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Ideals and Lie ideals of operators

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1. Introduction

Let $\mathfrak{H}$ denote an infinite dimensional (complex) Hilbert space and $\mathscr{B}(\mathfrak{H})$ the algebra of all (bounded, linear) operators on $\mathfrak{H}$. We say a linear manifold $\mathscr{L}$ in $\mathscr{B}(\mathfrak{H})$ is *unitarily invariant* if $U^*\mathscr{L}U\subseteqq\mathscr{L}$ for all unitaries U in $\mathscr{B}(\mathfrak{H})$. If $\mathscr{L}$ is such a manifold and $\mathfrak{K}$ is another Hilbert space of the same dimension as $\mathfrak{H}$, then we can "transport" $\mathscr{L}$ to a unitarily invariant manifold of operators acting on $\mathfrak{K}$ by taking any unitary transformation W from $\mathfrak{H}$ onto $\mathfrak{K}$ and setting $\mathscr{L}_{\mathfrak{K}}=W\mathscr{L}W^*$. That $\mathscr{L}_{\mathfrak{K}}$ is unitarily invariant, and that its definition is independent of the choice of W, follow from the fact that $\mathscr{L}$ is unitarily invariant. In particular, if we consider the case when $\mathfrak{K}=\mathfrak{H}\oplus\mathfrak{H}$, then $\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}}$ is a unitarily invariant manifold of operators which can be expressed as 2×2 operator matrices with entries in $\mathscr{B}(\mathfrak{H})$. Thus we can define the following two manifolds in $\mathscr{B}(\mathfrak{H})$:

(∗)

$$\mathscr{L}^c=\{T|_{\mathfrak{H}\oplus 0}\colon T\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}}\}=\left\{A\in\mathscr{B}(\mathfrak{H})\colon \begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}} \text{ for some } B, C, D\in\mathscr{B}(\mathfrak{H})\right\},$$

(∗∗)
$$\mathscr{I}_{\mathscr{L}}=\left\{B\in\mathscr{B}(\mathfrak{H})\colon \begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}}\right\}.$$

It was shown in [5] that $\mathscr{I}_{\mathscr{L}}$ is an ideal in $\mathscr{B}(\mathfrak{H})$, and that $[\mathscr{B}(\mathfrak{H}),\mathscr{I}_{\mathscr{L}}]\subseteqq\mathscr{L}\subseteqq\subseteqq\mathscr{I}_{\mathscr{L}}+\mathbf{C}I$. This fact covers a part of the following theorem, also proved in the same paper.

Theorem 1 ([5]). *Let $\mathscr{L}$ be a linear manifold in $\mathscr{B}(\mathfrak{H})$. Then the following conditions are equivalent:*

(1) *$\mathscr{L}$ is unitarily invariant;*

(2) *$\mathscr{L}$ is a Lie ideal in $\mathscr{B}(\mathfrak{H})$;*

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(3) *there exists an ideal* $\mathcal{I}$ *in* $\mathcal{B}(\mathfrak{H})$ *such that*

$$[\mathcal{B}(\mathfrak{H}), \mathcal{I}] \subseteqq \mathcal{L} \subset \mathcal{I}+\mathbf{C}I.$$

(The above results are shown in [5] only in the case where $\mathfrak{H}$ is separable; but, in fact, everything works in the non-separable case too. See remarks following Theorem 2.)

For a unitarily invariant manifold $\mathcal{L}$ in $\mathcal{B}(\mathfrak{H})$, the ideal $\mathcal{I}$ of the condition (3) above is uniquely determined by $\mathcal{L}$ (shown in Section 2), and will be called the *associate ideal* of $\mathcal{L}$. Among other results in Section 2, we show that $\mathcal{L}^c$ (defined by $(*)$) is either $\mathcal{I}_{\mathcal{L}}$ or $\mathcal{I}_{\mathcal{L}}+\mathbf{C}I$. A consequence, shown in Section 3, is the following useful characterization of ideals in $\mathcal{B}(\mathfrak{H})$: a linear manifold $\mathcal{L}$ in $\mathcal{B}(\mathfrak{H})$ is a proper ideal if and only if $\mathcal{L}$ is unitarily invariant, $I\notin\mathcal{L}$ and $\mathcal{L}^c\subseteqq\mathcal{L}$. Several applications of this result (or its variant) are given in Section 3.

In Section 4 we give some characterizations of ideals in C^*-algebras satisfying a certain condition, viz., we show that the ideals are precisely the linear manifolds $\mathcal{L}$ for which $P\mathcal{L}P\subseteqq\mathcal{L}$ for all projections in the algebra.

The proof of Theorem 1 previously mentioned in [5] uses the following weaker form of a theorem of Fillmore:

Theorem 2 ([3]). *Every operator in* $\mathcal{B}(\mathfrak{H})$ *is a linear combination of projections.*

The original proof of this result is quite complicated. We include an appendix to this paper in which we prove Theorem 1 in such a way that, not only do we obtain it without Theorem 2, but the latter theorem actually drops out as a bonus in the process. To generalize Theorem 1 for Hilbert spaces not necessarily separable, we need to extend a theorem of Calkin [1] to the non-separable case. Since this extension is by no means straightforward, we also include its proof in the appendix.

We use standard notation: $\mathcal{C}_2$ denotes the Hilbert—Schmidt class and $\mathcal{C}_p$ $(p>0)$ the ideal of operators such that $(T^*T)^{p/4}\in\mathcal{C}_2$. If $\mathcal{S}$, $\mathcal{T}$ are linear manifolds in $\mathcal{B}(\mathfrak{H})$, we write $\mathcal{S}\mathcal{T}$ (resp. $[\mathcal{S}, \mathcal{T}]$) for the linear span of all operators of the form ST (resp. $[S, T]=ST-TS$) where $S\in\mathcal{S}$ and $T\in\mathcal{T}$. All Hilbert spaces are assumed to be infinite dimensional, but they are not required to be separable unless otherwise stated.

2. The associate ideal of a Lie ideal in $\mathcal{B}(\mathfrak{H})$

Let $\mathcal{L}$ be a unitarily invariant manifold in $\mathcal{B}(\mathfrak{H})$. As mentioned in the introduction, the set

$$\mathcal{I}_{\mathcal{L}} = \left\{B\in\mathcal{B}(\mathfrak{H}):\ \begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\in\mathcal{L}_{\mathfrak{H}\oplus\mathfrak{H}}\right\}$$

forms an ideal. We call $\mathcal{I}_{\mathcal{L}}$ *the associate ideal* of $\mathcal{L}$ or *the ideal associated with* $\mathcal{L}$. There are several ways to describe this ideal, as the following proposition shows.

