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Principal tolerance trivial commutative semigroups

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Following I. CHAJDA [1] an algebra A is said to be (principal) tolerance trivial if every (principal) tolerance on A is a congruence. In [2] B. ZELINKA has shown that a commutative semigroup S is tolerance trivial if and only if either S is a group or card $S=2$.

In this paper we shall describe all commutative semigroups which are principal tolerance trivial. Non-defined terminology and notation may be found in [3] and [4].

Recall that a tolerance T on a commutative semigroup S is a reflexive and symmetric subsemigroup of the direct product $S\times S$. For $a, b\in S$ we denote by $T(a, b)$ the least tolerance on S containing (a, b), i.e. $T(a, b)$ is the principal tolerance on S generated by (a, b). We shall use the following notation: $(a, b)^m z=(a^m z, b^m z)$ for all $a, b, z\in S$ and for every positive integer m. The set of all idempotents of a commutative semigroup S is denoted by $E(S)$ and is partially ordered by: $e\leqq f$ if and only if $ef=e$. We write $e<f$ for $e\leqq f$ and $e\neq f$. We denote by G_e the maximal subgroup of S containing an idempotent e. The notation S^1 stands for S if S has an identity, otherwise it stands for S with an identity adjoined.

The following lemma is clear:

Lemma 1. *Let S be a commutative semigroup and $a, b\in S$, $a\neq b$. For $x, y\in S$, $x\neq y$, we have $(x, y)\in T(a, b)$ if and only if there exist $z\in S^1$ and a positive integer m such that either $(x, y)=(a, b)^m z$ or $(x, y)=(b, a)^m z$.*

Note 1. Let S be a zero semigroup, i.e. card $S^2=1$. Using Lemma 1 it is easy to show that S is principal tolerance trivial.

Note 2. Now, we give another example of a principal tolerance trivial commutative semigroup. Let G be a commutative periodic group and let A be a non-empty set. Suppose that $G\cap A=\emptyset$ and put $S=G\cup A$. Let a multiplication on S be defined as follows:

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a) If $e, f \in A$, then $ef=e$ for $e=f$ and $ef=h$ for $e \neq f$, where h denotes the identity of G.

b) If $e \in A$ and $g \in G$, then $eg=g=ge$.

c) If $g_1, g_2 \in G$, then the product $g_1 g_2$ is the same as in G.

It is easy to show that S is a commutative semigroup which is a semilattice of groups. Clearly $E(S)=A \cup \{h\}$, $G_e=\{e\}$ for all $e \in A$ and $G_h=G$.

Now, we shall prove that S is a principal tolerance trivial semigroup. Let $a, b \in S$, $a \neq b$. It suffices to show that the relation $T(a, b)$ is transitive.

Case 1. Suppose that $a, b \in A$. It follows from Lemma 1 that $T(a, b)= R \cup R^{-1} \cup \mathrm{id}_S$, where $R=\{(a, b), (b, h), (h, a)\}$. Clearly $T(a, b)$ is transitive.

Case 2. Suppose that $a \in A$ and $b \in G$. Evidently $T(a, b)=T(b, a)$. Let (x, y), $(y, z) \in T(a, b)$ and $x \neq y$, $y \neq z$. It follows from Lemma 1 that $(x, y)=(a, b)^m u$ or $(x, y)=(b, a)^m u$ for some $u \in S^1$ and some positive integer m. Analogously we have $(y, z)=(a, b)^n v$ or $(y, z)=(b, a)^n v$ for some $v \in S^1$ and some positive integer n.

Subcase 2a. Assume that $x=au$, $y=b^m u=av$ and $z=b^n v$. Then $z=b^n v= b^n av=b^{m+n}u$ and so, by Lemma 1, we have $(x, z)=(a, b)^{m+n} u \in T(a, b)$.

Subcase 2b. Assume that $x=au$, $y=b^m u=b^n v$ and $z=av$. If $u=v$, then $x=z$ and so $(x, z) \in T(a, b)$. We can suppose that $u \neq v$. If $u, v \in G$, then $b^m u=b^n v$ and so $uv^{-1}=b^{n-m}=b^r$ for some positive integer r, because the group G is periodic. By Lemma 1, we have $(x, z)=(u, v)=(b, a)^r v \in T(a, b)$. If $u \in G$ and $v \in S^1 \setminus G$, then $b^m u=b^n$ and so $u=b^{n-m}=b^r$ for some positive integer r. Hence we have $(x, z)= (u, a)=(b, a)^r$ for $v \in \{1, a\}$ and $(x, z)=(u, h)=(b, a)^r v$ for $v \notin \{1, a\}$. This gives in both cases $(x, z) \in T(a, b)$. Analogously we can prove that $u \in S^1 \setminus G$ and $v \in G$ imply $(x, z) \in T(a, b)$. Let $u, v \in S^1 \setminus G$. Then it is easy to show that $(x, z) \in \{a, h\} \times \{a, h\}$. Since G is periodic, there exists a positive integer k such that $b^k=h$ and so $(a, h)=(a, b)^k$. Therefore we have $(x, z) \in T(a, b)$.

