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# Best approximation of a normal operator in the trace norm

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**1. Introduction.** A problem that has received considerable attention is the classification of operators that have a unique best approximation among the nonnegative operators (a unique positive approximant) in one norm or another. For the operator norm this was done in [4] and, consequently, it solved a problem posed in [8]. Those results were generalized in [5], [9], [2] and other papers. The problem of approximation in trace norm was specifically excluded in [2], and it was noted how the methods given there failed in the case of the trace norm. This paper gives a characterization of those normal operators with a unique positive approximant in the trace norm. The result is a striking contrast to the characterizations given previously for other norms.

We are concerned throughout this paper with (bounded linear) operators on a separable Hilbert space $\mathfrak{H}$. For any operator $T$ we use the associated operator $|T|=(T^*T)^{1/2}$ and the Caratesian decomposition $T=B+iC$ with $B=(1/2)(T+T^*)$ and $C=(1/2i)(T-T^*)$. We refer to $B$ as $\operatorname{re} T$ and to $C$ as $\operatorname{im} T$. For a compact operator $T$ we let $s_1(T), s_2(T), \ldots$ denote the eigenvalues of $|T|$ in nonincreasing order repeated according to multiplicity. If we have

$$\sum_{j=1}^{\infty} s_j(T) < \infty$$

then we say that $T$ is trace class and the preceding sum is the trace norm, denoted $\|T\|_1$. If $T$ is not trace class then $\|T\|_1$ is defined to be infinity.

For a self-adjoint operator $B$ we define $B^+$ to be $\frac{1}{2}(|B|+B)$ and $B^-$ to be $\frac{1}{2}(|B|-B)$; we note that $B=B^+-B^-$ and $|B|=B^++B^-$. If $E(\cdot)$ is the spectral measure for $B$ then it follows from the usual operational calculus that $B^+=BE([0,\infty))$ and $B^-=BE((-\infty,0])$. If $T$ is a given operator and $P$ is a nonnegative operator

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such that $\infty > \|T-P\|_1$ and $\|T-R\|_1 \geqq \|T-P\|_1$ for every nonnegative operator $R$ then we say that $P$ is a *trace class positive approximant of* $T$.

We shall frequently use the following inequality for the trace class operator $T$ where $\{e_j\}$ is some orthonormal set:

$$\|T\|_1 \geqq \sum_j |\langle Te_j, e_j\rangle|.$$

This follows from the Corollary on p. 40 of [10].

**2. Preliminary results.** Of course, not all operators can be approximated by a nonnegative operator using the trace norm. The next theorem gives convenient conditions for recognizing when a given operator can be approximated.

Theorem 1. *For a given operator* $T=B+iC$, $B^*=B$, $C^*=C$, *the following conditions are equivalent:*

(i) *there exists a nonnegative operator* $P$ *such that* $T-P$ *is trace class;*

(ii) *the operator* $C$ *is trace class and the spectrum of* $B$, *denoted* $\sigma(B)$, *not in the interval* $[0, \infty)$ *consists of isolated eigenvalues, say* $\{\lambda_j\}$ *repeated according to multiplicity, such that* $\sum_j |\lambda_j| < \infty$;

(iii) *the operator* $(T-B^+)$ *is trace class.*

Proof. (i) *implies* (ii): Let $D$ be the trace class operator $T-P$ and note that $B=P+\mathrm{re}\, D$, $C=\mathrm{im}\, D$. According to Weyl's Theorem $B$ and $P$ have the same Weyl spectrum. (See [1], for example.) For any normal operator $A$ the Weyl spectrum coincides with the points of $\sigma(A)$ that are not isolated eigenvalues with finite multiplicity. (See [3, Theorem 3] or [1, Theorem 5.1].) It is elementary that $\mathrm{re}\, D$ and $\mathrm{im}\, D$ are trace class operators.

