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On asymptotic Toeplitz operators

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A symbol map is constructed for the C^*-algebra generated by the Toeplitz and compact operators on the space $H^2(\Sigma)$ associated with a semigroup Σ of a locally compact abelian group. As a consequence it follows that the essential range of the symbol is contained in the essential spectrum of the corresponding Toeplitz operator.

Let G be a locally compact abelian group with dual group $\hat{G}$, and let Σ denote a fixed sub-semigroup of $\hat{G}$ which is a Borel subset of $\hat{G}$. Let μ and $\hat{\mu}$ be the normalized Haar measures on G and $\hat{G}$, respectively. Let $L^2(G)$ and $L^2(\hat{G})$ be the corresponding Hilbert spaces of square-integrable functions. The Fourier transform $\mathcal{F}$ is an isometry from $L^2(G)$ onto $L^2(\hat{G})$. We denote by $H^2(\Sigma)$ the subspace of $L^2(G)$ consisting of the functions f for which $\mathcal{F}f$ is in $L^2(\Sigma)$, that is

$$H^2(\Sigma) = \{f \in L^2(G) \colon \mathcal{F}f \text{ is supported on } \Sigma\}.$$

Let P denote the orthogonal projection of $L^2(G)$ onto $H^2(\Sigma)$. If φ is a bounded measurable function on G, write M_φ for the multiplication operator defined on $L^2(G)$ by

$$M_\varphi f = \varphi f$$

and T_φ for the compression of M_φ defined on $H^2(\Sigma)$ by

$$T_\varphi f = PM_\varphi f = P(\varphi f).$$

The operator T_φ is called a *Toeplitz operator* with symbol φ.

The semigroup Σ induces a partial order $\leqq$ on $\hat{G}$ in which $\alpha \leqq \beta$ if $\beta\alpha^{-1}$ is in Σ. With this partial order, $\hat{G}$ and Σ are directed sets. Furthermore, if $\alpha \in \Sigma$ and $\alpha \leqq \beta$, then $\beta \in \Sigma$.

An operator T on $H^2(\Sigma)$ is called an *asymptotic Toeplitz operator* if the net $\{T_\alpha^* T T_\alpha \colon \alpha \in \Sigma\}$ converges strongly. The class of all asymptotic Toeplitz operators on $H^2(\Sigma)$ will be denoted by (AT).

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In case G is the unit circle group $\mathbf{T}$, $\hat{G}$ is the integers $\mathbf{Z}$, and Σ is the semigroup $\mathbf{Z}^+$ of non-negative integers, then T is an asymptotic Toeplitz operator if the sequence $\{U^{*n}TU^n\}_{n=1}^{\infty}$ converges strongly (here U is the unilateral shift on H^2). In [1] this asymptotic notion was defined and used to assign a symbol to any operator in the C^*-algebra generated by the Toeplitz and Hankel operators. In this paper the construction of the symbol map is carried out in the more general setting of locally compact abelian groups. As a consequence, it follows that the spectrum of M_φ is contained in the essential spectrum of T_φ.

Toeplitz operators on locally compact abelian groups were first studied by L. A. Coburn and R. G. Douglas [2]. One of their concerns was the C^*-algebra generated by the Toeplitz operators with symbol in the algebra of almost periodic functions. They proved that this algebra of operators modulo its commutator ideal is $*$-isomorphic to the algebra of almost periodic functions. Our results show that the C^*-algebra generated by all Toeplitz and compact operators is $*$-homomorphic to $L^\infty(G)$, and the kernel of this homomorphism is the ideal of operators T (in the algebra) such that $T_\alpha^* T T_\alpha \to 0$ $(\alpha \in \Sigma)$ strongly.

The following elementary facts about Toeplitz operators will be useful:

$$T_{\varphi+\psi} = T_\varphi + T_\psi, \quad T_\varphi^* = T_{\bar{\varphi}},$$
$$T_\alpha f = \alpha f \quad \text{if } \alpha \text{ is in } \Sigma.$$

For the rest of the paper we make the following assumptions: μ and $\hat{\mu}$ are σ-finite measures, $\hat{\mu}(\Sigma)>0$, $\hat{G}$ is generated by Σ (i.e. $\hat{G}=\Sigma\Sigma^{-1}$), Σ is not dense in $\hat{G}$.

From [2] we have that $\sigma(M_\varphi)$, the spectrum of M_φ, is contained in $\sigma(T_\varphi)$, and $\|T_\varphi\| = \|\varphi\|_\infty$ for φ in $L^\infty(G)$.

