Acta Sci. Math., 52 (1988), 21—28

Semigroups with a universally minimal left ideal

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A left ideal L of a semigroup S is called universally minimal if it is contained in every left ideal of S. In such a semigroup L is at the same time the kernel of S (i.e. the minimal two-sided ideal of S) and L itself is a left simple semigroup. We shall deal with the case that L is a left group.

For simplicity we introduce the following notation. A semigroup containing a universally minimal left ideal which is a left group will be called a *ULG-semigroup*. If L is a group, such semigroups are called *homogroups*. Let S be a semigroup and A an ideal of S. An endomorphism h of S onto A is called an *A-endomorphism* if h leaves the elements of A fixed.

In a forthcomming paper [5] I have been led in a quite natural way to the following class of semigroups: S is a ULG-semigroup with kernel L and S has an L-endomorphism. The main goal of this note is to show that such semigroups have a rather simple structure. Though there are several papers dealing with analogous (and even more general) questions (see, e.g. [1], [2], [3], [4]), I can find nowhere the results given below (at least not in an explicit formulation).

Throughout the paper we use the following notations. S is a ULG-semigroup, L is the kernel of S and $E=\{e_\nu|\nu\in M\}$ is the set of all idempotents of L (i.e. primitive idempotents of S). It is well-known that L can be written in the form $L=\bigcup_{\nu\in M} G_\nu$. Hereby each G_ν is a group (with identity element e_ν) and at the same time a minimal right ideal of S. We have $e_\alpha G_\nu=G_\alpha$, $G_\alpha G_\nu=G_\alpha$ (for any ν, $\alpha\in M$). Moreover each e_α ($\alpha\in M$) is a right identity of L.

In the sequel $|A|$ denotes the cardinality of A.

1. In order to make this note independent of [5] we give in Lemma 1 a modified version of a few results proved in [5].

Received May 18, 1984 and in revised form January 10, 1985.

Lemma 1. *Let S be a ULG-semigroup with kernel L and E the set of all idempotents of L. Then the following holds:*

a) *Any L-endomorphism of S can be written in the form* $x \mapsto xe_\alpha$ $(x \in S,\ e_\alpha \in E)$.

b) *If for some* $e_\alpha \in E$ *the mapping* $x \mapsto xe_\alpha$ *is an L-endomorphism of S, then* $x \mapsto xe_\nu$ *is an L-endomorphism of S for any* $e_\nu \in E$.

c) *The mapping* $x \mapsto xe_\alpha$ *is an L-endomorphism of S iff for any* $x \in S$ *we have* $|xE| = 1$.

Proof. a) Let h be an L-endomorphism of S and $x \in S$. Since $xe_\alpha \in L$, we have $h(xe_\alpha) = h(x) \cdot h(e_\alpha) = xe_\alpha$, i.e. $h(x)e_\alpha = xe_\alpha$. Since $h(x) \in L$ and e_α is a right identity of L, we have $h(x) = xe_\alpha$.

b) By assumption we have $xe_\alpha ye_\alpha = xye_\alpha$ for any $x, y \in S$. Putting $y = e_\nu$ we have in particular $xe_\alpha e_\nu e_\alpha = xe_\nu e_\alpha$. Since $e_\alpha e_\nu e_\alpha = e_\alpha$ and $e_\nu e_\alpha = e_\nu$, we have $xe_\alpha = xe_\nu$ for any $x \in S$. Hence $xe_\nu ye_\nu = xe_\alpha ye_\alpha = xye_\alpha = xye_\nu$, i.e. $x \mapsto xe_\nu$ is an L-endomorphism of S.

c) If $x \mapsto xe_\alpha$ is an L-endomorphism, we have [by b)] $xe_\alpha = xe_\nu$ for any $\nu \in M$, hence $xe_\alpha = xE$ so that $|xE| = 1$. Suppose conversely that $|xE| = 1$ for any $x \in S$ and consider the product $xe_\alpha ye_\alpha$ $(x, y \in S,\ e_\alpha \in E)$. The element ye_α is contained in L, hence there is a group $G_\gamma \subset L$ such that $ye_\alpha \in G_\gamma$. Therefore (if e_γ is the identity element of G_γ) $e_\gamma ye_\alpha = ye_\alpha$. By assumption $xe_\alpha = xe_\gamma$, hence $xe_\alpha ye_\alpha = xe_\gamma ye_\alpha = xye_\alpha$. The mapping $x \mapsto xe_\alpha$ is an L-endomorphism. This proves the statement c).

