Acta Sci. Math., 51 (1987), 457—460

A note on Schmüdgen's classes $\mathfrak{N}_1$ and $\mathfrak{N}_\infty^\infty$ of pairs generated by Toeplitz operators

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1. K. SCHMÜDGEN [1] introduced the following class of pairs of (unbounded) self-adjoint operators.

Definition 1. Let A, B be self-adjoint operators on a Hilbert space $\mathscr{H}$. The pair $\{A, B\}$ belongs to the class $\mathfrak{N}_1$ if there exists a dense linear manifold $\mathscr{D}$ in $\mathscr{H}$ such that

(i) $\mathscr{D} \subseteq \mathrm{Dom}\,(AB) \cap \mathrm{Dom}\,(BA)$ and $ABf = BAf$ for all $f \in \mathscr{D}$,

(ii) $A|\mathscr{D}$ and $B|\mathscr{D}$ are essentially self-adjoint.

Schmüdgen gives the following criterion for a pair $\{A, B\}$ to be in $\mathfrak{N}_1$. (In what follows $\mathscr{R}(\cdot)$ means "range of".)

Theorem 0 (Theorem 1.7 in [1]). *Suppose $\{A, B\} \in \mathfrak{N}_1$, $\alpha \in \mathbf{R} \setminus \sigma(A)$ and $\beta \in \mathbf{R} \setminus \sigma(B)$. Then the operators $X \overset{\mathrm{def}}{=} (A-\alpha)^{-1}$ and $Y \overset{\mathrm{def}}{=} (B-\beta)^{-1}$ satisfy the following conditions:*

$$\mathrm{Ker}\, X = \mathrm{Ker}\, Y = \{0\}, \tag{1}$$

$$\overline{\mathscr{R}([X, Y])} \cap \mathscr{R}(X) = \overline{\mathscr{R}([X, Y])} \cap \mathscr{R}(Y) = \{0\}. \tag{2}$$

Conversely, if X and Y are bounded self-adjoint operators satisfying (1) *and* (2), *then $\{X^{-1}+\alpha, Y^{-1}+\beta\} \in \mathfrak{N}_1$ for all $\alpha, \beta \in \mathbf{R}$.*

The main method in [1] to construct pairs belonging to $\mathfrak{N}_1$ is to consider pairs of the form $\{(\mathrm{Re}\, T)^{-1}, (\mathrm{Im}\, T)^{-1}\}$ for certain operators T. Among others Toeplitz operators with analytic symbols have been investigated in [1]. It was shown that Toeplitz operators with symbols which are cyclic for the backward shift do not generate a pair in $\mathfrak{N}_1$ ([1], Proposition 3.3). Moreover, the polynomials φ for which $\{(\mathrm{Re}\, T_\varphi)^{-1}, (\mathrm{Im}\, T_\varphi)^{-1}\} \in \mathfrak{N}_1$ are characterized in [1].

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*) Research supported by Naturwissenschaftlich-Theoretisches Zentrum (Karl-Marx-Universität, Leipzig).

Received October 16, 1984.

The aim of this note is to show that Schmüdgen's method works in fact for Toeplitz operators with arbitrary analytic (or antianalytic) symbols.

Suppose $\varphi\in H^\infty$. Let T_φ be the multiplication by φ on H^2. Let $X \overset{\text{def}}{=} \operatorname{Re} T_\varphi$ and $Y \overset{\text{def}}{=} \operatorname{Im} T_\varphi$. As usual, S^* is the backward shift, P_+ is the orthogonal projection of L^2 onto H^2 and $P_-=I-P_+$ is the projection onto H^2_- and $\vee\{...\}$ denotes the closed linear span of $\{...\}$.

Lemma 1. $\mathscr{R}([X, Y])=\vee\{(S^*)^n\varphi\colon n\geqq 1\}$.