Let $\{\lambda_j\}$ be an enumeration of the negative eigenvalues of $B$, repeated according to multiplicity, and let $\{e_j\}$ be an orthonormal sequence of eigenvectors with $e_j$ corresponding to $\lambda_j$. Note that $\|\mathrm{re}\, D\|_1 = \|P-B\|_1 \geqq \sum_j |\langle (P-B)e_j, e_j\rangle| = \sum_j (\langle Pe_j, e_j\rangle - \lambda_j) \geqq \sum_j -\lambda_j = \sum_j |\lambda_j|$.

(ii) *implies* (iii): Let $\{\lambda_j\}$ and $\{e_j\}$ have the same meaning as given in the first part. If $D$ is defined by $D=\sum_j \langle\cdot, e_j\rangle \lambda_j e_j$ then $\|D\|_1 = \sum_j |\lambda_j|$. Note that $B=B^+ + D$, since $B^- = BE((-\infty, 0])$ where $E(\cdot)$ is the spectral measure for $B$. We note that $T-B^+ = D+iC$; which proves (iii).

(iii) *implies* (i): This is obvious.

Next we show that if an operator can be approximated in trace norm by a nonnegative operator then it has a trace class positive approximant.

Theorem 2. *If the operator* $T$ *satisfies one of the conditions in Theorem* 1 *then* $T$ *has a trace class positive approximant.*

Proof. Recall that the conjugate space for the Banach space of compact operators on the underlying Hilbert space $\mathfrak{H}$ is the space of trace class operators on $\mathfrak{H}$. (See [10, p. 48], for example.) Recall that any closed sphere in the conjugate space is compact in the weak star topology. (See [6, p. 424], for example.) Let $R$ be a non-negative operator such that $(T-R)$ is trace class and let $\mathscr{B}$ denote the set of operators

$$\{T-P:\ P \geqq 0,\ \|T-P\|_1 \leqq \|T-R\|_1\}.$$

In order to show that $\mathscr{B}$ is weak star compact it suffices to show that $\mathscr{B}$ is weak star closed.

Let $\{T-R_\alpha:\ \alpha\in A\}$ be a net from $\mathscr{B}$ that converges to $T-P$ in the weak star topology; thus, $\lim_\alpha \operatorname{tr}(T-R_\alpha)X=\operatorname{tr}(T-P)X$ for every compact operator $X$. It suffices to show that $(T-P)$ belongs to $\mathscr{B}$. Let positive $\varepsilon$ and compact operator $X$ be given. Note that

$$\begin{aligned}|\operatorname{tr}(T-P)X| &= |\operatorname{tr}[(T-P)-(T-R_\alpha)+(T-R_\alpha)]X| \leqq \\ &\leqq |\operatorname{tr}(T-P)X-\operatorname{tr}(T-R_\alpha)X|+|\operatorname{tr}(T-R_\alpha)X| < \varepsilon+\|T-R_\alpha\|_1\|X\| \leqq \\ &\leqq \varepsilon+\|T-R\|_1\|X\|\end{aligned}$$

provided $\alpha>\beta$ where $\beta$ belongs to $A$ and depends on $\varepsilon$ and $X$. It follows from the preceding inequalities that

$$\|T-P\|_1 \leqq \varepsilon+\|T-R\|_1$$

for the arbitrarily chosen $\varepsilon$. Hence, $(T-P)$ belongs to $\mathscr{B}$ and, thus, $\mathscr{B}$ is weak star compact.

From elementary topology we know that any lower semicontinuous function defined on a compact set assumes its infimum. Thus, it suffices to show that $f(A)=\|A\|_1$ is lower semicontinuous on the space of trace class operators. Note that

$$\|A\|_1 = \sup\{|\operatorname{tr}(AX)|:\ X \text{ is a compact contraction}\}.$$

Since the supremum of any collection of lower semicontinuous function is lower semicontinuous, we conclude that $f(A)$ is lower semicontinuous on the compact set $\mathscr{B}$. This completes the proof.