Remark. To understand well the statement a) consider the ULG-semigroup S given by the multiplication table

	a	b	c
a	a	b	a
b	b	a	b
c	a	b	a.

Here $L = \{a, b\}$, $E = \{a\}$, hence S is a homogroup. S has an L-endomorphism φ_1: $x \mapsto xa$. Also φ_2: $x \mapsto xc$ is an endomorphism though here $c \notin E$. But φ_2 is the same endomorphism as φ_1. By c) whenever S has an L-endomorphism we can rewrite it in the form $x \mapsto xE$.

Needless to remark that the mapping $x \mapsto xe_\alpha$ need not be an endomorphism of S. But if it is an endomorphism, it is automatically an L-endomorphism. Hence the result of Lemma 1 can be reformulated as follows

Theorem 1. *Let S be a ULG-semigroup with kernel L. Then S has an L-endomorphism iff for any* $x \in S$ *we have* $|xE| = 1$.

The condition $|xE| = 1$ is a very simple one. If S is given by a multiplication table it can be immediately verified. But this condition does not reflect any structural property of S. The structure of such semigroups is given by Theorem 2. (A part of this theorem can be deduced from a result in [1].)

Theorem 2. *Let S be a ULG-semigroup with kernel L. Then S has an L-endomorphism iff S can be written as a union of disjoint right ideals of S each of which is a homogroup. The kernels of these homogroups are then isomorphic to one another.*

Proof. a) Suppose that S has an L-endomorphism. We use the notations introduced above. By Lemma 1 this endomorphism can be written in the form $x \mapsto xE$ $(x \in S)$. For any $\alpha \in M$ denote $R_\alpha = \{x \mid x \in S, xE \in G_\alpha\}$. Clearly $S = \bigcup_{\nu \in M} R_\nu$ and $R_\alpha \cap R_\beta = \emptyset$ if $\alpha \neq \beta$. Further $G_\alpha \subset R_\alpha$ (since $G_\alpha E = G_\alpha$).

We show that $R_\alpha R_\beta \subset R_\alpha$. Let $x \in R_\alpha$, $y \in R_\beta$, i.e., $xE \in G_\alpha$, $yE \in G_\beta$. Then $e_\beta yE = yE$ and $xyE = xe_\beta yE = xE \cdot yE \subset G_\alpha G_\beta = G_\alpha$. Hence $xy \in R_\alpha$, i.e. $R_\alpha R_\beta \subset R_\alpha$. In particular each R_α is a right ideal of S, since $R_\alpha S = R_\alpha \cdot [\bigcup_{\nu \in M} R_\nu] \subset R_\alpha$.

Finally we show that each R_α is a homogroup with kernel G_α. We have $G_\alpha \subset L \cap R_\alpha$, and since $G_\beta \cap R_\alpha = \emptyset$ for $\beta \neq \alpha$, this implies $G_\alpha = L \cap R_\alpha$. The intersection $L \cap R_\alpha$ is a two-sided ideal of R_α. Since it is a group, it is moreover the minimal two-sided ideal of R_α. Hence G_α is the kernel of R_α. This proves the first part of Theorem 2. Moreover it follows from the proof that the kernels of all R_α are isomorphic groups.

b) Suppose conversely that S is a ULG-semigroup with kernel L and S can be written as a union of disjoint right ideals of S in the form $S = \bigcup_{\mu \in N} R'_\mu$. Here we suppose that each R'_μ is a homogroup, hence the kernel of R'_μ is a group K_μ.

Write again $L = \bigcup_{\nu \in M} G_\nu$. Since $R'_\mu L \subset R'_\mu \cap L$, this latter intersection is not empty and it is a right ideal of S contained in L. Hence $L \cap R'_\mu$ is a union of some groups from the family $\{G_\nu\}_{\nu \in M}$. If a group $G_\varkappa$, $\varkappa \in M$, is contained in R'_μ, it is a minimal right ideal of R'_μ. Since a homogroup contains a unique minimal right ideal, we conclude $G_\varkappa = K_\mu$. Hence $L \cap R'_\mu$ contains exactly one group from the family $\{G_\nu\}_{\nu \in M}$ and we have $K_\mu = L \cap R'_\mu$. Otherwise expressed: To any R'_μ there exists an $\alpha \in M$ such that $L \cap R'_\mu = K_\mu = G_\alpha$.