Lemma 1. *If K is a compact operator on $H^2(\Sigma)$, then $K \in (\mathrm{AT})$ and $KT_\alpha \to 0$ $(\alpha \in \Sigma)$ strongly.*

Proof. It is enough to prove the last assertion. For this, let f be fixed in $H^2(\Sigma)$. Since K is a compact operator, and $KT_\alpha f = K(\alpha f)$, we only need to show that the net $\{\alpha f \colon \alpha \in \Sigma\}$ converges weakly to zero. For g in $H^2(\Sigma)$ we have $\bar{f}g \in L^1(G)$, and therefore

$$(g, \alpha f) = \int_G g \bar{\alpha} \bar{f} \, d\mu = \mathscr{F}(\bar{f}g)(\alpha)$$

for all α in Σ. By the Riemann—Lebesgue's Lemma ([4], Remark 28.42), given $\varepsilon > 0$ there exists a compact set F in $\hat{G}$ such that

$$|\mathscr{F}(\bar{f}g)(\sigma)| < \varepsilon \quad \text{for all } \sigma \text{ in } \hat{G} \setminus F.$$

Next we show that $\alpha\Sigma \subseteq \hat{G} \setminus F$ for some α in Σ. If this is not the case, then $\alpha\Sigma \cap F \neq \square$ for all α in Σ. Since $\alpha_1\alpha_2\Sigma \subseteq \alpha_i\Sigma$ for α_1, α_2 in Σ, it follows that the family $\{\alpha\Sigma \cap F \colon \alpha \in \Sigma\}$

has the finite intersection property. Hence there exists α_0 in the closure Σ^- of Σ such that $\alpha_0 \in \alpha\Sigma^-$ for all α in Σ, therefore $\alpha_0\Sigma^{-1} \subseteqq \Sigma^-$ and so $\alpha_0\hat{G} \subseteqq \Sigma^-$. This is a contradiction because $\alpha_0\hat{G}=\hat{G}$ and Σ is not dense in $\hat{G}$.

So far we have proved that there exists α in Σ such that $|\mathscr{F}(\bar{f}g)(\sigma)|<\varepsilon$ for all σ in $\alpha\Sigma$. This shows that $|(g, \beta f)|<\varepsilon$ for all β in Σ such that $\beta \geqq \alpha$.

Lemma 2. [2] *If E is a compact subset of $\hat{G}$, then there exists α in Σ such that $\alpha E \subseteqq \Sigma$.*

Proof. See [2], § 2.

Lemma 3. *Let A be an operator on $H^2(\Sigma)$ such that $T_\alpha^* A T_\alpha = A$ for all α in Σ. Then $M_\alpha^* APM_\alpha \to M_\varphi$ $(\alpha \in \Sigma)$ weakly, and $A=T_\varphi$ for some φ in $L^\infty(G)$. Furthermore, $M_\alpha^* PM \to I$ $(\alpha \in \Sigma)$ strongly.*

Proof. Let f be in $L^2(G)$ such that $\mathscr{F}f$ has compact support E in $\hat{G}$. From Lemma 2 there exists α_0 in Σ such that $\alpha E \subseteqq \Sigma$ for $\alpha \geqq \alpha_0$. If $\sigma \in \hat{G} \setminus \Sigma$ and $\alpha \geqq \alpha_0$, then $\alpha^{-1}\sigma \in \hat{G} \setminus E$, and so $\mathscr{F}(\alpha f)(\sigma) = (\mathscr{F}f)(\alpha^{-1}\sigma)=0$. Hence $\alpha f \in H^2(\Sigma)$ and $M_\alpha^* APM_\alpha f = M_\alpha^* A(\alpha f)$. If A is the identity, we conclude that $M_\alpha^* PM_\alpha f = f$ for $\alpha \geqq \alpha_0$. This completes the proof of the last assertion of the lemma.

Let $B_\alpha = M_\alpha^* APM_\alpha$ for α in Σ. Let g be in $L^2(G)$ such that $\mathscr{F}g$ has support contained in E. If $\alpha \geqq \alpha_0$, from above we have

$$(B_\alpha f, g) = \big(A(\alpha f), \alpha g\big) = \big(AT_{\alpha\alpha_0^{-1}}(\alpha_0 f), T_{\alpha\alpha_0^{-1}}(\alpha_0 g)\big) =$$

$$= \big(T^*_{\alpha\alpha_0^{-1}} AT_{\alpha\alpha_0^{-1}}(\alpha_0 f), \alpha_0 g\big) = \big(A(\alpha_0 f), \alpha_0 g\big)$$

because $\alpha\alpha_0^{-1} \in \Sigma$. Hence there exists an operator B on $L^2(G)$ such that $B_\alpha \to B$ $(\alpha \in \Sigma)$ weakly.