Subcase 2c. Assume that $x=b^m u$, $y=au=av$ and $z=b^n v$. Since b is a periodic element of G, there exists a positive integer r such that $b^{n-m}=b^r$. Thus we have $(x, z)=(b^m u, b^n v)=(a, b)^r b^m au \in T(a, b)$.

Subcase 2d. Assume that $x=b^m u$, $y=au=b^n v$ and $z=av$. Using the same method as in Subcase 2a we obtain that $(x, z) \in T(a, b)$.

Case 3. Suppose that $a, b \in G$. Let (x, y), $(y, z) \in T(a, b)$ and $x \neq y$, $y \neq z$. It follows from Lemma 1 that $x, y, z \in G$ and so $(x, z)=(x, y)(y^{-1}, y^{-1})(y, z) \in T(a, b)$.

Theorem. *A commutative semigroup S is principal tolerance trivial if and only if S satisfies one of the following conditions:*

(i) *S is group;*

(ii) *S is a zero semigroup;*

(iii) *S is of type defined in Note 2.*

Proof. Let S be a commutative semigroup. If S satisfies one of the conditions (i), (ii) or (iii), then S is principal tolerance trivial (see Notes 1 and 2).

Now, we shall prove the following lemmas, in which we shall suppose that the commutative semigroup S is principal tolerance trivial, $\operatorname{card} S^2 \geqq 2$ and S is not a group.

Lemma 2. *If* $a \in S \setminus a^2S$, *then* a^2 *is a zero in* S.

Proof. Let $a \in S \setminus a^2S$. Then $a \neq a^2$ and, by Lemma 1, we obtain (a, a^2), $(a^2, a^3) \in T(a, a^2)$. Since $T(a, a^2)$ is transitive, we have $(a, a^3) \in T(a, a^2)$. According to Lemma 1, there exists a $u \in S^1$ such that $(a, a^3) = (a, a^2)u$ and so $a^3 = a^2u = a^2$. Put $h = a^2$. Clearly $h^2 = h = ah$. Now, we shall show that $hx = h$ for all $x \in S$. Assume that $hb \neq h$ for some $b \in S$. If $hb = a$, then $a \in a^2S$, which is a contradiction. We have $hb \neq a$. It is clear that $(hb, h) = (hb, a)a$. According to Lemma 1, we have (a, hb), $(hb, h) \in T(a, hb)$ and so $(a, h) \in T(a, hb) = T(a, a^2b)$. It follows from Lemma 1 that $(a, h) = (a, hb)u$ for some $u \in S^1$. Hence we have $h = ah = ahbu = ahb = hb$, a contradiction. Therefore h is a zero in S.

Lemma 3. *Let S have a zero* 0 *and let* $a, b \in S$. *If* $a^2 = 0 = b^2$ *and* $a \neq 0 \neq b$, *then* $ab = 0$.

Proof. Assume that $ab \neq 0$. If $a = ab$, then $a = ab^2 = 0$, a contradiction. We have $a \neq ab$. By Lemma 1, we obtain (a, ab), $(ab, 0) \in T(a, ab)$, because $(ab, 0) = (a, ab)b$. Hence we have $(a, 0) \in T(a, ab)$. If $a = abu$ for some $u \in S^1$, then $ab = 0$, a contradiction. Lemma 1 implies that $(a, 0) = (a, ab)u$ for some $u \in S^1$. Then $ab = aub = 0$, a contradiction.

Lemma 4. *Let S have a zero* 0 *and let* $a, e \in S$. *If* $a^2 = 0$, $e^2 = e$ *and* $a \neq 0 \neq e$, *then* $ae = 0$.

Proof. Assume that $ae \neq 0$. We have $(e, 0), (0, ae) \in T(e, 0)$ and so $(e, ae) \in T(e, 0)$. If $e = ae$, then $e = a^2e = 0$, a contradiction. Hence we have $e \neq ae$ and so, by Lemma 1, $e = 0$ or $ae = 0$, which is a contradiction.