Conversely: Any $e_\beta \in E$ is contained in some R'_μ, hence G_β is contained in R'_μ. Since G_β is a right ideal of S, it is also a right ideal of R'_μ and (since G_β is a group) it is a minimal right ideal of R'_μ. Since R'_μ is a homogroup, G_β is the kernel of R'_μ.

We conclude $|M| = |N|$ and we may write $S = \bigcup_{\nu \in M} R'_\nu$. Also the kernels of all R'_ν are isomorphic groups.

If $x \in S$, then there is a unique R'_ν such that $x \in R'_\nu$. We denote this homogroup R'_ν by $R^{(x)}$. The kernel of $R^{(x)}$ will be denoted by $G^{(x)}$ and the identity element of $G^{(x)}$ by $e^{(x)}$. Note that $R^{(x)} e^{(x)} = e^{(x)} R^{(x)} = G^{(x)}$.

To prove that S has an L-endomorphism it is sufficient, by Theorem 1, to show that $x \cdot e_\alpha = x \cdot e^{(x)}$ for any $x \in S$, $e_\alpha \in E$. Now $x \cdot e_\alpha \in R^{(x)} \cdot L \subset R^{(x)} \cap L = G^{(x)}$. Taking into account that e_α is a right unit in L and $e^{(x)}$ is the unit element of the group $G^{(x)}$ (the kernel of $R^{(x)}$), we have

$$xe_\alpha = e^{(x)} x \cdot e_\alpha = e^{(x)} \cdot x = e^{(x)} \cdot xe^{(x)} = xe^{(x)}. \tag{1}$$

This proves our statement.

Example 1. Suppose that S is a ULG-semigroup with kernel L, S has an L-endomorphism and S is defined by its multiplication table. To find the right ideals R_α mentioned in Theorem 1 we may proceed as follows. We collect all "rows" of the multiplication table containing a fixed chosen $e_\alpha \in E$ (i.e. all sets $\{u, uS\}$ containing e_α). Then $R_\alpha = \bigcup_u \{u, uS\}$. Clearly R_α is a right ideal of S, it contains e_α, and it follows from the proof that it cannot contain any other idempotent of L.

Consider, e.g., the semigroup S given by the following multiplication table:

	a	b	c	d	f
a	a	a	c	c	a
b	b	b	d	d	b
c	c	c	a	a	c
d	d	d	b	b	d
f	a	a	c	c	a

Here $L = E = \{a, b\}$. S has an L-endomorphism since $|x \cdot \{a, b\}| = 1$ for any $x \in S$. The idempotent a is contained in $\{a, aS\}, \{c, cS\}, \{f, fS\}$. Hence $R^{(a)} = \{a, c, f\}$. Analogously $R^{(b)} = \{b, bS\} \cup \{d, dS\} = \{b, d\}$. Finally $S = R^{(a)} \cup R^{(b)}$.

We shall return to this procedure in Section 3.

2. In Theorem 2 the right ideals R_ν have the property that their kernels are isomorphic groups. The question arises whether there are some other limitations concerning the ideals R_ν. The answer is no. To any family of homogroups $\{Q_\nu\}$ with isomorphic kernels we can construct at least one ULG-semigroup which has an L-endomorphism. We give a special construction and we do not attempt to find all such semigroups.

More precisely we have:

Theorem 3. *Let L_0 be a left group. Write $L_0 = G_0 \times E_0$, where G_0 is a group and E_0 a left zero semigroup. Let $\{Q_\nu \mid \nu \in M\}$ be a family of disjoint homogroups whereby each Q_ν has a kernel isomorphic to G_0 and $|E_0| = |M|$. Then there exists a ULG-semigroup S having the following properties:*

1) $S = \bigcup_{\nu \in M} Q_\nu$.
2) *Each Q_ν is a right ideal of S.*
3) *The kernel L of S is isomorphic to L_0 and S has an L-endomorphism.*