For σ in Σ we have

$$(M_\sigma^* Bf, g) = \lim_\alpha (M_\sigma^* M_\alpha^* APM_\alpha f, g) = \lim_\alpha (M_{\alpha\sigma}^* APM_{\alpha\sigma} M_\sigma^* f, g) = (BM_\sigma^* f, g).$$

Therefore $M_\sigma B = BM_\sigma$ for all σ in $\hat{G}$. Since the subspace spanned by $\hat{G}$ is weak* dense in $L^\infty(G)$ ([4], Lemma 31.4), it follows that $M_\psi B = BM_\psi$ for all ψ in $L^\infty(G)$. Since the algebra of multiplication operators is maximal abelian, then $B=M_\varphi$ for some φ in $L^\infty(G)$. Finally, it is easy to see that $(T_\varphi f, g)=(Af, g)$ for f, g in $H^2(\Sigma)$.

Corollary 4. *If T is an asymptotic Toeplitz operator on $H^2(\Sigma)$, then $T_\alpha^* TT_\alpha \to T_\varphi$ $(\alpha \in \Sigma)$ strongly, for some φ in $L^\infty(G)$.*

Proof. If $T_\alpha^* TT_\alpha \to A$ $(\alpha \in \Sigma)$ strongly, then $T_\alpha^* AT_\alpha = A$ for all α in Σ. Now the result follows from Lemma 3.

We define the map Φ: (AT)$\to L^\infty(G)$ by $\Phi(T)=\varphi$ where $T_\alpha^* T T_\alpha \to T_\varphi$ $(\alpha \in \Sigma)$ strongly. The function φ is called the symbol of T. Lemma 1 shows that $\Phi(K)=0$ if K is a compact operator.

Corollary 5. *The class of asymptotic Toeplitz operators is a norm closed subspace. The map Φ is a linear contraction.*

Proof. Let $T\in$(AT) and $\Phi(T)=\varphi\in L^\infty(G)$. Then

$$\|T_\varphi f\| = \lim_\alpha \|T_\alpha^* T T_\alpha f\| \leqq \|T\|\,\|f\|.$$

Hence

$$\|\Phi(T)\| = \|\varphi\|_\infty = \|T_\varphi\| \leqq \|T\|.$$

Let $T_n\in$(AT) be such that $\|T-T_n\|\to 0$. Let $\varphi_n\in L^\infty(G)$ be such that $\Phi(T_n)=\varphi_n$. Since

$$\|\varphi_n-\varphi_m\|_\infty = \|\Phi(T_n-T_m)\| \leqq \|T_n-T_m\|,$$

there exists φ in $L^\infty(G)$ such that $\|\varphi_n-\varphi\|_\infty\to 0$. Now we have

$$\|T_\alpha^* T T_\alpha f - T_\varphi f\| \leqq \|T-T_n\|\,\|f\| + \|T_\alpha^* T_n T_\alpha f - T_{\varphi_n} f\| + \|\varphi_n-\varphi\|_\infty \|f\|,$$

therefore $T\in$(AT) and $\Phi(T)=\varphi$.

Corollary 6. *If K is a compact operator on $H^2(\Sigma)$, then $\|T_\varphi\|\leqq\|T_\varphi+K\|$ for φ in $L^\infty(G)$. Therefore the subspace $\{T_\varphi+K$: $\varphi\in L^\infty(G)$, K compact$\}$ is norm closed.*

Proof. By Lemma 1, $\Phi(T_\varphi+K)=\varphi$. Since Φ is a contraction,

$$\|T_\varphi\| = \|\varphi\|_\infty = \|\Phi(T_\varphi+K)\| \leqq \|T_\varphi+K\|.$$

Corollary 7. *If φ is in $L^\infty(G)$ and $H_\varphi=P^\perp M_\varphi|H^2(\Sigma)$, then $H_\varphi T_\alpha\to 0$ $(\alpha\in\Sigma)$ strongly.*

Proof. For f in $H^2(\Sigma)$ and α in Σ we have

$$H_\varphi T_\alpha f = P^\perp(\alpha\varphi f) = P^\perp M_\alpha(\varphi f).$$

Therefore $\|H_\varphi T_\alpha f\|=\|M_\alpha^* P^\perp M_\alpha(\varphi f)\|\to 0$ $(\alpha\in\Sigma)$ by Lemma 3.