Proposition 3. *Let $\mathscr{L}$ be a unitarily invariant manifold and $\mathscr{I}$ be an ideal in $\mathscr{B}(\mathfrak{H})$. Then the following conditions are equivalent:*

(1) *$\mathscr{I}$ is the associate ideal of $\mathscr{L}$;*
(2) $[\mathscr{B}(\mathfrak{H}), \mathscr{I}] \subseteqq \mathscr{L} \subseteqq \mathscr{I} + \mathbf{C}I$;
(3) $[\mathscr{B}(\mathfrak{H}), \mathscr{I}] \subseteqq \mathscr{L}$ *and* $[\mathscr{B}(\mathfrak{H}), \mathscr{L}] \subseteqq \mathscr{I}$;
(4) *$\mathscr{I}$ is the largest ideal among those ideals $\mathscr{J}$ satisfying $[\mathscr{B}(\mathfrak{H}), \mathscr{J}] \subseteqq \mathscr{L}$;*
(5) *$\mathscr{I}$ is the smallest ideal among those ideals $\mathscr{J}$ satisfying $\mathscr{L} \subseteqq \mathscr{J} + \mathbf{C}I$;*
(6) *$\mathscr{I}$ is the ideal generated by $[\mathscr{B}(\mathfrak{H}), \mathscr{L}]$;*
(7) $\mathscr{I} + \mathbf{C}I = \{T \in \mathscr{B}(\mathfrak{H}) \colon [\mathscr{B}(\mathfrak{H}), T] \subseteqq \mathscr{L}\}$.

For the proof of the above proposition, we need the following lemma.

Lemma 4. *Let $\mathscr{A}$ be an algebra with identity I and $\mathscr{B} = \mathscr{M}_2(\mathscr{A})$ be the algebra of all 2×2 matrices with entries in $\mathscr{A}$. Then, for two ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ in $\mathscr{B}$, $[\mathscr{B}, [\mathscr{B}, \mathscr{I}_1]] \subseteqq \mathscr{I}_2$ implies $\mathscr{I}_1 \subseteqq \mathscr{I}_2$.*

Proof. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathscr{I}_1$. Then

$$\left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right]\right] = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathscr{I}_1 \cap \mathscr{I}_2.$$

Hence $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathscr{I}_1$ and it suffices to show that $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathscr{I}_2$. Now $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ D & 0 \end{pmatrix} \in \mathscr{I}_1$ and, by using the same computation as above, we obtain $\begin{pmatrix} 0 & A \\ D & 0 \end{pmatrix} \in \mathscr{I}_2$. Hence

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & A \\ D & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \mathscr{I}_2. \quad \square$$

Proof of Proposition 3. (1)⇒(2) follows from Theorem 1. (2)⇒(3) is obvious. To show (3)⇒(6), let $\mathscr{J}$ be the ideal generated by $[\mathscr{B}(\mathfrak{H}), \mathscr{L}]$. Then we have $[\mathscr{B}(\mathfrak{H}), \mathscr{L}] \subseteqq \mathscr{J} \subseteqq \mathscr{I}$ and hence $[\mathscr{B}(\mathfrak{H}), [\mathscr{B}(\mathfrak{H}), \mathscr{I}]] \subseteqq [\mathscr{B}(\mathfrak{H}), \mathscr{L}] \subseteqq \mathscr{J}$. It follows from Lemma 4 (since $\mathscr{B}(\mathfrak{H})$ and $\mathscr{M}_2(\mathscr{B}(\mathfrak{H}))$ are isomorphic algebras) that $\mathscr{I} \subseteqq \mathscr{J}$. Therefore $\mathscr{I} = \mathscr{J}$. Similarly we can show that (3)⇒(4) and (2)⇒(5). Since the ideal $\mathscr{J}$ described by either (4), (5) or (6) is unique and since the associate ideal fits into each of these descriptions, we have (4)⇒(1), (5)⇒(1) and (6)⇒(1). Thus conditions (1) to (6) are equivalent.

Finally, let $\mathscr{I}_0$ be the ideal associated with $\mathscr{L}$ and

$$\mathscr{S} = \{T \in \mathscr{B}(\mathfrak{H}) \colon [\mathscr{B}(\mathfrak{H}), T] \subseteqq \mathscr{L}\}.$$

Then it is easy to see that $\mathscr{S}$ is unitarily invariant and $\mathscr{I}_0 \subseteqq \mathscr{S}$. Let $\mathscr{J}$ be the associate ideal of $\mathscr{S}$. Since

$$[\mathscr{B}(\mathfrak{H}), [\mathscr{B}(\mathfrak{H}), \mathscr{J}]] \subseteqq [\mathscr{B}(\mathfrak{H}), \mathscr{S}] \subseteqq \mathscr{L} \subseteqq \mathscr{I}_0 + \mathbf{C}I$$

it follows from Lemma 4 that $\mathcal{J}\subseteqq\mathcal{I}_0$. Therefore we have $\mathcal{J}\subseteqq\mathcal{I}_0\subseteqq\mathcal{S}\subseteqq\mathcal{J}+\mathbf{C}I$. Hence $\mathcal{J}=\mathcal{I}_0$ and $\mathcal{S}=\mathcal{J}+\mathbf{C}I=\mathcal{I}_0+\mathbf{C}I$. We have proved (1)⇒(7). Conversely, if $\mathcal{S}=\mathcal{I}+\mathbf{C}I$, then $\mathcal{I}+\mathbf{C}I=\mathcal{I}_0+\mathbf{C}I$ and hence $\mathcal{I}=\mathcal{I}_0$. Therefore (7)⇒(1) follows. □

For brevity, in the rest of this section, we replace the term "unitarily invariant linear manifold in $\mathcal{B}(\mathfrak{H})$" by its synonym "Lie ideal in $\mathcal{B}(\mathfrak{H})$".

By definition, the associate ideal $\mathcal{I}$ of a Lie ideal $\mathcal{L}$ in $\mathcal{B}(\mathfrak{H})$ is obtained by taking the upper right corners of 2×2 matrices in $\mathcal{L}_{\mathfrak{H}\oplus\mathfrak{H}}$. The next result says, if we take the upper left corners instead, then either $\mathcal{I}$ or $\mathcal{I}+\mathbf{C}I$ is produced.

Proposition 5. *If $\mathcal{L}$ is a Lie ideal and $\mathcal{I}$ is its associate ideal, then either $\mathcal{L}^c=\mathcal{I}$ or $\mathcal{L}^c=\mathcal{I}+\mathbf{C}I$.*

Proof. From Theorem 1, we have $\mathcal{L}\subseteqq\mathcal{I}+\mathbf{C}I$. It is elementary that if $\mathcal{J}$ is an ideal in $\mathcal{B}(\mathfrak{H})$, then $\mathcal{J}^c=\mathcal{J}$. Hence we have $\mathcal{L}^c\subseteqq\mathcal{I}^c+\mathbf{C}I=\mathcal{I}+\mathbf{C}I$.

Now we show $\mathcal{I}\subseteqq\mathcal{L}^c$. Let $T\in\mathcal{I}$. Then $\begin{pmatrix}0 & T\\ 0 & 0\end{pmatrix}\in\mathcal{L}_{\mathfrak{H}\oplus\mathfrak{H}}$. Let W be the unitary operator on $\mathfrak{H}\oplus\mathfrak{H}$ given by the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix}I & I\\ -I & I\end{pmatrix}$. Then, since $\mathcal{L}_{\mathfrak{H}\oplus\mathfrak{H}}$ is unitarily invariant, we have

$$2W^*\begin{pmatrix}0 & T\\ 0 & 0\end{pmatrix}W=\begin{pmatrix}T & T\\ -T & -T\end{pmatrix}\in\mathcal{L}_{\mathfrak{H}\oplus\mathfrak{H}}$$

and hence $T\in\mathcal{L}^c$.