Theorems 1 and 2 might lead the reader to conjecture that $B^+$ is always a trace class positive approximant for $T=B+iC$, $B^*=B$, $C^*=C$. Such a conjecture is false, as we demonstrate. Define $T$ by

$$T = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

and note that $B^+=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$. It is routine to determine that the spectrum of $|T-B^+|$ is $\{((3+\sqrt{5})/2)^{1/2}, ((3-\sqrt{5})/2)^{1/2}\}$ and, consequently, $\|T-B^+\|_1=\operatorname{tr}|T-B^+|=\sqrt{5}$. Since $\|T\|_1=2$, we see that the zero operator is closer to $T$ than $B^+$ is.

**3. Main results.** Note that $T$ in the counterexample at the end of the preceding section is not a normal operator. If $T$ is normal then there is a simple trace class positive approximant.

Theorem 3. *If $A=B+iC$, $B^*=B$, $C^*=C$, is a normal operator satisfying one of the conditions in Theorem 1 then $B^+$ is a trace class positive approximant for $A$.*

Proof. It is clear that $-B^-+iC=A-B^+$ is a normal operator, and by Theorem 1 it is trace class. Clearly $B^-$ and $C$ are commuting self-adjoint trace class operators and there is an orthonormal basis, say $\{e_j\}$, that diagonalizes both operators. Let $z_j$ be the eigenvalue of $-B^-+iC$ corresponding to the eigenvector $e_j$ for each $j$. Since $B^+=BE([0,\infty))$ where $E(\cdot)$ is the spectral measure for $B$, it is routine to see that $B^+e_j=0$ for every $j$. Thus, we have $\langle Ae_j, e_j\rangle = \langle(-B^-+iC)e_j, e_j\rangle=z_j$.

For any nonnegative operator $R$ we note that

$$\|A-R\|_1 \geqq \sum_j |\langle(A-R)e_j, e_j\rangle| = \sum_j |\langle -Re_j e_j\rangle + z_j| \geqq \sum |z_j| = \|A-B^+\|_1.$$

The preceding inequality proves that $B^+$ is a trace class positive approximant of $A$.

It follows from the main theorem in [2] that $B^+$ is the unique positive approximant in the Schatten $p$-norm $\|\cdot\|_p$, with $p\geqq 2$, for the normal operator $A=B+iC$, $B^*=B$, $C^*=C$. The next lemma shows that no statement like the preceding is true when the norm used is $\|\cdot\|_1$.

Lemma 4. *Let $\alpha$, $\beta$, $\gamma$ and $\delta$ be positive numbers and define $A$ by*

$$A=\begin{bmatrix}\alpha+i\gamma & 0\\ 0 & \beta-i\delta\end{bmatrix}.$$

*Two trace class positive approximants of $A$ are*

$$(\operatorname{re} A)^+ = \begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} \quad and \quad R=\begin{bmatrix}\alpha & \varepsilon\\ \varepsilon & \beta\end{bmatrix}$$

*where $\varepsilon$ is chosen to satisfy $\gamma\delta\geqq\varepsilon^2>0$ and $\alpha\beta\geqq\varepsilon^2$.*

Proof. By Theorem 3 we know that $(\operatorname{re} A)^+$ is a trace class positive approximant; thus, the $\|\cdot\|_1$-distance between $A$ and the nonnegative $2\times 2$ matrices is

$$\|A-(\operatorname{re} A)^+\|_1 = \left\|\begin{bmatrix}i\gamma & 0\\ 0 & -i\delta\end{bmatrix}\right\|_1 = \gamma+\delta.$$

