfrak{M}_{-\infty}$ are non-simply invariant. By Corollary 2.2 we can describe both $\mathfrak{M}_\infty$ and $\mathfrak{M}_{-\infty}$. However, if $\mathfrak{M}$ is simply invariant, more can be said.

Proposition 2.4. *Let $\mathfrak{M}$ be a simply invariant subspace. Then $\mathfrak{M}_{-\infty}=q_1\mathbf{H}^2$ and $\mathfrak{M}_\infty=q_2\mathbf{H}^2$, where q_1 and q_2 are unimodular.*

Proof. By Proposition 2.1, $z\mathfrak{M}\subsetneqq\mathfrak{M}$, so $\mathfrak{M}_\infty\subsetneqq\mathfrak{M}\subsetneqq\mathfrak{M}_{-\infty}$. By Corollary 2.2,

$$\mathfrak{M}_{-\infty} = \chi_{E_1}q_1\mathbf{H}^2\oplus\chi_{E_2}L^2, \quad \text{with} \quad \chi_{E_1}+\chi_{E_2} \leqq 1, \quad |q_1| = 1,$$

and

$$\mathfrak{M}_\infty = \chi_{F_1}q_2\mathbf{H}^2\oplus\chi_{F_2}L^2, \quad \text{with} \quad \chi_{F_1}+\chi_{F_2} \leqq 1, \quad |q_2| = 1.$$

Since $\mathfrak{M}_\infty\subset\mathfrak{M}\subset\mathfrak{M}_{-\infty}$, it follows that $\chi_{F_1}+\chi_{F_2}\leqq\chi_{E_1}+\chi_{E_2}$ and $\chi_{F_2}\leqq\chi_{E_2}$. Since $\bar{z}^k w^n\in A_n$ for all $k\geqq 0$ and $A_n\mathfrak{M}\subset\mathfrak{M}$, we see that $\bar{z}^k w^n\mathfrak{M}\subset\mathfrak{M}$ for all $k\geqq 0$. Therefore, $w^n\mathfrak{M}_{-\infty}\subset\mathfrak{M}$, thus $w^n\mathfrak{M}_{-\infty}=w^n z^k\mathfrak{M}_{-\infty}\subset z^k\mathfrak{M}$ for all $k\geqq 0$, so $w^n\mathfrak{M}_{-\infty}\subset$ $\subset\bigcap_{k\geq 0} z^k\mathfrak{M}=\mathfrak{M}_\infty$. Consequently, $w^n\chi_{E_2}L^2\subset\chi_{F_2}L^2$, and so $\chi_{E_2}=\chi_{F_2}$. Likewise, $\chi_{E_1}=\chi_{F_1}$, because $w^n\chi_{E_1}q_1\mathbf{H}^2\subset\chi_{F_1}q_2\mathbf{H}^2$. Thus we find that $\mathfrak{M}_{-\infty}\ominus\mathfrak{M}_\infty=$ $=\chi_{E_1}(q_1\mathbf{H}^2\ominus q_2\mathbf{H}^2)$ which, in turn, is contained in $\chi_{E_1}q_1(\mathbf{H}^2\ominus w^n\mathbf{H}^2)$, since $w^n\mathfrak{M}_{-\infty}\subset\mathfrak{M}_\infty$. Set $\mathfrak{M}_0=\mathfrak{M}\ominus\mathfrak{M}_\infty$. Then since $z\mathfrak{M}_\infty=\mathfrak{M}_\infty$, but $z\mathfrak{M}\subsetneqq\mathfrak{M}$, it follows that $z\mathfrak{M}_0\subsetneqq\mathfrak{M}_0$. If f is a nonzero function in $\mathfrak{M}_0\ominus z\mathfrak{M}_0$, then for all $k>0$, we have $0=(f, z^k f)=\iint_{\mathbf{T}^2}|f(e^{i\theta}, e^{i\varphi})|^2 e^{-ik\theta}\,d\theta\,d\varphi$. Since $|f|$ is real, this implies that $\int_{\mathbf{T}}|f(e^{i\theta}, e^{i\varphi})|^2\,d\varphi$ is constant, a.e., in θ. Since f is nonzero and χ_{E_1} is a function of θ alone, we conclude that $\chi_{E_1}=1$. Thus $\mathfrak{M}_{-\infty}=q_1\mathbf{H}^2$ and $\mathfrak{M}_\infty=q_2\mathbf{H}^2$, as promised.