Proof. First note that, for any $h\in H^2$, $[T_\varphi^*, T_\varphi]h=(P_+\bar\varphi\varphi-\varphi P_+\bar\varphi)h=P_+\varphi P_-\bar\varphi h=H_{\bar\varphi}^* H_{\bar\varphi}h$, where $H_{\bar\varphi}\colon H^2\to H^2_-$ $(H_{\bar\varphi}h=P_-\bar\varphi h)$ is the Hankel operator with symbol $\bar\varphi$. Hence we have $\overline{\mathscr{R}([X,Y])}=\overline{\mathscr{R}([T_\varphi^*, T_\varphi])}=\overline{\mathscr{R}(H_{\bar\varphi}^* H_{\bar\varphi})}=\overline{\mathscr{R}(H_{\bar\varphi}^*)}=\overline{P_+\varphi H^2_-}=\vee\{P_+\bar z^n\varphi\colon n\geqq 1\}$. Now the assertion follows.

According to Beurling's theorem, the S^*-invariant subspace $\vee\{(S^*)^n\varphi\colon n\geqq 1\}$ has the form $H^2\ominus\Theta H^2$ with a certain inner function Θ or $\Theta=0$. We introduce the bounded analytic functions φ_+ and φ_- by

$$\varphi_\pm(z)=\frac{1}{2}\,\Theta(z)\big(\varphi(z)\pm\overline{\varphi(z)}\big)\quad\text{for}\quad |z|=1.$$

$\Theta\bar\varphi$ is indeed analytic, because $(\bar\varphi\Theta, \bar z^n)=(\Theta, (S^*)^n\varphi)=0$ for $n\geqq 1$.

Theorem 1. $\{(\operatorname{Re} T_\varphi)^{-1}, (\operatorname{Im} T_\varphi)^{-1}\}\in\mathfrak{N}_1$ *if and only if* φ_+ *and* φ_- *are non-zero outer functions.*

Proof. Let us note at first that for $\Theta=0$ we have by Lemma 1 $\overline{\mathscr{R}([X,Y])}=H^2$, i.e. condition (2) in Theorem 0 is not fulfilled. Hence we may assume that Θ is a non-zero function. Since the only bounded self-adjoint Toeplitz operator with non-trivial kernel is the zero operator, the conditions $X\neq 0$ and $Y\neq 0$ imply $\operatorname{Ker} X=\operatorname{Ker} Y=\{0\}$, i.e. condition (1) in Theorem 0.

We show that $(H^2\ominus\Theta H^2)\cap\mathscr{R}(X)=\{0\}$ iff φ_+ is outer. Since $Xf=\frac{1}{2}P_+(\varphi+\bar\varphi)f=P_+\Theta\varphi_+ f$, we have

$$P_+\bar\Theta Xf=P_+\bar\Theta P_+\Theta\varphi_+ f=P_+\bar\varphi_+ f=T_{\varphi_+}^* f.$$

Therefore,

$$(H^2\ominus\Theta H^2)\cap\mathscr{R}(X)=\{Xf\colon Xf\perp\Theta H^2\}=\{Xf\colon P_+\bar\Theta Xf=0\}=$$
$$=X\operatorname{Ker} T_{\varphi_+}^*=X\big(H^2\ominus\overline{\mathscr{R}(T_{\varphi_+})}\big)=X(H^2\ominus\varphi_+^i H^2),$$

where φ_+^i is the inner part of φ_+. Since $\operatorname{Ker} X=\{0\}$, $(H^2\ominus\Theta H^2)\cap\mathscr{R}(X)=\{0\}$ if and only if φ_+ is outer. Similarly it follows that $(H^2\ominus\Theta H^2)\cap\mathscr{R}(Y)=\{0\}$ if and only if φ_- is outer. By Theorem 0, this completes the proof of Theorem 1.

Corollary 1. *If* $\varphi\in H^\infty$ *is* S^**-cyclic, then* $\{(\operatorname{Re} T_\varphi)^{-1}, (\operatorname{Im} T_\varphi)^{-1}\}\notin\mathfrak{N}_1$.

Proof. Note that φ and $S^*\varphi$ are S^*-cyclic simultaneously. In this case $\Theta=0$ and $\varphi_+=\varphi_-=0$.