Proof. Denote the kernel of Q_ν by H_ν and denote the identity element of H_ν by e_ν. Suppose that $1 \in M$. For every $\nu \in M$ let φ_ν be a fixed chosen isomorphism of H_1 onto H_ν. Define the mapping $\varphi_{\alpha\beta}\colon H_\alpha \to H_\beta$ by $\varphi_{\alpha\beta} = \varphi_\alpha^{-1}\varphi_\beta$. Then $\varphi_{\alpha\beta}$ is an isomorphism and $\varphi_{\alpha\alpha}$ is the identity mapping of H_α onto H_α. For any $a \in H_\alpha$ we have

$$(a\varphi_{\alpha\beta})\varphi_{\beta\gamma} = (a\varphi_\alpha^{-1}\varphi_\beta)\varphi_\beta^{-1}\varphi_\gamma = a\varphi_\alpha^{-1}\varphi_\gamma = a\varphi_{\alpha\gamma}.$$

In this way we get a set of mappings $\{\varphi_{\mu\nu}\}$ where $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ for any $\alpha, \beta, \gamma \in M$.

Note finally: Since $\varphi_{\alpha\beta}$ is an isomorphism, we have $(e_\alpha)\varphi_{\alpha\beta} = e_\beta$.

1) We now use the set of these mappings to define on $S = \bigcup_{\nu \in M} Q_\nu$ a multiplication (denoted by $*$). For $\alpha \neq \beta$ and $x \in Q_\alpha$, $y \in Q_\beta$, we define

$$x * y = (e_\alpha x) \cdot (e_\beta y)\varphi_{\beta\alpha},$$

while inside of each Q_α the multiplication remains unaltered.

The definition implies $x * y \in H_\alpha \cdot H_\alpha = H_\alpha$, hence for $\alpha \neq \beta$, $Q_\alpha * Q_\beta \subset H_\alpha$. Since $H_\alpha \subset Q_\alpha$, $(H_\beta)\varphi_{\beta\alpha} = H_\alpha$, we have $H_\alpha * H_\beta = H_\alpha$ and therefore for $\alpha \neq \beta$,

$$Q_\alpha * Q_\beta = Q_\alpha * H_\beta = H_\alpha * H_\beta = H_\alpha * Q_\beta = H_\alpha. \tag{2}$$

In order to show that S is a semigroup we have to check associativity.

a) Suppose first $\alpha \neq \beta$, $\beta \neq \gamma$ and $x \in Q_\alpha$, $y \in Q_\beta$, $z \in Q_\gamma$.
In the following we use: $x * y \in H_\alpha$ implies $e_\alpha(x * y) = x * y$ and $u * v \in H_\beta$ implies $e_\beta(u * v) = u * v$. We have:

$$x * (y * z) = x * [e_\beta y \cdot (e_\gamma z)\varphi_{\gamma\beta}] = e_\alpha x \cdot [e_\beta y \cdot (e_\gamma z)\varphi_{\gamma\beta}]\varphi_{\beta\alpha} =$$

$$= e_\alpha x \cdot (e_\beta y)\varphi_{\beta\alpha} \cdot (e_\gamma z)\varphi_{\gamma\alpha} = (x * y) \cdot (e_\gamma z)\varphi_{\gamma\alpha} = e_\alpha(x * y) \cdot (e_\gamma z)\varphi_{\gamma\alpha} = (x * y) * z.$$

b) Suppose next. $\alpha \neq \beta$, $\beta = \gamma$, and $x \in Q_\alpha$, $y \in Q_\beta$, $z \in Q_\beta$.
In the following we use $e_\beta y \in H_\beta$, hence $e_\beta y = e_\beta y e_\beta$. We have:

$$x * (y * z) = x * (yz) = e_\alpha x \cdot (e_\beta yz)\varphi_{\beta\alpha} = (e_\alpha x)(e_\beta y e_\beta z)\varphi_{\beta\alpha} =$$

$$= (e_\alpha x) \cdot (e_\beta y)\varphi_{\beta\alpha} \cdot (e_\beta z)\varphi_{\beta\alpha} = (x * y) \cdot (e_\beta z)\varphi_{\beta\alpha} = e_\alpha(x * y) \cdot (e_\beta z)\varphi_{\beta\alpha} = (x * y) * z.$$

c) Suppose finally $\alpha = \beta$, $\beta \neq \gamma$, and $x \in Q_\alpha$, $y \in Q_\alpha$, $z \in Q_\gamma$.

$$x * (y * z) = x * \left[(e_\alpha y) \cdot (e_\gamma z)\varphi_{\gamma\alpha}\right] = x \cdot (e_\alpha y) \cdot (e_\gamma z)\varphi_{\gamma\alpha}.$$