Remark 2.5. When $\mathfrak{M}=H_n^2$, we see that $\mathfrak{M}_{-\infty}=\mathbf{H}^2$ while $\mathfrak{M}_\infty=w^n\mathbf{H}^2$.

3. Simply invariant subspaces

Suppose that $\mathfrak{N}$ is a simply invariant subspace such that $w^l\mathbf{H}^2=\mathfrak{N}_\infty\subset\mathfrak{N}\subset$ $\subset\mathfrak{N}_{-\infty}=\mathbf{H}^2$ where $1\leqq l\leqq n$. Then applying Lax's generalization of Beurling's theorem, we find that $\mathfrak{N}$ has a very special form. Specifically, using [3, VI.3, p. 60], we see that there is a $j\leqq l$ and there are functions $f_{ik}\in\mathscr{L}^2$, $1\leqq i\leqq j$, $0\leqq k\leqq l-1$, such that

a) $\sum_{k=0}^{l-1} f_{ij}\overline{f_{mk}} = \delta_{im}, \quad 1 \leqq i, m \leqq j,$ and

b) $\mathfrak{N} = [z;\ f_1, \ldots, f_j]_2 \oplus w^l \mathbf{H}^2$

where $f_i = \sum_{k=0}^{l-1} f_{ik} w^k$, $1 \leqq i \leqq j$, and where $[z; f_1, \ldots, f_j]_2$ denotes the smallest subspace containing $f_1, f_2, \ldots, f_j$ that is invariant under multiplication by z. For instance, it is clear that

$$H_l^2 = [z;\ 1, w, \ldots, w^{l-1}]_2 \oplus w^l \mathbf{H}^2 \quad \text{and} \quad H_{l,0}^2 = [z;\ z, w, \ldots, w^{l-1}]_2 \oplus w^l \mathbf{H}^2.$$

If, now, F is a unimodular function and if $\mathfrak{M} = F\mathfrak{N}$, where $\mathfrak{N}$ is of the above form, then $\mathfrak{M}$ is easily seen to be simply invariant, but of course, $\mathfrak{M}$ need no longer be nestled between some $w^l\mathbf{H}^2$ and $\mathbf{H}^2$. Our goal, Theorem 3.2, is to show that every simply invariant subspace can be expressed in this way as $F\mathfrak{N}$.

Proposition 3.1. *Let $\mathfrak{M}$ be a simply invariant subspace for A_n and (for $n \geqq 2$) assume that $A_{n-1}\mathfrak{M} \not\subset \mathfrak{M}$. Then $\mathfrak{M} = F\mathfrak{N}$ where F is a unimodular function on $\mathbf{T}^2$ and $\mathfrak{N}$ is a simply invariant subspace such that $\mathfrak{N}_\infty = w^n \mathbf{H}^2$ and $\mathfrak{N}_{-\infty} = \mathbf{H}^2$.*