We have shown that $\mathcal{I}\subseteqq\mathcal{L}^c\subseteqq\mathcal{I}+\mathbf{C}I$ from which it follows that either $\mathcal{L}^c=\mathcal{I}$ or $\mathcal{L}^c=\mathcal{I}+\mathbf{C}I$. □

Now we consider some "permanence properties" of Lie ideals and their associate ideals. First we state the following obvious fact without proof in order to put it into record.

Proposition 6. *If $\{\mathcal{L}_j\}$ is a family of Lie ideals in $\mathcal{B}(\mathfrak{H})$, then the intersection $\bigcap_j\mathcal{L}_j$ and the sum $\sum_j\mathcal{L}_j$ are Lie ideals also. If, furthermore, $\mathcal{I}_j$ is the associate ideal of $\mathcal{L}_j$ for each j, then the associate ideals of $\bigcap_j\mathcal{L}_j$ and $\sum_j\mathcal{L}_j$ are $\bigcap_j\mathcal{I}_j$ and $\sum_j\mathcal{I}_j$ respectively.*

The next "permanence property" is less obvious and more interesting.

Proposition 7. *If $\mathcal{L}_1$, $\mathcal{L}_2$ are Lie ideals in $\mathcal{B}(\mathfrak{H})$ with $\mathcal{I}_1$, $\mathcal{I}_2$ as their associate ideals, then $[\mathcal{L}_1,\mathcal{L}_2]$ is a Lie ideal with $\mathcal{I}_1\mathcal{I}_2$ as its associate ideal.*

Remark. It is easy to check that $\mathcal{I}_1\mathcal{I}_2$ is an ideal in $\mathcal{B}(\mathfrak{H})$. Since every ideal in $\mathcal{B}(\mathfrak{H})$ is self-adjoint, we have $\mathcal{I}_1\mathcal{I}_2=\mathcal{I}_2\mathcal{I}_1$.

Proof. It is easy to check that $[\mathscr{L}_1, \mathscr{L}_2]$ is a Lie ideal. Let $\mathscr{I}$ be the ideal associated with $[\mathscr{L}_1, \mathscr{L}_2]$. From $\mathscr{L}_j \subseteqq \mathscr{I}_j + \mathbf{C}I\ (j=1, 2)$ we have $[\mathscr{L}_1, \mathscr{L}_2] \subseteqq \mathscr{I}_1\mathscr{I}_2 + \mathbf{C}I$. Hence, by (1)⇔(5) in Proposition 3, we have $\mathscr{I} \subseteqq \mathscr{I}_1\mathscr{I}_2$. Next, suppose that $A_j \in \mathscr{I}_j$ $(j=1, 2)$. Then we have

$$\begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix} \in (\mathscr{L}_1)_{\mathfrak{H}\oplus\mathfrak{H}}, \quad \begin{pmatrix} 0 & 0 \\ A_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in (\mathscr{L}_2)_{\mathfrak{H}\oplus\mathfrak{H}}$$

and hence

$$\begin{pmatrix} A_1A_2 & 0 \\ 0 & -A_2A_1 \end{pmatrix} = \left[\begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ A_2 & 0 \end{pmatrix}\right] \in [\mathscr{L}_1, \mathscr{L}_2]_{\mathfrak{H}\oplus\mathfrak{H}}.$$

Therefore it follows from Proposition 5 that $A_1A_2 \in \mathscr{I}$. □

Remark. In case $\mathscr{L}_1$ and $\mathscr{L}_2$ actually are ideals, i.e. $\mathscr{L}_1 = \mathscr{I}_1$ and $\mathscr{L}_2 = \mathscr{I}_2$, we have an easier proof as follows. We have to show $[\mathscr{B}(\mathfrak{H}), \mathscr{I}_1\mathscr{I}_2] \subseteqq [\mathscr{I}_1, \mathscr{I}_2] \subseteqq \subseteqq \mathscr{I}_1\mathscr{I}_2 + \mathbf{C}I$. The second inclusion is obvious. The first follows from the identity $[T, AB] = [TA, B] + [BT, A]$.

Example. Let $\mathscr{C}_1^0$ denote the set of trace class operators on $\mathfrak{H}$ of trace zero. Then, by using some properties of the trace function, we have $[\mathscr{C}_2, \mathscr{C}_2] \subseteqq \mathscr{C}_1^0$. G. Weiss [17] has shown that $[\mathscr{C}_2, \mathscr{C}_2] \neq \mathscr{C}_1^0$. Using this result, it is observed in [5, Remark 1] that $[\mathscr{B}(\mathfrak{H}), \mathscr{C}_1] \neq \mathscr{C}_1^0$. Now we claim that for no Lie ideal $\mathscr{L}$ do we have $\mathscr{C}_1^0 = [\mathscr{B}(\mathfrak{H}), \mathscr{L}]$ or $\mathscr{C}_1^0 = [\mathscr{L}, \mathscr{L}]$. Since, if the associate ideal of $\mathscr{L}$ is $\mathscr{I}$, the associate ideals of $\mathscr{C}_1^0$, $[\mathscr{B}(\mathfrak{H}), \mathscr{L}]$ and $[\mathscr{L}, \mathscr{L}]$ are $\mathscr{C}_1$, $\mathscr{I}$ and $\mathscr{I}^2$ respectively, $\mathscr{C}_1^0 = [\mathscr{B}(\mathfrak{H}), \mathscr{L}]$ would imply that $\mathscr{I} = \mathscr{C}_1$ and $\mathscr{C}_1^0 = [\mathscr{L}, \mathscr{L}]$ would imply $\mathscr{I}^2 = \mathscr{C}_1$, i.e. $\mathscr{I} = \mathscr{C}_2$, both contradictory to the results previously mentioned. We do not know whether we can have $\mathscr{C}_1^0 = [\mathscr{I}, \mathscr{J}]$ for distinct ideals $\mathscr{I}$ and $\mathscr{J}$.

3. A characterization of operator ideals and its applications

We now turn our attention to the main theme of this paper: characterizations of operator ideals.

Proposition 8. *A linear manifold $\mathscr{L}$ in $\mathscr{B}(\mathfrak{H})$ is either an ideal or $\mathscr{I} + \mathbf{C}I$ for some ideal $\mathscr{I}$ if and only if $\mathscr{L}$ is unitarily invariant and $\mathscr{L}^c \subseteqq \mathscr{L}$.*

Proof. Suppose that $\mathscr{L}$ is a unitarily invariant manifold in $\mathscr{B}(\mathfrak{H})$ and $\mathscr{L}^c \subseteqq \mathscr{L}$. It follows from Theorem 1 that $\mathscr{L}$ is a Lie ideal and $\mathscr{L} \subseteqq \mathscr{I} + \mathbf{C}I$ where $\mathscr{I}$ is its associate ideal. From Proposition 5, we have $\mathscr{I} \subseteqq \mathscr{L}^c$. Now we have $\mathscr{I} \subseteqq \mathscr{L}^c \subseteqq \subseteqq \mathscr{L} \subseteqq \mathscr{I} + \mathbf{C}I$. Hence either $\mathscr{L} = \mathscr{I}$ or $\mathscr{L} = \mathscr{I} + \mathbf{C}I$.