Now since $e_\alpha x \in H_\alpha$ we have $e_\alpha x = e_\alpha x e_\alpha$ and $e_\alpha xy = e_\alpha x e_\alpha y$. Also since $x e_\alpha y \in H_\alpha$ we have $e_\alpha x e_\alpha y = x e_\alpha y$. Hence $e_\alpha xy = x e_\alpha y$. We may write therefore:

$$x * (y * z) = e_\alpha xy \cdot (e_\gamma z)\varphi_{\gamma\alpha} = (xy) * z = (x * y) * z.$$

This proves that S is a semigroup.

2) The relation (2) implies $Q_\alpha * Q_\beta = H_\alpha \subset Q_\alpha$ for $\alpha \neq \beta$ and $Q_\alpha^2 \subset Q_\alpha$ (for any $\alpha \in M$). Next

$$Q_\alpha * S = Q_\alpha * [\bigcup_{\nu \in M} Q_\nu] \subset Q_\alpha,$$

so that each Q_α is a right ideal of S. Denote $L = \bigcup_{\mu \in M} H_\mu$, then by (2)

$$S * L = [\bigcup_{\nu \in M} Q_\nu] * [\bigcup_{\mu \in M} H_\mu] = \bigcup_{\nu \in M} H_\nu = L,$$

$$L * S = [\bigcup_{\mu \in M} H_\mu] * [\bigcup_{\nu \in M} Q_\nu] = \bigcup_{\mu \in M} H_\mu = L.$$

Hence L is a two-sided ideal of S.

To prove that L is a left group it is sufficient to show that for any $y \in S$ we have $L * y = L$. Now $y \in S$ implies $y \in Q_\beta$ for some $\beta \in M$. Denote $(e_\beta y)\varphi_{\beta\nu} = y_\nu \in H_\nu$. We have

$$L * y = [\bigcup_{\nu \in M} H_\nu] * y = \bigcup_{\nu \in M} [H_\nu * y] = \bigcup_{\nu \in M} [H_\nu \cdot (e_\beta y)\varphi_{\beta\nu}] =$$

$$= \bigcup_{\nu \in M} [H_\nu \cdot y_\nu] = \bigcup_{\nu \in M} H_\nu = L.$$

This proves that S is a ULG-semigroup with kernel L and clearly L is isomorphic to L_0.

3) It remains to show that S has an L-endomorphism. Denote by E the set of all idempotents contained in L. It is sufficient to show that for $x \in S$ we have $|x * E| = 1$. If $x \in S$ we have $x \in Q_\alpha$ for some $\alpha \in M$. Let $e_\gamma \in E$. Then

$$x * e_\gamma = (e_\alpha x) \cdot (e_\gamma)\varphi_{\gamma\alpha} = e_\alpha x e_\alpha.$$

The right-hand side is independent of e_γ, hence $|x * E| = 1$. This proves Theorem 3.

3. The procedure described in Example 1 can be carried out in any ULG-semigroup (even if S has not an L-endomorphism). To any minimal right ideal G_ν of a ULG-semigroup S there is a largest right ideal R_ν^* of S (containing G_ν) such that R_ν^* is a homogroup. This right ideal consists of all "rows" $\{u, uS\}$ containing e_ν but no other idempotent of E. If $e_\alpha \neq e_\beta$, then $R_\alpha^* \cap R_\beta^* = \emptyset$. The union $S^* = \bigcup_{\nu \in M} R_\nu^*$ is a right ideal of S. If S does not have an L-endomorphism, then S^* is a proper subset of S.

Lemma 2. *The set S^* consists exactly of those elements $x \in S$ for which $|xE| = 1$.*

Proof. a) Let $x \in S^*$, hence $x \in R_\alpha^*$ with suitably chosen $\alpha \in M$. We have $xE \subset R_\alpha^* L \subset R_\alpha^* \cap L = G_\alpha$. Note that in the homogroup R_α^* we have $xe_\alpha = e_\alpha x$ (for any $x \in R_\alpha^*$).