Lemma 2. *If* $\vee\{(S^*)^n\varphi:\ n\geqq 1\}=H^2\ominus\Theta H^2$, *then* $\Theta\bar{\varphi}$ *and* Θ *have no common inner divisor.*

Proof. Let ϑ be a common inner divisor of $\Theta\bar{\varphi}$ and Θ and let $\Theta'\overset{\text{def}}{=}\Theta\bar{\vartheta}$. Then $\Theta'\bar{\varphi}\in H^2$ and $((S^*)^n\varphi,\Theta' f)=(\bar{z}^n\bar{f},\Theta'\bar{\varphi})=0$ for $n\geqq 1$ and $f\in H^2$. Therefore, $\Theta' H^2\subseteqq \Theta H^2$, i.e., $\bar{\vartheta}=\bar{\Theta}\Theta'\in H^2$ and ϑ is a constant function.

If Θ is a finite Blaschke product, then φ is meromorphic in $\bar{\mathbf{C}}$, the function $\bar{\bar{\varphi}}$ defined by $\bar{\bar{\varphi}}(z)=\overline{\varphi(1/\bar{z})}$ is meromorphic too, and $\varphi_\pm(z)=\frac{1}{2}\,\Theta(z)\big(\varphi(z)\pm\bar{\bar{\varphi}}(z)\big)$ for $|z|\leqq 1$.

Corollary 2. *Let Θ be a finite Blaschke product. Then,* $\{(\operatorname{Re} T_\varphi)^{-1},(\operatorname{Im} T_\varphi)^{-1}\}\in\mathfrak{N}_1$ *if and only if* $\varphi^2(z)\neq\bar{\bar{\varphi}}^2(z)$ *for every* $z\in\mathbf{C}$, $|z|\neq 1$.

Proof. Suppose that $\varphi^2(z)=\bar{\bar{\varphi}}^2(z)$ for some $z\in\mathbf{C}$, $|z|\neq 1$. Since $\varphi^2(1/\bar{z})=\overline{\bar{\bar{\varphi}}^2(z)}=\overline{\varphi^2(z)}=\bar{\bar{\varphi}}^2(1/\bar{z})$, we can assume without loss of generality that $|z|<1$. Hence $\varphi_+\varphi_-$ has a zero inside the unit circle. Therefore it is not outer.

Suppose now that φ_+ (or φ_-) is not outer. Then it has a zero, say z_0, inside the unit circle (see the remark just before Corollary 2). According to Lemma 2, $\Theta(z_0)\neq 0$ and therefore $\varphi(z_0)+\bar{\bar{\varphi}}(z_0)=0$ (or $\varphi(z_0)-\bar{\bar{\varphi}}(z_0)=0$, resp.), i.e. $\varphi^2(z_0)=\bar{\bar{\varphi}}^2(z_0)$.

2. In [2] the study of commuting unbounded self-adjoint operators was continued. The more general classes $\mathfrak{N}_{rs}$ are introduced in [2]. Here we only need the class $\mathfrak{N}_\infty^\infty$.

Definition 2. Let A, B be self-adjoint operators on a Hilbert space $\mathscr{H}$. The pair $\{A,B\}$ is in the class $\mathfrak{N}_\infty^\infty$ if there exists a dense linear manifold $\mathscr{D}$ in $\mathscr{H}$ such that

(i)′ $\mathscr{D}\subseteqq\operatorname{Dom}(A^jB^k)\cap\operatorname{Dom}(B^kA^j)$ and $A^jB^kf=B^kA^jf$ for all $f\in\mathscr{D}$ and all $j,k=0,1,\dots$;

(ii)′ $A^k|\mathscr{D}$ and $B^k|\mathscr{D}$ are essentially self-adjoint for all $k\geqq 1$.

For polynomial symbols it was shown in [2, Theorem 4.1] that all pairs $\{(\operatorname{Re} T_\varphi)^{-1},(\operatorname{Im} T_\varphi)^{-1}\}\in\mathfrak{N}_1$ are in fact in the class $\mathfrak{N}_\infty^\infty$. Using the same method as in [2] we prove this assertion for arbitrary analytic symbols.