For the proof of the "only if" part, it suffices to note that, if $\mathscr{I}$ is an ideal in $\mathscr{B}(\mathfrak{H})$, then $\mathscr{I}_{\mathfrak{H}\oplus\mathfrak{H}}$ consists of all 2×2 operator matrices with entries in $\mathscr{I}$. □

The following immediate consequence of the above proposition is a useful characterization of ideals in $\mathscr{B}(\mathfrak{H})$.

Proposition 9. *A linear manifold $\mathscr{L}$ in $\mathscr{B}(\mathfrak{H})$ is a proper ideal if and only if it is unitarily invariant, $\mathscr{L}^c \subseteq \mathscr{L}$ and $I \notin \mathscr{L}$.*

Remark. The "if" part of Proposition 9, under the additional assumption that $\mathscr{L}$ contains all Hilbert—Schmidt operators, was obtained by Sourour [16].

Next we give a few applications, labelled as examples, of the above two propositions. In many cases, it is convenient to think of $\mathscr{L}^c$ in the following way. Take any subspace $\mathfrak{M}$ in $\mathfrak{H}$ with $\dim \mathfrak{M} = \dim \mathfrak{M}^\perp (= \dim \mathfrak{H})$ and let

$$\mathscr{L}^{\mathfrak{M}} = \{\text{compression of } T \text{ to } \mathfrak{M}\colon\ T \in \mathscr{L}\}.$$

Then $\mathscr{L}^{\mathfrak{M}}$ is a unitarily invariant manifold and $(\mathscr{L}^{\mathfrak{M}})_{\mathfrak{H}} = \mathscr{L}^c$. Thus, roughly speaking, $\mathscr{L}^c$ can be obtained by taking the compression of $\mathscr{L}$ to a subspace $\mathfrak{M}$ with $\dim \mathfrak{M} = \dim \mathfrak{M}^\perp$ and then transporting it back to $\mathfrak{H}$.

Example 1. Let $\mathscr{S}$ be a linear manifold of numerical sequences converging to zero. We consider the set $\mathscr{I}$ of those operators T such that, for each orthonormal sequence $\{e_n\}$ in $\mathfrak{H}$, the sequence $\{(Te_n, e_n)\}_{n=1}^{\infty}$ is in $\mathscr{S}$. Then it is easy to see that $\mathscr{I}$ is a unitarily invariant manifold which does not contain I. By the obvious fact that an orthonormal sequence in a subspace is also an orthonormal sequence in the whole Hilbert space, one can see the validity of the inclusion $\mathscr{I}^c \subseteq \mathscr{I}$. By Proposition 9, it follows that $\mathscr{I}$ is an ideal.

If we take $\mathscr{S} = l^p$, the set of all numerical sequences $\{\lambda_j\}$ such that $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$, then the corresponding ideal $\mathscr{I}$ turns out to be the $\mathscr{C}^p$-class of operators. If π_j is a sequence of positive numbers decreasing to zero such that $\sum_{n=1}^{\infty} \pi_j = \infty$, and if $\mathscr{S}$ is the set of numerical sequences $\{\lambda_j\}$ satisfying $\sum_{j=1}^{\infty} \pi_j |\lambda_j| < \infty$, then the corresponding ideal $\mathscr{I}$ is σ_π which is defined in [7].

Example 2. An operator T in $\mathscr{B}(\mathfrak{H})$ (here $\mathfrak{H}$ is assumed to be separable) is said to be universally absolutely bounded if, for every orthonormal basis $\{e_n\}$ in $\mathfrak{H}$, the matrix

$$\begin{pmatrix} |(Te_1, e_1)| & |(Te_2, e_1)| & \dots \\ |(Te_1, e_2)| & |(Te_2, e_2)| & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

represents a bounded operator on l^2. Let $\mathscr{U}$ be the set of all universally bounded operators. Clearly $\mathscr{U}$ is a unitarily invariant manifold, $I\in\mathscr{U}$ and, for an infinite dimensional subspace $\mathfrak{M}$ of $\mathfrak{H}$, each operator in $\mathscr{U}^{\mathfrak{M}}$ is also universally absolutely bounded. Hence it follows from Proposition 8 that $\mathscr{U}=\mathscr{I}+\mathbf{C}I$ for some ideal $\mathscr{I}$. In fact, HALMOS and SUNDER [10] showed that $\mathscr{U}=\mathscr{C}_2+\mathbf{C}I$. Our discussion here can be used to shorten their proof.

Example 3. For $p>0$, let $\mathscr{U}_p$ be the set of all those operators $T\in\mathscr{B}(\mathfrak{H})$ ($\mathfrak{H}$ is separable) satisfying the condition that the matrix

$$\begin{pmatrix} |(Te_1, e_1)|^p & |(Te_2, e_1)|^p \dots \\ |(Te_1, e_2)|^p & |(Te_2, e_2)|^p \dots \\ \dots & \dots \quad \dots \end{pmatrix}$$

represents a bounded operator on l^2 for every orthonormal basis $\{e_n\}$ of $\mathfrak{H}$. From the inequality $(a+b)^p\leqq 2^p(a^p+b^p)$ $(a, b\geqq 0)$ we see that $\mathscr{U}_p$ is a linear manifold. It is easy to check that $\mathscr{U}_p$ is unitarily invariant, $I\in\mathscr{U}_p$ and $\mathscr{U}_p^c\subseteqq\mathscr{U}_p$. Hence, by Proposition 8, $\mathscr{U}_p=\mathscr{I}_p+\mathbf{C}I$ for some ideal $\mathscr{I}_p$. For $p\geqq 2$, it follows from a classical result of Schur (which says, for two $n\times n$ matrices (a_{kj}) and (b_{kj}), $\|(a_{kj}b_{kj})\|\leqq \leqq\|(a_{kj})\|\,\|(b_{kj})\|$; see [15]) that $\mathscr{U}_p=\mathscr{B}(\mathfrak{H})$. As we have mentioned in Example 2, $\mathscr{U}_1=\mathscr{C}_2+\mathbf{C}I$. We do not know how to describe $\mathscr{I}_p$ in an explicit way when $1<p<2$ or $0<p<1$.

Example 4. Let $\mathscr{T}$ be the set of all those operators in $\mathscr{B}(\mathfrak{H})$ ($\mathfrak{H}$ is separable) which, in any matrix representation, allow triangular truncation. More precisely, $T\in\mathscr{T}$ if and only if, for an arbitrary orthonornal basis $\{e_n\}$ in $\mathfrak{H}$, the triangular matrix

$$\begin{pmatrix} (Te_1, e_1) & (Te_2, e_1) & (Te_3, e_1)\dots \\ 0 & (Te_2, e_2) & (Te_3, e_2)\dots \\ 0 & 0 & (Te_3, e_3)\dots \\ & & \ddots \end{pmatrix}$$

represents a bounded operator on l^2. Then it is easy to see that $\mathscr{T}$ is a unitarily invariant manifold and $I\in\mathscr{T}$. A little reflexion on forming submatrices reveals that $\mathscr{T}^c\subseteqq\mathscr{T}$. Hence it follows from Proposition 8 that $\mathscr{T}=\mathscr{I}+\mathbf{C}I$ for some ideal $\mathscr{I}$. It follows from a result of MACAEV (see [7]) that $\mathscr{I}$ contains all those operator T with their s-numbers $\{s_n(\tau)\}$ satisfying $\sum_{n=1}^{\infty} n^{-1}s_n(T)<\infty$.