Proof. By Proposition 2.4, $\mathfrak{M}_{-\infty} = q_1 \mathbf{H}^2$ and $\mathfrak{M}_\infty = q_2 \mathbf{H}^2$, where $|q_1| = |q_2| = 1$. Since $q_2\mathbf{H}^2 \subset q_1 \mathbf{H}^2$, we must have $\bar{q}_1 q_2 \in \mathbf{H}^2$ and $q_1 \bar{q}_2 w^n \in \mathbf{H}^2$ (recall that $w^n \mathfrak{M}_{-\infty} \subset \subset \mathfrak{M}_\infty$). Set $q = \bar{q}_1 q_2$, so that $q \in \mathbf{H}^2$ and $w^n \bar{q} \in \mathbf{H}^2$. Therefore $q = \sum_{k=0}^{n} c_k w^k$, where $c_k \in \mathscr{L}^2$. Since $|q| = 1$, we have $q = \sum_{k=0}^{n} a_k \chi_{E_k} w^k$, where each a_k is a function of z alone, $|a_k| = 1$ a.e. on E_k, $0 \leqq k \leqq n$, and $\sum_{k=0}^{n} \chi_{E_k} = 1$. Since $q_1 q\mathbf{H}^2 \subset \mathfrak{M} \subset q_1 \mathbf{H}^2$, we see that $\chi_{E_0} q_1 \mathbf{H}^2 = \chi_{E_0} q_1 q \mathbf{H}^2 \subset \chi_{E_0} \mathfrak{M} \subset \chi_{E_0} q_1 \mathbf{H}^2$, and therefore, $\chi_{E_0} \mathfrak{M} = = \chi_{E_0} q_1 q \mathbf{H}^2 \subset q_1 q \mathbf{H}^2 = \mathfrak{M}_\infty$. Now we may assume that $\chi_{E_0} \not\equiv 1$, for otherwise $\mathfrak{M} = \mathfrak{M}_\infty$ and so $\mathfrak{M}$ is not simply invariant. Moreover, $\chi_{E_0} \mathfrak{M} \subset \mathfrak{M}$ and, if $\chi_{E_0} \not\equiv 0$, then it is easy to see that $\bar{z}\mathfrak{M} \subset \mathfrak{M}$, so that $\mathfrak{M}$ is not simply invariant. (Indeed, on the basis of the Wold decomposition for an isometry, it is straightforward to show that if a subspace $\mathfrak{M}$ is invariant for a unitary operator U and if $\mathfrak{M}$ is also invariant for some nontrivial spectral projection of U, then $\mathfrak{M}$ reduces U. In our special situation, χ_{E_0} is a spectral projection for multiplication by z since χ_{E_0} is a function of z alone.) Thus $\chi_{E_0} \equiv 0$. Put $D = \{f \in L^\infty \colon f\mathfrak{M} \subset \mathfrak{M}\}$. Then $H_n^\infty \subset D$ and $q\mathbf{H}^\infty \subset D$, since $q\mathbf{H}^\infty \mathfrak{M} \subset q_1 q \mathbf{H}^2 = q_2 \mathbf{H}^2 \subset \mathfrak{M}$. Hence $w\mathscr{H}^\infty$ and (since $\bar{a}_k \chi_{E_1} \in \mathbf{H}^\infty$) $w\chi_{E_1}\mathscr{L}^\infty$ are both contained in D. Since $\mathscr{L}^\infty$ is isomorphic to $L^\infty(\mathbf{T})$ with $\mathscr{H}^\infty$ corresponding to $H^\infty(\mathbf{T})$, it follows that if $\chi_{E_1} \not\equiv 0$, then $[\mathscr{H}^\infty + \chi_{E_1}\mathscr{L}^\infty]_* = \mathscr{L}^\infty$. But then $w\mathscr{L}^\infty \subset D$ and $H_1^\infty \subset D$. Thus $A_{n-1}\mathfrak{M} \subset \mathfrak{M}$, a contradiction. Thus, $\chi_{E_1} \equiv 0$. One shows similarly that $\chi_{E_2} = \ldots = \chi_{E_{n-1}} \equiv 0$ and $\chi_{E_n} \equiv 1$. Therefore, $q = w^n a_n$. Set $F = q_1$ and $\mathfrak{N} = \bar{q}_1 \mathfrak{M}$ to complete the proof.