Thus, it suffices to show that $\|A-R\|_1\leqq\gamma+\delta$. Straightforward computations show that the spectrum of $|A-R|=[(A-R)^*(A-R)]^{1/2}$ is

$$\{2^{-1/2}(\delta^2+\gamma^2+2\varepsilon^2\pm((\delta^2+\gamma^2+2\varepsilon^2)^2-4[(\gamma^2+\varepsilon^2)(\delta^2+\varepsilon^2)-\varepsilon^2(\gamma+\delta)^2])^{1/2})^{1/2}\}.$$

It follows that

$$\|A-R\|_1^2 = (\operatorname{tr}|A-R|)^2 = \delta^2+\gamma^2+2\varepsilon^2+\sqrt{4[(\gamma^2+\varepsilon^2)(\delta^2+\varepsilon^2)-\varepsilon^2(\gamma+\delta)^2]} =$$
$$= \delta^2+\gamma^2+2\gamma\delta = (\delta+\gamma)^2.$$

This proves the lemma.

Theorem 5. *Let $A$ be a normal operator satisfying one of the conditions in Theorem 1. If the eigenvalues of $A$ include $z=\alpha+i\gamma$ and $w=\beta-i\delta$ with $\alpha, \gamma, \beta, \delta>0$ then $A$ does not have a unique trace class positive approximant.*

Proof. Write $A$ as an orthogonal direct sum $A_0 \oplus A_1$ such that the spectrum of $A_0$ is the set $\{z, w\}$. Clearly the direct sum of trace class positive approximants of $A_0$ and $A_1$, respectively, is a trace class positive approximant for $A$. It follows from Lemma 4 that we can construct multiple approximants for $A_0$ and, hence, for $A$.

Before we are done we shall prove the converse of the preceding theorem. First, we must accumulate some appropriate basic results. The next lemma gives another circumstance in which $(\operatorname{re} T)^+$ is a trace class positive approximant of $T$.

Lemma 6. *Let $T=B+iC$, $B^*=B$, $C^*=C$, be an operator satisfying one of the conditions in Theorem 1. If $B \geqq 0$ then $B$ is a trace class positive approximant for $T$.*

Proof. Let $\{e_j\}$ be an orthonormal basis of eigenvectors for $C$ and let $\lambda_j$ be the eigenvalue corresponding to $e_j$ for each $j$. If $R$ is any nonnegative operator then we have

$$\|T-R\|_1 \geqq \sum_j |\langle (T-R)e_j, e_j\rangle| = \sum_j [(\langle (B-R)e_j, e_j\rangle)^2+\lambda_j^2]^{1/2} \geqq$$
$$\geqq \sum_j |\lambda_j| = \|C\|_1 = \|T-B\|_1.$$

This proves the lemma.

By strengthening the hypothesis of the preceding lemma we get a uniqueness result.

Theorem 7. *Let $T=B+iC$, $B^*=B$, $C^*=C$, be an operator satisfying one of the conditions of Theorem 1. If $B \geqq 0$ and $C \geqq 0$ then $B$ is the unique trace class positive approximant for $T$.*

Proof. Choose $\{e_j\}$ and $\lambda_j$ as in the proof of Lemma 6, and note that $B$ is a trace class positive approximant of $T$ according to that lemma. For $R$ any trace class positive approximant of $T$ we have

$$\|T-R\|_1 \geqq \sum_j |\langle (T-R)e_j, e_j\rangle| \geqq \Big|\sum_j \langle (T-R)e_j, e_j\rangle\Big| =$$
$$= \Big|\sum_j \langle (B-R)e_j, e_j\rangle + i\sum_j \lambda_j\Big| \geqq \sum_j \lambda_j = \|C\|_1 = \|T-R\|_1.$$

Since equality must hold throughout the preceding inequalities, we have

$$\|T-R\|_1 = \left|\sum_j \langle (T-R)e_j, e_j\rangle\right| = |\mathrm{tr}\,(T-R)|.$$

By the last part of Theorem 8.6 of [7, pp. 104—105], we conclude that $e^{-i\theta}(T-R)$ is a nonnegative operator for $\theta = \arg \mathrm{tr}\,(T-R)$. The equality of the third and fourth lines in the earlier inequalities shows that $\mathrm{tr}\,(T-R) = i \sum_j \lambda_j$. Thus, we know that $-i(T-R) = -i(B-R)+C$ is a nonnegative operator. This implies that $B-R=0$, which is the desired conclusion.