Example 5. Let (X, m) be a separable σ-finite measure space which is not purely atomic. We say that an operator T on $\mathfrak{H}=\mathscr{L}^2(X, m)$ is an integral operator if $Tx(s)=\int k(s, t)x(t)\,dm(t)$ a.e. $(x\in\mathfrak{H})$ for some measurable function k on $X\times X$. Proposition 8 can be used to give a simplified proof of a result due to KOROTKOV: if U^*TU is an integral operator for every unitary U, then $T\in\mathscr{C}_2+\mathbf{C}I$. For details, we refer to SOUROUR [16].

4. Characterization of ideals in certain classes of C^*-algebras

In the present section, we give some characterizations of ideals in certain general C^*-algebras which share some "noncommutative" features with $\mathscr{B}(\mathfrak{H})$.

For the next two results, we consider those unital C^*-algebras $\mathscr{A}$ which satisfy the following condition:

(C) Every unitary element in $\mathscr{A}$ can be expressed as a product of a scalar and several symmetries (i.e. hermitian unitaries) in $\mathscr{A}$.

That $\mathscr{B}(\mathfrak{H})$ satisfies condition (C) is a consequence of the following result of Halmos and Kakutani [9]:

Theorem 10. *Each operator on an infinite dimensional Hilbert space is a product of four symmetries.*

This result was generalized by Fillmore [3] to properly infinite von Neumann algebras. Note that if $\mathscr{A}$ is a commutative C^*-algebra and $\dim \mathscr{A} \geqq 2$, then condition (C) fails.

The notion of unitarily invariant manifolds in $\mathscr{B}(\mathfrak{H})$ can be extended to general C^*-algebras in a straightforward manner: in a C^*-algebra $\mathscr{A}$, a linear manifold $\mathscr{L}$ is unitarily invariant if and only if $U^*\mathscr{L}U \subseteqq \mathscr{L}$ for all unitary elements U in $\mathscr{A}$. The following result characterizes unitarily invariant manifolds in a C^*-algebra satisfying condition (C).

Proposition 11. *A linear manifold $\mathscr{L}$ in a C^*-algebra $\mathscr{A}$ satisfying* (C) *is unitarily invariant if and only if $(I-P)\mathscr{L}P \subseteqq \mathscr{L}$ for all projections P in $\mathscr{A}$.*

Proof. Suppose that $\mathscr{L}$ is unitarily invariant. Let P be a projection in $\mathscr{A}$ and $T \in \mathscr{L}$. Then both $U=I-2P$ and $V=P+i(I-P)$ are unitary and hence $T_1 \equiv \frac{1}{2}(T-U^*TU) \in \mathscr{L}$ and

$$(I-P)TP = \frac{1}{2}(T_1 - iV^*T_1V) \in \mathscr{L}.$$

Conversely, suppose that $(I-P)\mathscr{L}P \subseteqq \mathscr{L}$ for all projections P. Let S be a symmetry in $\mathscr{A}$. Then $S=2P-I$ for some projection P. Hence, for $T \in \mathscr{L}$,

$$STS = T - 2\big(PT(I-P) + (I-P)TP\big) \in \mathscr{L}.$$

By condition (C) we see that $\mathscr{L}$ is unitarily invariant, since $\mathscr{A}$ is linearly spanned by unitaries. □

Proposition 12. *A linear manifold $\mathscr{L}$ in a C^*-algebra $\mathscr{A}$ satisfying* (C) *is an ideal if and only if $P\mathscr{L}P \subseteqq \mathscr{L}$ for all projections P in $\mathscr{A}$.*

Proof. Suppose that $P\mathscr{L}P\subseteqq\mathscr{L}$ for all projections P. Let $T\in\mathscr{L}$ and S be a symmetry so that $S=2P-I$ for some projection P. Then

$$STS=2(PTP+(I-P)T(I-P))-T\in\mathscr{L}.$$

Hence, by condition (C), $\mathscr{L}$ is unitarily invariant. Therefore, by Proposition 11, $(I-P)\mathscr{L}P\subseteqq\mathscr{L}$ for all projections P in $\mathscr{A}$. Now, for a symmetry $S=2P-I$ and $T\in\mathscr{L}$, we have

$$TS=2(PTP+(I-P)TP)-T\in\mathscr{L}.$$

By condition (C) again, we have $TU\in\mathscr{L}$ for each $T\in\mathscr{L}$ and each unitary U. Since unitary elements span $\mathscr{A}$ linearly, we have $\mathscr{L}\mathscr{A}\subseteqq\mathscr{L}$. In the same way we can show that $\mathscr{A}\mathscr{L}\subseteqq\mathscr{L}$. Hence $\mathscr{L}$ is an ideal of $\mathscr{A}$. □

A linear manifold $\mathscr{L}$ in a C^*-algebra $\mathscr{A}$ is said to be a *Jordan ideal* if $AX+XA\in\mathscr{L}$ for all $A\in\mathscr{L}$ and $X\in\mathscr{A}$. It is shown in [5, Theorem 3] that Jordan ideals in $\mathscr{B}(\mathfrak{H})$ are just associative ideals. This result can be generalized for a class of C^*-algebras wider than $\mathscr{B}(\mathfrak{H})$:

Corollary 13. *If $\mathscr{L}$ is a Jordan ideal in a C^*-algebra $\mathscr{A}$ which satisfies condition* (C), *then $\mathscr{L}$ is an associative ideal.*

Proof. Let P be a projection in $\mathscr{A}$ and $T\in\mathscr{L}$. Then

$$P(PT+TP)+(PT+TP)P=2PTP+(PT+TP)\in\mathscr{L}$$

and hence $PTP\in\mathscr{L}$. Now the corollary follows from Proposition 12. □

Sourour has informed the authors that, in case $\mathscr{A}=\mathscr{B}(\mathfrak{H})$, Proposition 12 can be deduced in the following way. Assume that $\mathscr{L}$ is a linear manifold such that $P\mathscr{L}P\subseteqq\mathscr{L}$ for all projections P. For $T\in\mathscr{L}$ and a projection P we have $TP+PT=T+PTP-(I-P)T(I-P)\in\mathscr{L}$. By the fact that projections span $\mathscr{B}(\mathfrak{H})$ linearly (Theorem 2), we see that $\mathscr{L}$ is a Jordan ideal. Now it follows from [5, Theorem 3] that $\mathscr{L}$ is an associative ideal.

The condition (C) in Proposition 12 is essential. For example, if $\mathscr{A}=C[0, 1]$, then there is no proper projection in $\mathscr{A}$ and hence the inclusion $P\mathscr{L}P\subseteqq\mathscr{L}$ is automatically satisfied for every linear manifold $\mathscr{L}$ in $\mathscr{A}$; but of course there are linear manifolds in $\mathscr{A}$ which are not ideals.

In the next result, we let $\mathscr{B}$ be a C^*-algebra with the identity I, $\mathscr{A}=\mathscr{M}_2(\mathscr{B})$ and P_0 be the projection in $\mathscr{A}$ given by the matrix $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$.