Theorem 3.2. *Let $\mathfrak{M}$ be a simply invariant subspace for A_n. Then $\mathfrak{M}=F\mathfrak{N}$ for some unimodular function F and a simply invariant subspace $\mathfrak{N}$ such that $\mathfrak{N}_{-\infty}=\mathbf{H}^2$ and $\mathfrak{N}_\infty=w^l\mathbf{H}^2$ for some l, $1\leqq l\leqq n$. Moreover, $\mathfrak{M}\cap Fw^{l-1}\mathbf{H}^2=Fw^{l-1}qH_1^2$, where q is a unimodular function in $\mathscr{L}^\infty$.*

Proof. Let l, $1\leqq l\leqq n$, be the smallest integer such that $A_l\mathfrak{M}\subset\mathfrak{M}$. Proposition 3.1 then establishes the first part of the theorem. Now, $\mathfrak{M}\cap Fw^{l-1}\mathbf{H}^2=$ $=F(\mathfrak{N}\cap w^{l-1}\mathbf{H}^2)$, and $\mathfrak{N}\cap w^{l-1}\mathbf{H}^2=q\mathscr{H}^2w^{l-1}\oplus w^l\mathbf{H}^2$, because $\bar{w}^{l-1}((\mathfrak{N}\cap w^{l-1}\mathbf{H}^2)\ominus$ $\ominus w^l\mathbf{H}^2))$ is a simply invariant subspace of $\mathscr{L}^2$ under multiplication by z. Therefore the second part of the theorem follows.

The following corollary is of course well known since A_1 is a Dirichlet algebra. However, our methods provide an alternate proof.

Corollary 3.3. *If $n=1$ and $\mathfrak{M}$ is a simply invariant subspace, then $\mathfrak{M}=FH_1^2$ for some unimodular function F.*

Proof. Obviously, l must be 1 in this case, so that $\mathfrak{N}=\mathfrak{N}\cap\mathbf{H}^2=qH_1^2$, which implies that $\mathfrak{M}=F\mathfrak{N}=FH_1^2$.

Corollary 3.4. *Let $\mathfrak{M}$ be a simply invariant subspace for A_n. Then* $\dim(\mathfrak{M}\ominus z\mathfrak{M})=1$ *if and only if $\mathfrak{M}=FH_1^2$ for some unimodular function F.*

Proof. The sufficiency is clear. By Theorem 3.2, $\mathfrak{M}=\tilde{F}\mathfrak{N}$ for some unimodular function $\tilde{F}$ and a simply invariant subspace $\mathfrak{N}$ such that $\mathfrak{N}_{-\infty}=\mathbf{H}^2$ and $\mathfrak{N}_\infty=w^l\mathbf{H}^2$ for some l, $1\leqq l\leqq n$. We claim that $l=1$. This will give the desired result, as in the proof of the previous corollary. Since $\dim(\mathfrak{M}\ominus z\mathfrak{M})=1$, we also have $\dim(\mathfrak{N}\ominus z\mathfrak{N})=1$, so that $\mathfrak{N}\ominus z\mathfrak{N}=[\mathbf{C}f]_2$ for some function $f=\sum_{k=0}^{l} f_k w^k$, where $f_k\in\mathscr{L}^2$ $(0\leqq k\leqq l)$. Since $\mathfrak{N}\supset\mathfrak{N}_\infty=w^l\mathbf{H}^2$, f must be orthogonal to w^l and therefore $f_l=0$. Moreover, $fw^{l-1}=f_0w^{l-1}+w^lg$, where $g\in\mathbf{H}^2$, so that $f_0w^{l-1}\in\mathfrak{N}\ominus\mathfrak{N}_\infty$. Now, $\mathfrak{N}\ominus\mathfrak{N}_\infty=[\bigcup_{i\geqq 0} z^if]_2=[z;f]_2$, and there exists a sequence $\{g_m\}\subset\mathscr{H}^\infty$ such that $g_mf\to f_0w^{l-1}$ in L^2. By projecting onto $w^{l-1}\mathscr{L}^2$ we get: $g_mf_{l-1}\to f_0$. Assume that $l>1$. Then $g_m\sum_{k=0}^{l-2} f_kw^k=g_m(f-f_{l-1}w^{l-1})\to 0$, and in particular, $g_mf_0\to 0$. However, by the second part of Theorem 3.2 we must have $f_0w^{l-1}\in w^{l-1}qH_1^2$, or $f_0w^{l-1}=w^{l-1}qh$, where $|q|=1$, $q\in\mathscr{L}^\infty$ and $h\in\mathscr{H}^2$. Therefore $|f_0|=|h|$ a.e. If $f_0=0$ a.e. then $\mathfrak{N}_{-\infty}\subset w\mathbf{H}^2$, so that $|f_0|>0$ on a set of positive measure. That forces $|h|>0$ a.e. and then $|f_0|>0$ a.e. If $\{g_{m_i}\}$ is a subsequence such that $g_{m_i}f_0\to 0$ a.e., the previous observation implies that $g_{m_i}\to 0$ a.e., so that $g_{m_i}f_{l-1}\to 0$ a.e., or $f_0=0$ a.e.. This contradiction establishes the original claim and completes the proof.