Theorem 2. *For arbitrary* $\varphi\in H^\infty$ *the following are equivalent:*

$$\{(\operatorname{Re} T_\varphi)^{-1},\ (\operatorname{Im} T_\varphi)^{-1}\}\in\mathfrak{N}_1,\tag{3}$$

$$\{\operatorname{Re} T_\varphi)^{-1},\ (\operatorname{Im} T_\varphi)^{-1}\}\in\mathfrak{N}_\infty^\infty.\tag{4}$$

Lemma 3. $Q_{rs} \overset{\text{def}}{=} \bigvee\{\mathscr{R}(X^j Y^k[X, Y]): j<r,\ k<s\} = H^2 \ominus \Theta^{r+s-1} H^2$.

Proof. Since the subspace $H^2 \ominus \Theta H^2$ is invariant under the operator T_φ^*, it is sufficient to show that

$$\bigvee\{\mathscr{R}(T_\varphi^k[X, Y]):\ k < n\} = H^2 \ominus \Theta^n H^2.$$

We prove this assertion by induction. By Lemma 1 this is true in case $n=1$. Suppose that

$$\bigvee\{\mathscr{R}(T_\varphi^k[X, Y]):\ k < n\} = H^2 \ominus \Theta^n H^2 \overset{\text{def}}{=} K_n.$$

Then

$$\bigvee\{\mathscr{R}(T_\varphi^k[X, Y]):\ k < n+1\} = \bigvee\{T_\varphi K_n, K_n\}.$$

If $f \perp \bigvee\{T_\varphi K_n, K_n\}$, then $f=\Theta^n g$ and $P_+\bar{\varphi} f=\Theta^n h$, for some $g\in H^2$, $h\in H^2$. Hence $\Theta h=\bar{\Theta}^{n-1} P_+\bar{\varphi}\Theta^n g=\bar{\Theta}^{n-1}(\Theta\bar{\varphi})\Theta^{n-1} g=(\Theta\bar{\varphi})g$. According to Lemma 2, $\Theta\bar{\varphi}$ and Θ have no common inner divisor. Thus $g\in\Theta H^2$ and $f\in\Theta^{n+1}H^2$. Therefore, $K_{n+1}\subseteq\bigvee\{T_\varphi K_n, K_n\}$. On the other hand, $(\varphi K_n, \Theta^{n+1}H^2)=(K_n, \Theta^n(\Theta\bar{\varphi})H^2)=0$. Hence $\bigvee\{T_\varphi K_n, K_n\}=K_{n+1}$ which completes the induction proof.

Proof of Theorem 2. Since (4)⇒(3) is obvious by definition we only have to prove the implication (3)⇒(4). Suppose that (3) is fulfilled. Then, by Theorem 1, φ_+ and φ_- are outer. To prove (4), we apply Corollary 1.9 in [2]. By this Corollary, it is sufficient to verify the following two conditions:

(x) $\quad Xf\in Q_{r+1,s} \Rightarrow f\in Q_{rs}$ for all $r \geqq 0$, $s \geqq 0$ and all $f\in H^2$,

(y) $\quad Yf\in Q_{r,s+1} \Rightarrow f\in Q_{r,s}$ for all $r \geqq 0$, $s \geqq 0$ and all $f\in H^2$.

Let $Xf\in Q_{r+1,s}=H^2\ominus\Theta^n H^2$, $n=r+s$, i.e., $Xf=P_+\Theta\bar{\varphi}_+ f\perp\Theta^n H^2$. Hence $0=(P_+\Theta\bar{\varphi}_+ f, \Theta^n H^2)=(f, \varphi_+\Theta^{n-1}H^2)$. Therefore, since φ_+ is outer, $f\in H^2\ominus \Theta^{n-1}H^2=Q_{rs}$. In a similar way we see that (y) is satisfied if φ_- is outer. This completes the proof.

References

[1] K. Schmüdgen, On commuting unbounded self-adjoint operators. I, *Acta Sci. Math.*, **47** (1984), 131—146.

[2] K. Schmüdgen, J. Friedrich, On commuting unbounded self-adjoint operators. II, *Integral Equations Operator Theory*, **7** (1984), 815—867.

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