Proposition 14. *A linear manifold $\mathscr{L}$ in $\mathscr{A}$ is an ideal if and only if $\mathscr{L}$ is unitarily invariant and $P_0\mathscr{L}P_0\subseteqq\mathscr{L}$.*

Proof. Suppose $\mathscr{L}$ is unitarily invariant and $P_0\mathscr{L}P_0\subseteq\mathscr{L}$. Let

$$\mathscr{I}=\left\{B\in\mathscr{B}:\begin{pmatrix}0&B\\0&0\end{pmatrix}\in\mathscr{L}\right\}.$$

If U is a unitary element in $\mathscr{B}$ and $B\in\mathscr{I}$, then

$$\begin{pmatrix}I&0\\0&U\end{pmatrix}^*\begin{pmatrix}0&B\\0&0\end{pmatrix}\begin{pmatrix}I&0\\0&U\end{pmatrix}=\begin{pmatrix}0&BU\\0&0\end{pmatrix}\in\mathscr{L},$$

$$\begin{pmatrix}U&0\\0&I\end{pmatrix}\begin{pmatrix}0&B\\0&0\end{pmatrix}\begin{pmatrix}U&0\\0&I\end{pmatrix}^*=\begin{pmatrix}0&UB\\0&0\end{pmatrix}\in\mathscr{L},$$

and hence BU and UB are in $\mathscr{I}$. Since unitary elements in $\mathscr{B}$ span the whole algebra $\mathscr{B}$, we see that $\mathscr{I}$ is an ideal in $\mathscr{B}$. Now let $\mathscr{J}$ be the set of all 2×2 matrices with entries in $\mathscr{I}$. Then $\mathscr{J}$ is an ideal in $\mathscr{A}$.

Let $T=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ be an element in $\mathscr{L}$. We are going to show that $T\in\mathscr{J}$. For this purpose, we introduce the following unitary elements in $\mathscr{A}$:

$$U=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad V=\begin{pmatrix}I&0\\0&iI\end{pmatrix},\quad J=\begin{pmatrix}0&I\\I&0\end{pmatrix},\quad W=\frac{1}{\sqrt{2}}\begin{pmatrix}I&I\\-I&I\end{pmatrix}.$$

Then we have

$$S\equiv\frac{1}{2}(T-U^*TU)=\begin{pmatrix}0&B\\C&0\end{pmatrix}\in\mathscr{L},\quad \frac{1}{2}(S-iV^*SV)=\begin{pmatrix}0&B\\0&0\end{pmatrix}\in\mathscr{L}$$

and

$$\frac{1}{2}J(S+iV^*SV)J=\begin{pmatrix}0&C\\0&0\end{pmatrix}\in\mathscr{L}.$$

Hence, $B, C\in\mathscr{I}$. We also have

$$2W^*P_0TP_0W=\begin{pmatrix}A&A\\A&A\end{pmatrix}\in\mathscr{L},\quad 2W^*P_0(JTJ)P_0W=\begin{pmatrix}D&D\\D&D\end{pmatrix}\in\mathscr{L}.$$

By the previous argument, we have $A, D\in\mathscr{I}$.

Next we show that $\mathscr{J}\subseteq\mathscr{L}$ and let $T=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathscr{I}$. From the definition of $\mathscr{J}$ we know A, B, C and D are in $\mathscr{I}$, or, in other words,

$$S_1=\begin{pmatrix}0&A\\0&0\end{pmatrix},\quad S_2=\begin{pmatrix}0&B\\0&0\end{pmatrix},\quad S_3=\begin{pmatrix}0&C\\0&0\end{pmatrix},\quad S_4=\begin{pmatrix}0&D\\0&0\end{pmatrix}$$

are in $\mathscr{L}$. We have to show that

$$T_1=\begin{pmatrix}A&0\\0&0\end{pmatrix},\quad T_2=\begin{pmatrix}0&B\\0&0\end{pmatrix},\quad T_3=\begin{pmatrix}0&0\\C&0\end{pmatrix},\quad T_4=\begin{pmatrix}0&0\\0&D\end{pmatrix}$$

are in $\mathscr{L}$. This can be seen from the following identities:

$$T_1=2P_0(WS_1W^*)P_0,\quad T_2=S_2,\quad T_3=JS_3J,\quad T_4=2J(P_0WS_4W^*P_0)J,$$

where J and W are the unitary operators previously defined. □

Ideas similar to those in the above proof appear in [12].

Corollary 15. *If P_0 is a projection in $\mathscr{B}(\mathfrak{H})$ with* $\dim P_0\mathfrak{H}=\dim(I-P_0)\mathfrak{H}$ *and if $\mathscr{L}$ is a unitarily invariant manifold in $\mathscr{B}(\mathfrak{H})$ satisfying $P_0\mathscr{L}P_0\subseteqq\mathscr{L}$, then $\mathscr{L}$ is an ideal in $\mathscr{B}(\mathfrak{H})$.*

Proof. This follows from the previous proposition and the fact that $\mathscr{B}(\mathfrak{H})$ and $\mathscr{M}_2(\mathscr{B}(\mathfrak{H}))$ are isomorphic C^*-algebras. □

Example. Let $\mathscr{L}\subseteqq\mathscr{K}(\mathfrak{H})$ be an ideal in $\mathscr{K}(\mathfrak{H})$, i.e., for $X\in\mathscr{K}(\mathfrak{H})$ and $A\in\mathscr{L}$, we have $XA\in\mathscr{L}$ and $AX\in\mathscr{L}$. In general, $\mathscr{L}$ is not necessarily an ideal in $\mathscr{B}(\mathfrak{H})$. Among other things, it was shown in [6] that if $\mathscr{L}$ is also a Lie ideal and $\mathscr{L}$ is countably generated as an ideal of $\mathscr{K}(\mathfrak{H})$, then $\mathscr{L}$ is also an ideal of $\mathscr{B}(\mathfrak{H})$. This result can be proved in the following alternative way.

Let $\mathscr{I}$ be the linear span of operators of the form XAY, where $A\in\mathscr{S}$ and $X, Y\in\mathscr{K}(\mathfrak{H})$. It is easy to see that $\mathscr{I}$ is an ideal in $\mathscr{B}(\mathfrak{H})$ and $\mathscr{I}\subseteqq\mathscr{L}$. On the other hand, it follows from a lemma in [6] that there is a projection P_0 in $\mathscr{B}(\mathfrak{H})$ such that $\dim P_0\mathfrak{H}=\dim(I-P_0)\mathfrak{H}$ and P_0S, $SP_0\in\mathscr{I}$ for all S in $\mathscr{S}$. Notice that each element S in $\mathscr{L}$ can be expressed as a finite sum:

$$S = D+\Sigma_j(\alpha_jA_j+B_jX_j+Y_jC_j)$$

where $\alpha_j\in\mathbf{C}$; $A_j, B_j, C_j\in\mathscr{S}$; $X_j, Y_j\in\mathscr{K}(\mathfrak{H})$ and $D\in\mathscr{I}$. For such a sum, we have

$$P_0SP_0 = P_0DP_0+\Sigma_j\big(\alpha_jP_0(A_jP_0)+(P_0B_j)XP_0+P_0Y_j(C_jP_0)\big)\in\mathscr{I}$$

since $\mathscr{I}$ is an ideal in $\mathscr{B}(\mathfrak{H})$ and the operators D, A_jP_0, P_0B_j, C_jP_0 are all in $\mathscr{I}$. Now it follows from Corollary 15 and $\mathscr{I}\subseteqq\mathscr{L}$ that $\mathscr{L}$ is an ideal in $\mathscr{B}(\mathfrak{H})$.