Lemma 5. *S is regular.*

Proof. Suppose that S is not regular. From Lemma 2 it follows that S has a zero 0. Since $\operatorname{card} S^2 \geqq 2$ by hypothesis, therefore there exist $a, b \in S$ such that $ab \neq 0$. According to Lemmas 2 and 3, a or b is a regular element of S. This implies that there exists an idempotent $e \neq 0$ in S. Evidently, S has an element $c \neq 0$, which is not regular. It follows from Lemma 2 that $c^2 = 0$ and Lemma 4 implies that $ce = 0$. Clearly $c \neq e$, and according to Lemma 1, we have (c, e), $(e, 0) \in T(e, c)$, because $(e, 0) = (e, c)e$. Thus $(c, 0) \in T(e, c)$. If $c = eu$ for some $u \in S^1$, then $0 = ce = c$, a contradiction. Hence, by Lemma 1, we obtain $(c, 0) = (c, e)u$ for some $u\in S^1$. Then $c=cu=cu^2$ and so $u^2\neq 0$. Lemma 2 implies that u is regular, which means that $u=u^2v$ for some $v\in S$. Hence we obtain $uv\neq 0$ and $(uv)^2=uv$. According to Lemma 4, we have $cuv=0$ and so $c=cu=(cuv)u=0$, a contradiction.

Lemma 6. *If* $e\leqq f<g$ *for* $e, f, g\in E(S)$, *then* $e=f$.

Proof. Assume that $e<f$. Then $e<g$ and $(f, e)=(g, e)f$. It follows from Lemma 1 that $(f, e), (e, g)\in T(e, g)$ and so $(f, g)\in T(e, g)$. By Lemma 1, we have either $f=ez$ or $g=ez$ for some $z\in S^1$. If $f=ez$, then $e=ef=f$, a contradiction. If $g=ez$, then analogously $e=eg=g$, a contradiction.

Lemma 7. *$E(S)$ is of the type defined in Note 2.*

Proof. It follows from Lemma 5 that $E(S)\neq\emptyset$. If $\operatorname{card} E(S)=1$, then S is a group, which is a contradiction. Hence we have $\operatorname{card} E(S)\geqq 2$. Our statement follows from Lemma 6.

Lemma 8. *S is periodic.*

Proof. It follows from Lemma 5 that S is a semilattice of maximal subgroups G_e $(e\in E(S))$. Suppose that there exists a $c\in S$ which is not periodic. Then $c\in G_e$ for some $e\in E(S)$. Clearly $c\neq e$. It follows from Lemma 7 that there exists an $f\in E(S)$ such that either $f<e$ or $e<f$.

Case 1. $f<e$. According to Lemma 1, we have (c, f), $(f, c^2)\in T(f, c)$ and so $(c, c^2)\in T(f, c)$. It follows from Lemma 1 that either $c=fu$ or $c^2=fu$ for some $u\in S^1$. Then either $e=fuc^{-1}$ or $e=fu(c^{-1})^2$ (c^{-1} denotes the inverse element of c in G_e). This gives in both cases $e=ef=f$, a contradiction.

Case 2. $e<f$. Then we have $(c, e)=(c, f)e$ and so, by Lemma 1, we obtain $(f, c), (c, e)\in T(f, c)$. By hypothesis we have $(f, e)\in T(f, c)$. Lemma 1 implies that either $(f, e)=(f, c)^m u$ or $(f, e)=(c, f)^m u$ for some $u\in S^1$ and some positive integer m. If $f=fu$ and $e=c^m u$, then $e=ef=c^m uf=c^m f=(c^m e)f=c^m$ and so c is periodic, a contradiction. If $f=c^m u$, then $e=ef=ec^m u=c^m u=f$, a contradiction.

Lemma 9. *If* $h<e$, $e, h\in E(S)$, *then* $\operatorname{card} G_e=1$.

Proof. Assume that there exists a $c\in G_e$ such that $c\neq e$. It follows from Lemma 8 that $c^k=e$ for some positive integer k. By Lemma 1, we have $(c, h), (h, e)\in T(h, c)$ and so $(c, e)\in T(h, c)$. It follows from Lemma 1 that either $c=hu$ or $e=hu$ for some $u\in S^1$. If $c=hu$, then $e=c^k=hu^k$ and so $h=he=e$, a contradiction. If $e=hu$, then analogously we have $h=e$, a contradiction.

The proof of Theorem follows from Lemmas 5, 6, 7, 8 and 9.

Corollary 1. *A semilattice is principal tolerance trivial if and only if its length is not greater than two.*

It is known (see [5] and [6]) that the set $\mathscr{L}(S)$ of all tolerances on a semigroup S forms a complete algebraic lattice with respect to set inclusion.

Corollary 2. *Let S be a tolerance trivial commutative semigroup. Then the lattice $\mathscr{L}(S)$ is modular.*

Proof. If S is a commutative group, then $\mathscr{L}(S)$ is the lattice of all congruences on S and so $\mathscr{L}(S)$ is modular. If S is a zero semigroup, then $\mathscr{L}(S)$ is the lattice of all reflexive and symmetric relations on S and so $\mathscr{L}(S)$ is distributive. If S is of the type defined in Note 2, then it follows from Theorem 1 of [7] that $\mathscr{L}(S)$ is modular.

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