The next theorem gives another situation where $(\mathrm{re}\,T)^+$ is the unique trace class positive approximant for $T$.

Theorem 8. *Let $A=B+iC$, $B^*=B$, $C^*=C$, be a normal operator satisfying one of the conditions in Theorem* 1. *If $B \leqq 0$ then the zero operator* 0 *is the unique trace class positive approximant of $A$.*

Proof. Let $\{e_j\}$ be an orthonormal basis consisting of eigenvectors of $A$ and let $z_j$ be the eigenvalue corresponding to $e_j$ for each $j$. Note that $\mathrm{re}\, z_j \leqq 0$ for each $j$. If $R$ is any nonnegative operator then we have

$$\|A-R\| \geqq \sum_j |\langle (A-R)e_j, e_j\rangle| = \sum_j |\langle z_j - (Re_j, e_j\rangle| =$$

$$= \sum_j [(\langle Re_j, e_j\rangle - \mathrm{re}\, z_j)^2 + (\mathrm{im}\, z_j)^2]^{1/2} \geqq$$

$$\geqq \sum_j [(\mathrm{re}\, z_j)^2 + (\mathrm{im}\, z_j)^2]^{1/2} = \sum_j |z_j| = \|A\|_1.$$

This proves that 0 is a trace class positive approximant of $A$.

Furthermore, if $R$ is any trace class positive approximant of $A$ then equality holds in each of the preceding inequalities. It follows that $\langle Re_j, e_j\rangle = 0$ for each $j$ and hence, $R$ must be 0. The uniqueness is proved.

Before we can exploit Theorems 7 and 8 we need an elementary observation about matrices of operators.

Lemma 9. *If $R=\begin{pmatrix} 0 & D \\ D^* & B\end{pmatrix}$ is a nonnegative operator on $\mathfrak{H}_0 \oplus \mathfrak{H}_1$ then $B \geqq 0$ and $D=0$.*

Proof. Assume that there exists some $f$ in $\mathfrak{H}_1$ such that $Df \neq 0$, and define $e$ by $e = (-\gamma/\|Df\|^2)Df$ where $\gamma$ is an arbitrary positive number. Note that

$$\left\langle \begin{bmatrix} 0 & D \\ D^* & B\end{bmatrix} \begin{bmatrix} e \\ f\end{bmatrix}, \begin{bmatrix} e \\ f\end{bmatrix}\right\rangle = -2\gamma + \langle Bf, f\rangle.$$

This contradicts the nonnegativity of $R$ and, thus, it shows that $D=0$.

Since

$$\left\langle R\begin{bmatrix}0\\f\end{bmatrix}, \begin{bmatrix}0\\f\end{bmatrix}\right\rangle \geqq 0$$

for any $f$ in $\mathfrak{H}_1$, it is clear that $B \geqq 0$.

Using the results of 7, 8 and 9 we can prove a partial converse for Theorem 5.

Theorem 10. *Let $A=B+iC$, $B^*=B$, $C^*=C$, be a normal operator satisfying one of the conditions in Theorem* 1. *If the spectrum of $A$, denoted $\sigma(A)$, is contained in $\{z$: either $\operatorname{im} z \geqq 0$ or $\operatorname{re} z \leqq 0\}$ then $B^+$ is the unique trace class positive approximant for $A$.*

Proof. Let $E(\cdot)$ be the spectral measure for $A$ and define $E_0$, $E_1$, $A_0$ and $A_1$ by $E_0=E(\{z\colon \operatorname{re} z \leqq 0\})$, $E_1=E(\{z\colon \operatorname{re} z \geqq 0,\ \operatorname{im} z \geqq 0\})$, $A_0=AE_0$, $A_1=AE_1$. The hypothesis concerning $\sigma(A)$ shows that $A=A_0 \oplus A_1$. According to Theorem 8, 0 is the unique trace class positive approximant of $A_0$; according to Theorem 7, the unique trace class positive approximant of $A_1$ is $(\operatorname{re} A_1)$. It suffices to show that $B^+=0 \oplus \operatorname{re} A_1$ is the unique trace class positive approximant for $A$.