Lemma 8. *Let $T=T_{\varphi_1}T_{\varphi_2}\dots T_{\varphi_n}$ with $\varphi_i\in L^\infty(G)$. Then T is an asymptotic Toeplitz operator and $\Phi(T)=\varphi_1\varphi_2\dots\varphi_n$.*

Proof. For φ in $L^\infty(G)$ and H_φ as defined in Corollary 7 we have

$$M_\varphi = \begin{pmatrix} T_\varphi & * \\ H_\varphi & * \end{pmatrix}.$$

with respect to the decomposition $L^2(G)=H^2(\Sigma)\oplus H^2(\Sigma)^{\perp}$. If φ and ψ are in $L^\infty(G)$, then $M_{\psi\varphi}=M_\psi M_\varphi$ and therefore (multiply matrices and compare upper left corners) $T_{\psi\varphi}-T_\psi T_\varphi=AH_\varphi$ for some operator A. Applying this last equality to the telescoping sum

$$T-T_{\varphi_1\varphi_2\dots\varphi_n}=T_{\varphi_1}T_{\varphi_2\dots\varphi_n}-T_{\varphi_1(\varphi_2\dots\varphi_n)}+T_{\varphi_1}(T_{\varphi_2}T_{\varphi_3\dots\varphi_n}-T_{\varphi_2(\varphi_3\dots\varphi_n)})+$$
$$+T_{\varphi_1}T_{\varphi_2}(T_{\varphi_3}T_{\varphi_4\dots\varphi_n}-T_{\varphi_3(\varphi_4\dots\varphi_n)})+\dots+T_{\varphi_1}T_{\varphi_2}\dots T_{\varphi_{n-2}}(T_{\varphi_{n-1}}T_{\varphi_n}-T_{\varphi_{n-1}\varphi_n})$$

we conclude that each of the $n-1$ summands on the right can be written as BH_φ for some operator B and some φ in $L^\infty(G)$. From Corollary 7 we have that $BH_\varphi T_\alpha\to 0$ $(\alpha\in\Sigma)$ strongly. Therefore $(T-T_{\varphi_1\varphi_2\dots\varphi_n})T_\alpha\to 0$ $(\alpha\in\Sigma)$ strongly. From this it follows that $T\in(\mathrm{AT})$ and $\Phi(T)=\varphi_1\varphi_2\dots\varphi_n$.

Theorem 9. *Let $\mathcal{A}$ be the C^*-algebra generated by the Toeplitz and compact operators on $H^2(\Sigma)$. Then $\mathcal{A}$ is contained in the class of asymptotic Toeplitz operators. Furthermore, the restriction of Φ to $\mathcal{A}$ is a $*$-homomorphism.*

Proof. Let $\mathcal{A}_0$ be the linear manifold generated by the compact operators and all the finite products of Toeplitz operators. Clearly $\mathcal{A}_0$ is an algebra which is closed under the operation of taking adjoint, and the norm closure of $\mathcal{A}_0$ is equal to $\mathcal{A}$. Since (AT) is a subspace, from Lemmas 1 and 8 it follows that $\mathcal{A}_0$ is contained in (AT), and the restriction of Φ to $\mathcal{A}_0$ is clearly a $*$-homomorphism. Since (AT) is norm closed, then $\mathcal{A}\subseteq(\mathrm{AT})$, and the proof is complete.

Remark. In general, (AT) is not an algebra, it is not even closed under adjointion (cf. [1]).

Corollary 10. *If φ is in $L^\infty(G)$, then the spectrum of M_φ is contained in the essential spectrum of T_φ.*

Proof. Since the spectrum of M_φ is the essential range of φ, it will be enough to show that if T_φ is a Fredholm operator, then φ has an inverse in $L^\infty(G)$. Let $\mathcal{A}$ be the C^*-algebra defined in Theorem 9. If $\mathbf{K}$ is the closed ideal of compact operators on $H^2(\Sigma)$, then $\mathcal{A}/\mathbf{K}$ is a C^*-algebra. If T_φ is Fredholm, then $[T_\psi]$ is invertible in $\mathcal{A}/\mathbf{K}$, so there exists S in $\mathcal{A}$ such that $T_\varphi S-I$ is compact. Therefore $\Phi(T_\varphi S-I)=0$. Since Φ is a homomorphism on $\mathcal{A}$, $\varphi\cdot\Phi(S)\equiv 1$ a.e. $[\mu]$. Since $\Phi(S)$ is in $L^\infty(G)$, then φ is invertible in $L^\infty(G)$.

Remark. In Corollary 10 it is actually proved that the spectrum of M_φ is contained in the intersection of the left essential spectrum and the right essential spectrum of T_φ.

Remark. From Theorem 9 we have that $TS-ST$ is in $\ker\Phi$ for any S and T in $\mathcal{A}$. Therefore the commutator ideal of $\mathcal{A}$ is contained in $\ker\Phi$. For Toeplitz operators on the unit circle this inclusion is an equality [1]. Is this true in general?

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