4. Weak-$*$ closed superalgebras

The following theorem generalizes [5, Theorem 4] (see [5, Example 3.(1)]).

Theorem 4.1. *Let B be a weak-$*$ closed subalgebra of L^∞ containing H_n^∞. Then either $B\subset \mathbf{H}^\infty$, or $B=\chi_E\mathbf{H}^\infty+(1-\chi_E)L^\infty$, for some measurable set E with $\chi_E\in\mathscr{L}^\infty$. If $B\subset\mathbf{H}^\infty$ then $\bigcap_{k\geq 0} z^kB=w^l\mathbf{H}^\infty$ for some l, $1\leq l\leq n$.*

Proof. Put $B_{-\infty}=[\bigcup_{k\geq 0}\bar{z}^kB]_*$ and $B_\infty=\bigcap_{k\geq 0} z^kB$. Then $B_\infty\subset B\subset B_{-\infty}$. By [5, Lemma 1] and Corollary 2.2, $B_\infty=\chi_{E_1}q_1\mathbf{H}^\infty+\chi_{E_2}L^\infty$, where $\chi_{E_1}\in\mathscr{L}^\infty$, $\chi_{E_1}+\chi_{E_2}=1$, and $B_{-\infty}=\chi_{F_1}q_2\mathbf{H}^\infty+\chi_{F_2}L^\infty$, with $\chi_{F_1}\in\mathscr{L}^\infty$ and $\chi_{F_1}+\chi_{F_2}=1$. As in the case of invariant subspaces of L^2, $w^nB_{-\infty}\subset B_\infty$. Thus $w^n\chi_{F_2}L^\infty\subset\chi_{E_2}L^\infty$, and this implies $\chi_{E_2}=\chi_{F_2}$, because $\chi_{E_2}L^\infty\subset\chi_{F_2}L^\infty$. Since $B_{-\infty}$ is also an algebra and $q_2\in B_{-\infty}$, we get $q_2B_{-\infty}\subset B_{-\infty}$. Thus $B_{-\infty}\subset\bar{q}_2B_{-\infty}$. This implies that $\chi_{E_1}B_{-\infty}\subset\subset\bar{q}_2\chi_{E_1}q_2\mathbf{H}^\infty=\chi_{E_1}\mathbf{H}^\infty$. In particular, $\chi_{E_1}B\subset\chi_{E_1}\mathbf{H}^\infty$. Put $D=\chi_{E_1}B+\chi_{E_2}\mathbf{H}^\infty$. Then D is a weak-$*$ closed superalgebra of H_n^∞ and $D\subset\mathbf{H}^\infty$. We shall consider two cases:

Case 1: $B\not\subset\mathbf{H}^\infty$. In this case $\chi_{E_2}\not\equiv 0$. Consequently (as in the proof of Proposition 3.1) $[\mathscr{H}^\infty+\chi_{E_2}\mathscr{L}^\infty]_*=\mathscr{L}^\infty$. We have $\mathscr{H}^\infty\subset B$, hence $D\supset\mathscr{H}^\infty+\chi_{E_2}\mathscr{L}^\infty$, and so $D\supset\mathscr{L}^\infty$. This implies $D\supset\mathbf{H}^\infty$, which yields $D=\mathbf{H}^\infty$. Now $\chi_{E_1}B==\chi_{E_1}D=\chi_{E_1}\mathbf{H}^\infty$. On the other hand, $\chi_{E_2}L^\infty=\chi_{E_2}B_\infty\subset\chi_{E_2}B\subset\chi_{E_2}B_{-\infty}\subset\chi_{E_2}L^\infty$. Consequently $\chi_{E_2}B=\chi_{E_2}L^\infty$, and so we can conclude $B=\chi_{E_1}\mathbf{H}^\infty+(1-\chi_{E_1})L^\infty$.

Case 2: $B\subset\mathbf{H}^\infty$. In this case $\chi_{E_2}\equiv 0$. Since $w^n\mathbf{H}^\infty\subset B\subset\mathbf{H}^\infty$ and $B_\infty=q_1\mathbf{H}^\infty$, $q_1=\sum_{j=0}^{n}\chi_{S_j}w^j$, where $\chi_{S_j}\in\mathscr{L}^\infty$, $0\leq j\leq n$, and $\sum_{j=0}^{n}\chi_{S_j}=1$. If $\chi_{S_0}\not\equiv 0$, then $B=\mathbf{H}^\infty$ because $\chi_{S_0}=q_1\chi_{S_0}\in B$ and $zB\subset B$. If k is the first integer such that $\chi_{S_k}\not\equiv 0$ then $B\supset w^kH^\infty$ and $B_\infty=w^k\mathbf{H}^\infty$. For, if $\chi_{S_k}\equiv 1$ then $B\supset w^k\mathbf{H}^\infty$ trivially. If $\chi_{S_k}\not\equiv 1$ then $B\supset w^k\mathbf{H}^\infty$ because $w^k\chi_{S_k}\in B$ and $zB\subset B$. By the hypothesis on k, $q_1=w^k$ and therefore $B_\infty=w^k\mathbf{H}^\infty$.

When $n=2$ in the above theorem, more can be said about B.

Theorem 4.2. *Let B be a weak-$*$ closed subalgebra of $\mathbf{H}^\infty$ containing H_2^∞, and assume that $\bigcap_{k\geq 0} z^kB=w^2\mathbf{H}^\infty$. Then $B=\mathscr{H}^\infty\oplus w\bar{q}\mathscr{H}^\infty\oplus w^2\mathbf{H}^\infty$, where q is an inner function.*

Proof. Consider $B_0=B\cap\mathscr{L}^\infty$. B_0 is a weak-$*$ closed subalgebra of $\mathscr{L}^\infty$ containing $\mathscr{H}^\infty$; moreover, if $B_0=\mathscr{L}^\infty$ then $\mathscr{L}^\infty\subset\bigcap_{k\geq 0} z^kB$, a contradiction. Therefore $B_0=\mathscr{H}^\infty$, i.e., $\mathscr{H}^\infty\subset B$. Let P_1 be the orthogonal projection from $\mathbf{H}^2$ onto $w\mathscr{H}^2$. Since $\bigcap_{k\geq 0} z^kB=w^2\mathbf{H}^\infty$ and $\mathscr{H}^\infty\subset B$, it follows that $P_1B:=\{P_1f\colon f\in B\}\subset B$, and that $B=\mathcal{H}^\infty \oplus P_1 B \oplus w^2\mathbf{H}^\infty$. Moreover, $P_1 B = w\mathfrak{M}_1$, where $\mathfrak{M}_1 := \{f\in\mathcal{L}^\infty : wf\in B\}$ is an $\mathcal{H}^\infty$-submodule of $\mathcal{L}^\infty$; $\mathfrak{M}_1$ is, therefore, of the form $\mathfrak{M}_1=\bar{q}\mathcal{H}^\infty$, for some unimodular function $q\in\mathcal{L}^\infty$. Since $\mathcal{H}^\infty\subset\mathfrak{M}_1$, we easily get that q is inner. Thus, $P_1 B = w\bar{q}\mathcal{H}^\infty$.

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