Let now e_γ be any element of E. Then $xe_\gamma \in G_\alpha$ implies $(xe_\gamma)e_\alpha = e_\alpha(xe_\gamma)$. This implies $xe_\gamma = e_\alpha(xe_\gamma) = (e_\alpha x)e_\gamma = (xe_\alpha)e_\gamma = x(e_\alpha e_\gamma) = xe_\alpha$. Hence $xe_\gamma = xe_\alpha$; therefore $xE = xe_\alpha$, i.e., $|xE| = 1$ for any $x \in S^*$.

b) Suppose conversely that $x \in S - S^*$. We have to show that $|xE| \geqq 2$. The right ideal $\{x, xS\}$ contains at least two idempotents of E, say e_α, e_β $(e_\alpha \neq e_\beta)$. (Note that any right ideal of a ULG-semigroup contains at least one minimal right ideal hence some of the groups $\{G_\gamma\}$.) Write $\{x, xS\} = \{e_\alpha, e_\beta, S_1\}$, where S_1 is a subset of S. (We do not exclude that S_1 contains some further elements of E.) Multiplying by E we have

$$\{xE, xSE\} = \{e_\alpha E, e_\beta E, S_1 E\}.$$

Since $SE = L, e_\alpha E = e_\alpha$, $e_\beta E = e_\beta$, we have

$$\{xE, xL\} = \{e_\alpha, e_\beta, L_1\},$$

where L_1 is a subset of L. Finally since $xE \subset xL$ we get

$$xL = \{e_\alpha, e_\beta, L_1\}.$$

Hence there are two elements $g \in L, g_1 \in L$, such that

(3) $$xg = e_\alpha,$$

(4) $$xg_1 = e_\beta.$$

Since $L = \bigcup_{\nu \in M} G_\nu$, there are two indices $\gamma, \delta \in M$ such that $g \in G_\gamma$, $g_1 \in G_\delta$. Denote by g^{-1} the element of G_γ for which $gg^{-1} = e_\gamma$ and by g_1^{-1} the element of G_δ for which $g_1 g_1^{-1} = e_\delta$. Then (3) and (4) imply

$$xgg^{-1} = e_\alpha g^{-1}, \quad xg_1 g_1^{-1} = e_\beta g_1^{-1},$$

hence

$$xe_\gamma = e_\alpha g^{-1} \in e_\alpha L = G_\alpha, \quad xe_\delta = e_\beta g_1^{-1} \in e_\beta L = G_\beta.$$

Since $G_\alpha \cap G_\beta = \emptyset$, the elements xe_γ, xe_δ are different elements (contained in L). Hence xE contains at least two different elements (namely xe_γ, xe_δ) so that $|xE| \geqq 2$. This proves Lemma 2.

The semigroup S^* (being a union of right ideals of S) is a right ideal of S. But we easily show that S^* is also a left ideal of S (hence a two-sided ideal of S). Suppose that $x \in S^*$, i.e. $|xE| = 1$. Then for any $s \in S$ $(sx)E = s(xE)$ and since xE is a unique element (contained in L), we conclude $|(sx)E| = 1$, i.e. $sx \in S^*$, hence $SS^* \subset S^*$.

We have proved:

Theorem 4. *Let S be a ULG-semigroup with kernel L. Denote by E the set of all idempotents of L. Then there exists a unique largest subsemigroup S^* of S containing*

L such that S^* has an L-endomorphism. The semigroup S^* is a two-sided ideal of S and it can be characterised by the following two equivalent conditions:

a) *S^* is the set of all $x \in S$ such that $|xE|=1$.*

b) *S^* is the union of (disjoint) largest right ideals of S each of which is a homogroup.*

Remark. The emphasis in the second characterization is on the fact that the right ideals in $S^* = \bigcup_{\alpha \in M} R_\alpha^*$ are right ideals of S (and not merely of S^*).

Example 2. Consider the ULG-semigroup S given by the multiplication table

	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	a	b	c	d
d	a	a	a	d.

Here $L=E=\{a, b\}$. The semigroup S has no L-endomorphism. The largest right ideal R_a^* containing the idempotent a which is a homogroup is $R_a^*=\{a, d\}$. Next R_b^* is $\{b\}$ itself. We have $S^*=\{a, d\} \cup \{b\}$. The element c cannot be contained in a right ideal which is a homogroup, since $\{c, cS\}$ contains both idempotents a and b.

It is worth noting that $R_a^*=\{a, d\}$ is a homogroup, but not the largest homogroup containing a. The largest homogroup containing a is the subsemigroup $\{a, d, c\}$. (Of course this semigroup is not a right ideal of S.)

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