We use Theorem 8.7 of [7, pp. 105—106] in the first inequality below. If $R$ is a nonnegative operator then we have

$$\begin{aligned}\|A-R\|_1 &\geqq \|E_0(A-R)E_0\|_1+\|E_1(A-R)E_1\|_1 = \\ &= \|A_0-E_0RE_0\|_1+\|A_1-E_1RE_1\|_1 \geqq \\ &\geqq \|A_0\|_1+\|A_1-\operatorname{re} A_1\|_1 = \|A-B^+\|_1.\end{aligned}$$

The preceding computation shows that $B^+$ is a trace class positive approximant for $A$. Furthermore, if $R$ is any trace class positive approximant for $A$ then $E_0RE_0=0$, $E_1RE_1=\operatorname{re} A_1$ by the uniqueness of the approximants of $A_0$ and $A_1$. It now follows from Lemma 9 that $R=0 \oplus \operatorname{re} A_1=B^+$, which proves the theorem.

Using Theorems 5 and 10 we characterize the normal operators that have a unique trace class positive approximant.

Theorem 11. *Let $A=B+iC$, $B^*=B$, $C^*=C$, be a normal operator that satisfies one of the conditions in Theorem* 1. *There is a unique trace class positive approximant for $A$ if and only if $\sigma(A)$ is contained in one or the other of the two sets $\{z$: either $\operatorname{im} z \geqq 0$ or $\operatorname{re} z \leqq 0\}$, $\{z$: either $\operatorname{im} z \leqq 0$ or $\operatorname{re} z \leqq 0\}$.*

Proof. If $\sigma(A)$ is contained in the first set then it is immediate from Theorem 10 that $B^+$ is the unique trace class positive approximant of $A$. If $\sigma(A)$ is contained in the second set then $\sigma(A^*)$ is contained in the first set and $B^+$ is the unique trace class positive approximant of $A^*$. For any nonnegative operator $R$ we have $\|A^*-R\|_1=\|A-R\|_1$ (by Lemma 8 of [10, p. 39], for example). It follows that $B^+$ is the unique trace class positive approximant for $A$.

If $A$ has a unique trace class positive approximant then Theorem 5 shows that $A$ does not have eigenvalues in each of the sets $\{z\colon \operatorname{im} z<0\}$ and $\{z\colon \operatorname{im} z>0\}$. Thus, the eigenvalues of $A$ are contained in one or the other of the two sets $\{z\colon \text{either } \operatorname{im} z\geqq 0 \text{ or } \operatorname{re} z\leqq 0\}$, $\{z\colon \text{either } \operatorname{im} z\leqq 0 \text{ or } \operatorname{re} z\leqq 0\}$. According to Theorem 1, $A-B^+$ is trace class and so $A$ is a compact perturbation of $B^+$, that is $A=B^++(A-B^+)$. By Weyl's theorem $A$ and $B^+$ have the same Weyl spectrum. For each of these normal operators the Weyl spectrum consists of the points that are not isolated eigenvalues with finite multiplicity. Clearly the Weyl spectrum of $B^+$ (and, hence, the Weyl spectrum of $A$) is contained in the interval $[0, \infty)$. Since both the Weyl spectrum of $A$ and the eigenvalues of $A$ are contained in one of the desired sets, we conclude that $\sigma(A)$ is contained in one or the other of the sets indicated in the statement of the theorem.

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