Finally, we have the following characterization of ideals in $\mathscr{B}(\mathfrak{H})$.

Proposition 16. *If $\mathscr{L}$ is a unitarily invariant manifold in $\mathscr{B}(\mathfrak{H})$ consisting of compact operators and if $T\in\mathscr{L}$ implies $|T|=(T^*T)^{1/2}\in\mathscr{L}$, then $\mathscr{L}$ is an ideal.*

Proof. Again, let $\mathscr{I}$ be the ideal of those operators B such that $\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}}$. Then, from Theorem 1, we have $\mathscr{L}\subseteqq\mathscr{I}$. Next we show that $\mathscr{I}\subseteqq\mathscr{L}$. Since every ideal in $\mathscr{B}(\mathfrak{H})$ is linearly spanned by its positive elements, it suffices to show that positive elements in $\mathscr{I}_{\mathfrak{H}\oplus\mathfrak{H}}$ are in $\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}}$. So let $T=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ be a positive element in $\mathscr{I}_{\mathfrak{H}\oplus\mathfrak{H}}$. Then A, B, C and D are in $\mathscr{I}$. Hence

$$\begin{pmatrix}0 & B\\ C & 0\end{pmatrix} = \begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & C\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}},$$

$$\begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix} = \left|\begin{pmatrix}0 & D\\ 0 & 0\end{pmatrix}\right|\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}},\quad \begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix} = \left|\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\right|\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{H}}.$$

Hence we obtain $T\in\mathscr{L}_{\mathfrak{H}\oplus\mathfrak{S}}$. □

Appendix

In this appendix we give a transparent proof of Theorem 1 and Theorem 2 based on an idea in [4]. The main tool we use in this proof is Halmos—Kakutani's Theorem (Theorem 10): every unitary operator can be expressed as a product of not more than four symmetries. This theorem can be deduced constructively by the following three short steps: first express it as a direct sum of countably many blocks such that each block has the same dimension as the Hilbert space; then, using this expression, write the operator as a product of two bilateral shifts (of infinite rank); finally, write each bilateral shift as a product of two symmetries. (For details, we refer to [9].) From this argument we see that the symmetries involved can be chosen in such a way that their eigen-subspaces have the same dimension as the underlying Hilbert space.

In order to reveal the essential part of our argument in proving Theorem 1, we consider a more general situation. We let $\mathscr{B}$ be a unital C^*-algebra, $\mathscr{A}=\mathscr{M}_2(\mathscr{B})$ and $\mathscr{S}$ be the set of all those symmetries of the form $U^*\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}U$, where I is the identity in $\mathscr{B}$ and U is a unitary element in $\mathscr{A}$. We consider the following condition:

(C′) each unitary element in $\mathscr{A}$ is a product of finitely many elements in $\mathscr{S}$ and a scalar.

It follows from our previous remark that for $\mathscr{A}=\mathscr{B}(\mathfrak{H})$, condition (C′) is satisfied.

In the following three lemmas, we always assume that $\mathscr{A}$ is the C^*-algebra described in the previous paragraph and condition (C′) is satisfied. Furthermore, we assume that $\mathscr{L}$ is a unitarily invariant manifold in $\mathscr{A}$,

$$\mathscr{J}=\left\{B\in\mathscr{B}:\ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\in\mathscr{L}\right\}$$

and

$$\mathscr{I}=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathscr{A}:\ A, B, C \text{ and } D \text{ are in } \mathscr{J}\right\}.$$

By using the same argument as that in the proof of Proposition 14, we see that $\mathscr{J}$ is an ideal in $\mathscr{B}$ and $\mathscr{I}$ is an ideal in $\mathscr{A}$.

Lemma A. *With the above assumption, we have* $[\mathscr{I}, \mathscr{A}]\subseteq\mathscr{L}$.

Proof. It suffices to show that $[\mathscr{I}, U]\subseteq\mathscr{L}$ for all unitary elements U in $\mathscr{A}$. First we note that if $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathscr{I}$, then

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\right]=2\begin{pmatrix} 0 & -B \\ C & 0 \end{pmatrix}=2\left(\begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\in\mathscr{L}$$

since $\mathscr{L}$ is unitary and $B, C \in \mathscr{J}$. Now if $T \in \mathscr{I}$ and $S = W^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} W \in \mathscr{S}$ (W is unitary), then

$$[T, S] = W^* \left[WTW^*, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right] W \in \mathscr{L}.$$

Now we consider an arbitrary unitary element U in $\mathscr{A}$ and show that $[T, U] \in \mathscr{L}$. By condition (C′), U can be written as a product $\lambda S_1 S_2 \dots S_n$ where $S_1, \dots, S_n \in \mathscr{S}$ and $\lambda \in \mathbf{C}$. We proceed by induction on n. Let $V = S_2 S_3 \dots S_n$. Then

$$[T, U] = \lambda [TS_1, V] + \lambda [VT, S_1].$$

Since $VT \in \mathscr{I}$ and $TS_1 \in \mathscr{I}$, we have $[VT, S_1] \in \mathscr{L}$ by our previous argument and $[TS_1, V] \in \mathscr{L}$ by our induction assumption. Therefore $[T, U] \in \mathscr{L}$. □

Lemma B. *The linear span of $\mathscr{S}$ includes $[\mathscr{A}, \mathscr{A}]$.*

Proof. Let $\mathscr{L}_0$ be the linear span of $\mathscr{S}$. Then $\mathscr{L}_0$ is unitarily invariant. Let $\mathscr{J}_0$ and $\mathscr{I}_0$ be the ideals defined from $\mathscr{L}_0$ in the same way as $\mathscr{J}, \mathscr{I}$ defined from $\mathscr{L}$. Since $\frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \in \mathscr{L}_0$, we see that $I \in \mathscr{J}_0$ and hence $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \in \mathscr{I}_0$. Therefore $\mathscr{I}_0 = \mathscr{A}$. Thus, by Lemma A, $[\mathscr{A}, \mathscr{A}] = [\mathscr{A}, \mathscr{I}_0] \subset \mathscr{L}_0$. □

Lemma C. $[\mathscr{L}, [\mathscr{A}, \mathscr{A}]] \subseteqq \mathscr{I} \cap \mathscr{L}$.

Proof. It follows from Lemma B that it suffices to show $[\mathscr{L}, \mathscr{S}] \subseteqq \mathscr{I} \cap \mathscr{L}$. If $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathscr{L}$, then, by an argument similar to that in the proof of Proposition 14, we can show that $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ are in $\mathscr{I} \cap \mathscr{L}$ and hence

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right] = 2 \begin{pmatrix} 0 & -B \\ C & 0 \end{pmatrix} \in \mathscr{I} \cap \mathscr{L}.$$

If $S = W^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} W \in \mathscr{S}$, where W is unitary, and if $T \in \mathscr{L}$, then

$$[T, S] = W^* \left[WTW^*, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right] W \in \mathscr{I} \cap \mathscr{L}. \quad \square$$

Proof of Theorem 2. Apply Lemma B to the case $\mathscr{A} = \mathscr{B}(\mathfrak{H})$ and note that $[\mathscr{B}(\mathfrak{H}), \mathscr{B}(\mathfrak{H})] = \mathscr{B}(\mathfrak{H})$, (see [8]). □

Proof of Theorem 1. By Halmos—Kakutani's Theorem and Theorem 2, we can easily deduce the equivalence of (1) and (2). (For details, see [5].) That (3) implies (1) is obvious. It remains to show (1)⇒(3). By Lemma A, Lemma C and the fact that $[\mathscr{B}(\mathfrak{H}), \mathscr{B}(\mathfrak{H})] = \mathscr{B}(\mathfrak{H})$, we have $[\mathscr{B}(\mathfrak{H}), \mathscr{I}] \subseteqq \mathscr{L}$ and $[\mathscr{B}(\mathfrak{H}), \mathscr{L}] \subseteqq \mathscr{I}$.

Now Theorem 1 follows from the following theorem of CALKIN [1]:

Theorem D. *If $\mathcal{I}$ is a proper ideal in $\mathcal{B}(\mathfrak{H})$, $T\in\mathcal{B}(\mathfrak{H})$ and if $[T,\mathcal{B}(\mathfrak{H})]\subseteq\mathcal{I}$, then $T\in\mathcal{I}+\mathbf{C}I$.*

Calkin only showed this theorem for the case when $\mathfrak{H}$ is separable. Now we prove this theorem under the assumption that $\mathfrak{H}$ is nonseparable.

Let $\mathcal{M}(\mathfrak{H})$ be the unique maximal ideal in $\mathcal{B}(\mathfrak{H})$. (Thus, for an operator S on $\mathfrak{H}$, $S\in\mathcal{M}(\mathfrak{H})$ if and only if there exists a projection E in $\mathcal{B}(\mathfrak{H})$ such that $ESE=S$ and $\dim E\mathfrak{H}<\dim\mathfrak{H}$.) Let $\mathcal{C}(\mathfrak{H})$ be the "Calkin algebra" $\mathcal{B}(\mathfrak{H})/\mathcal{M}(\mathfrak{H})$. Let t be the canonicali mage of T in $\mathcal{C}(\mathfrak{H})$.

Since $\mathcal{I}$ is self-adjoint, with no loss of generality, we may assume $T=T^*$. Let $T=\int\lambda\, dE_\lambda$ be the spectral decomposition of T. Note that $\lambda\in\sigma(t)$ if and only if, for all $\varepsilon>0$, $\dim E(\lambda-\varepsilon,\lambda+\varepsilon)\mathfrak{H}=\dim\mathfrak{H}$.

First we demonstrate that $\sigma(t)$ is a singleton. Assume the contrary: we have $\lambda_1,\lambda_2\in\sigma(t)$ with $\lambda_1\neq\lambda_2$. Choose $\varepsilon>0$ such that the intervals $[\lambda_1-\varepsilon,\lambda_1+\varepsilon]$ and $[\lambda_2-\varepsilon,\lambda_2+\varepsilon]$ are disjoint. Let $\mathfrak{H}_j=E[\lambda_j-\varepsilon,\lambda_j+\varepsilon]\mathfrak{H}$ $(j=1,2)$ and $\mathfrak{K}=\mathfrak{H}\ominus(\mathfrak{H}_1\oplus\mathfrak{H}_2)$. Then $\dim\mathfrak{H}_1=\dim\mathfrak{H}_2=\dim\mathfrak{H}$. Let U be a unitary transformation from $\mathfrak{H}$ onto $\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{K}$ such that $U\mathfrak{H}_1=\mathfrak{H}\oplus0\oplus0$, $U\mathfrak{H}_2=0\oplus\mathfrak{H}\oplus0$ and $U\mathfrak{K}=0\oplus0\oplus\mathfrak{K}$. Then

$$U^{-1}TU=\left(\begin{array}{cc|c} T_1 & 0 & \\ 0 & T_2 & \\ \hline & & * \end{array}\right)$$

for some hermitian operators T_1 and T_2 in $\mathcal{B}(\mathfrak{H})$ with disjoint spectra. By a well-known result of ROSENBLUM [14], there exists an operator A in $\mathcal{B}(\mathfrak{H})$ such that $T_1A-AT_2=I$. Let

$$X=U\left(\begin{array}{cc|c} 0 & A & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right)U^{-1}.$$

Then

$$TX-XT=U\left(\begin{array}{cc|c} 0 & I & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right)U^{-1}\in\mathcal{I}$$

and hence $I\in\mathcal{I}$. Therefore $\mathcal{I}=\mathcal{B}(\mathfrak{H})$, a contradiction to our assumption that $\mathcal{I}$ is proper.

We have $T=S+\lambda I$ for some $\lambda\in\mathbf{C}$ and self-adjoint operator S in $\mathcal{M}(\mathfrak{H})$. Choose a projection E in $\mathcal{B}(\mathfrak{H})$ such that $ESE=S$ and $\dim E\mathfrak{H}=\dim(I-E)\mathfrak{H}$. Let W be a unitary transformation from $\mathfrak{H}$ onto $\mathfrak{H}\oplus\mathfrak{H}$ such that $W(E\mathfrak{H})=\mathfrak{H}\oplus0$ and $W((I-E)\mathfrak{H})=0\oplus\mathfrak{H}$. Then $WSW^{-1}=\begin{pmatrix} S_0 & 0\\ 0 & 0\end{pmatrix}$ for some $S_0\in\mathcal{B}(\mathfrak{H})$. Let

$V=W^{-1}\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}W$. Then we have

$$SV-VS=W^{-1}\begin{pmatrix}0 & S_0\\ 0 & 0\end{pmatrix}W\in\mathscr{I}$$

and hence $S_0\in\mathscr{I}$. Therefore $S\in\mathscr{I}$. □

By using the above three lemmas, Fillmore's extension [3] of Halmos and Kakutani's Theorem and the fact that $[\mathscr{A},\mathscr{A}]=\mathscr{A}$ for a properly infinite von Neumann algebra [13], we can show the following two results.

Theorem 1′. *Let $\mathscr{L}$ be a linear manifold in a properly infinite von Neumann algebra. Then the following conditions are equivalent:*

(1) *$\mathscr{L}$ is unitarily invariant;*
(2) *$\mathscr{L}$ is a Lie ideal in $\mathscr{A}$, i.e., $[\mathscr{L},\mathscr{A}]\subseteqq\mathscr{L}$;*
(3) *there is an ideal $\mathscr{I}$ in $\mathscr{A}$ such that $[\mathscr{A},\mathscr{I}]\subseteqq\mathscr{L}$ and $[\mathscr{A},\mathscr{L}]\subseteqq\mathscr{I}$.*

Theorem 2′ [13]. *Every element in a properly infinite von Neumann algebra is a linear combination of projections.*

As in Section 2, in a properly infinite von Neumann algebra, we can define the associate ideals of Lie ideals. Also we can show that conditions (1), (3), (4), (6) in Proposition 3 are equivalent in this general situation.

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